Asymptotic expansions of the solutions of the heat equations with generalized functions $(\mathcal{S}_r^r)'$ initial value

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Definition 1 We denote by $(\mathcal{S}_r^r)'(\mathbb{R}^d)$ the dual space of the Gel'fand-Shilov space $\mathcal{S}_r^r(\mathbb{R}^d)$.

Main Theorem 1 *Let $U(x,t) \in C^\infty(\mathbb{R}^d \times (0,\infty))$ satisfy the following conditions:*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0\ , \ \text{in } \mathbb{R}^d \times (0,\infty).$$

$\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ *s.t.* $|U(x,t)|$

$$\leq C_\varepsilon \exp[\varepsilon(|x|^{\frac{1}{r}} + (1/t)^{1/(2r-1)})], \quad x \in \mathbb{R}^d,\ 0 < t < 1.$$

or

Every $\varepsilon > 0$, there is a positive constant $C_{\varepsilon,t}$ such that

$$|U(x,t)| \leq C_{\varepsilon,t} e^{\varepsilon|x|^2},\ x \in \mathbb{R}^d,\ t > 0,$$

Then $U(x,t)$ has the following asymptotic expansions:

$$U(x,t) \sim \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_x^k u, \ \left(u \in \left(\mathcal{S}_{\frac{1}{2}}^{\frac{1}{2}}\right)'(\mathbb{R}^d), \text{ such that } u = \lim_{t\to 0} U(x,t)\right).$$

Namely, for any even N,

$$\lim_{t\to 0} |\langle U(x,t), \varphi\rangle - \sum_{k=0}^{\frac{N}{2}} \frac{t^k}{k!} \langle \Delta_x^k u, \varphi\rangle |t^{-\frac{N}{2}} = 0,\ \varphi \in \mathcal{S}_{\frac{1}{2}}^{\frac{1}{2}}(\mathbb{R}^d),$$

where $\Delta_x = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$.

As an antecedent result about this investigation, K.Yoshino and Y.Oka also obtained the similar result for tempered distributions $\mathcal{S}'$, the Fourier-hyperfunctions$(\mathcal{S}_1^1)'$ and hyperfunctions $A'(K)$ (See [4] and [5]).

Moreover by choice of the space of a test function φ, we partially find the convergence of the asymptotic series as follows:

Proposition 1 *Let $u \in \mathcal{S}_p^p$, $p \geq 1/2$ and $\varphi \in \mathcal{S}_{1/2}^{1/2}$. Then the asymptotic series*

$$\sum_{n=0}^{\infty} \frac{\langle \Delta^n u, \varphi\rangle}{n!} t^n$$

converges in $\{|t| < 1/(2e^2)\rho^2 B^2\}$.

By using A. D. Sokal's result([3]), we obtain the fallowing result:

Proposition 2 *Let $U(x,t)$ satisfies the following conditions:*
(1) $U(x,t) \in C^\infty(\mathbb{R}^d \times (0,\infty))$,

(2) $\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0$,

(3) $\forall T > 0$, $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ *s.t.*

$$|U(x,t)| \leq C_\varepsilon \exp[\varepsilon(|x| + (1/t))], \ x \in \mathbb{R}^d, \ 0 < t < T, \ (r = 1).$$

Moreover for any $\varphi \in \mathcal{S}_1^1(\mathbb{R}^d)$, *we put*

$$b_n := \frac{\langle \Delta^n u, \varphi \rangle}{n!}, \ (u = \lim_{t \to 0} U(x,t))$$

and

$$f_B(\zeta) := \sum_{n=0}^{\infty} \frac{b_n}{n!} \zeta^n.$$

then we obtain

$$\langle U(x,t), \varphi \rangle = \frac{1}{t} \int_0^\infty f_B(\zeta) e^{-\zeta/t} d\zeta, \ t \in D_R,$$

where $D_R = \{t \in \mathbb{C} \mid Re\, t^{-1} > R^{-1}\}$.

References

[1] C. Dong and T. Matsuzawa, *$\mathcal{S}$-space of Gel'fand-Shilov and differential equations,* Japan. J. Math. Vol. 19, No.2, (1994), 227-239.

[2] I. M. Gel'fand and G. E. Shilov, Generalized Functions Vol. 2, *Academic press,* 1958.

[3] A. D. Sokal, *An improvement of Watson's theorem on Borel summability,* J. Math. Phys. 21 (2), February (1980), 261-263.

[4] K. Yoshino and Y. Oka, *Asymptotic expansions of the solutions to the heat equations with generalized functions initial value,* Complex Analysis and Potential Theory, Proceedings of the Conference Satellite to ICM 2006 (Gebze Institute of Technology, Turkey) (Tahir Aliyev Azeroğlu and Promarz M. Tamrazov eds.), World Scientific Publishing Co. Pte. Ltd (2007), 198-206.

[5] K.Yoshino and Y. Oka, *Asymptotic expansions of the solutions to the heat equations with hyperfunctions initial value,* Commun. Korean Math. Soc. 23, No. 4, (2008), 555-565

A construction of instanton-type solutions for Painlevé hierarchy by using multiple-scale analysis

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Abstract

In this talk, by using the multiple-scale analysis, we construct the instanton-type solutions of the first Painlevé hierarchy. We first recall the explicit form of the first Painlevé hierarchy $(P_{\mathrm{I}})_m$ ($m = 1, 2, \ldots,$) with a large parameter η (> 0)

$$\begin{cases} \eta^{-1}\dfrac{du_j}{dt} = 2v_j, & j = 1, 2, \ldots, m, \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \ldots, m, \end{cases} \tag{1}$$

where u_j, v_j are unknown functions with the conventional assumption $u_{m+1} \equiv 0$ and w_j denotes a polynomial of u_k and v_l defined by the following recursive relation

$$w_j = \frac{1}{2}\sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm} t. \tag{2}$$

Here c_j is a constant and δ_{jm} stands for the Kronecker's delta. If $m = 1$, $(P_{\mathrm{I}})_1$ is equivalent to the traditional first Painlevé equation with a large parameter η.

The instanton-type solutions of $(P_{\mathrm{I}})_m$ was first constructed by Y.Takei. ([T1],[T2]). Y. Takei constructed instanton-type solutions by using singular perturbative reduction of a Hamiltonian system to its Birkhoff normal form. On the other hand, by using the multiple-scale analysis, T. Aoki constructed instanton-type solutions to the second member of the first Painlevé hierarchy $(P_{\mathrm{I}})_2$ in [A]. In this talk, we generalize the method given in [A] so that it may be applied to $(P_{\mathrm{I}})_m$.

References

[A] T.Aoki, Multiple-Scale Analysis for Higher-Order Painlevé Equations, RIMS Kôkyûroku Bessatsu B2, 2007, 89-97.

[Ko] T.Koike, On the Hamiltonian structures of the second and the forth Painlevé hierarchies and degenerate Garnier systems, RIMS Kôkyûroku Bessatsu B2, 2007, 99-127.

[KKNT] T.Kawai, T.Koike, Y.Nishikawa and Y.Takei, On the Stokes geometry of higher order Painlevé equations, Astérisque, Vol. 297, 2004, 117-166.

[KT] T.Kawai and Y. Takei, Half of the Toulose Project Part 5 is completed - Structure theorem for instanton-type solutions of $(P_{\mathrm{J}})_m$ (J = I, II or IV) near a simple p-turning point of the first kind, RIMS Kôkyûroku Bessatsu B5, 2008, 99-115.

[T1] Y.Takei, Instanton-type formal solutions for the first Painlevé hierarchy, Algebraic Analysis of Differential Equations, Springer-Verlag, 2008, 307-319.

[T2] Y.Takei, Toward the exact WKB analysis for instanton-type solutions of Painlevé hierarchies, RIMS Kôkyûroku Bessatsu B2, 2007, 247-260.

WKB analysis for a holonomic system satisfied by the Pearcey integral

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In this talk, we consider the following integral:

$$u = \int \exp\left\{\eta\left(t^4 + x_2t^2 + x_1t\right)\right\} dt, \tag{1}$$

where η is a large parameter and the path of integration is taken to be rapidly decresing at infinity. This integral is called Pearcey integral.

The Pearcey integral (1) satisfies the following system of partial differential equations:

$$\begin{cases} \left(4\dfrac{\partial^3}{\partial x_1^3} + 2x_2\eta^2\dfrac{\partial}{\partial x_1} + x_1\eta^3\right)u = 0, \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)u = 0. \end{cases} \tag{2}$$

A WKB solution for (2) was constructed by Aoki ([A]).

In the meanwhile, it is well known that for fixed x_2 the integral (1) satisfies a third order differential equation discussed by Berk, Nevins and Roberts ([BNR]):

$$\left(4\frac{d^3}{dx_1^3} + 2x_2\eta^2\frac{d}{dx_1} + x_1\eta^3\right)u = 0. \tag{3}$$

This differential equation (3) has a virtual turning point and a new Stokes curve ([AKT, BNR]). By Aoki-Kawai-Takei ([AKT]), the connection formula for a new Stokes curve is obtained through the single valuedness of WKB solutions near a crossing at Stokes curves.

The purpose of this talk is to show that a connection formula for a new Stokes curve is obtained from a connection formula by considering exact WKB analysis for a holonomic system satisfied by the Pearcey integral in two variables.

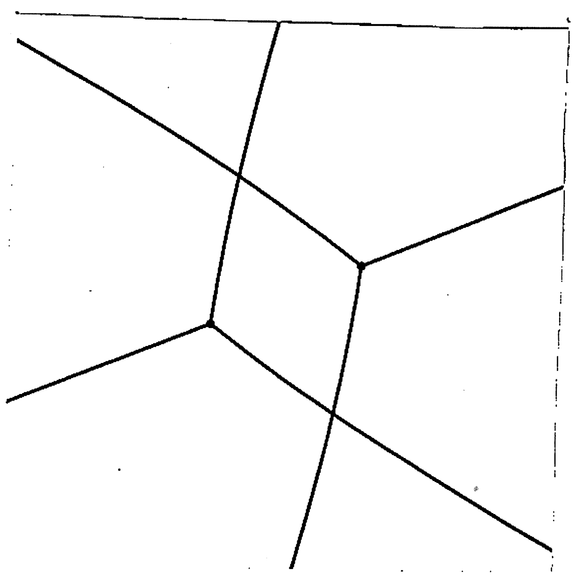
References

[A] T.Aoki. Toward the exact WKB analysis for a holonomic system. RIMS Kokyuroku, **1433**, 2005, pp. 1-8. (In Japanese.)

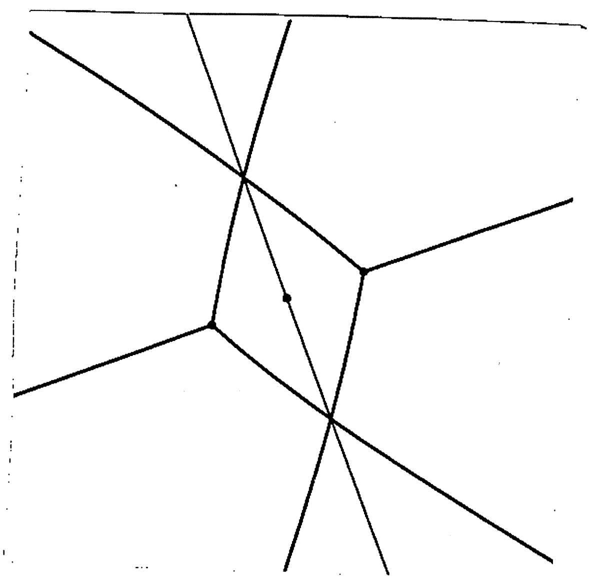
[AKT] T.Aoki, T.Kawai and Y.Takei, New turning points in the exact WKB analysis for higher order ordinary differential equations, Analysis algébrique des perturbations singulières, I ;Methodes resurgentes, Hermann, 1994, pp 69-84.

[BNR] H.L.Berk, W.M.Nevins and K.V.Roverts, New Stokes' line in WKB theory, J. Math. Phys., **23**, 1982, pp 988-1002.

Stokes geometry of (3) $\left(x_2 = \frac{1}{10}(1-i)\right)$



without virtual turning point and new Stokes curve



with virtual turning point and new Stokes curve

On the characterization of Stokes graphs for second order Fuchsian equations

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Let us consider a second order differential equation

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q(x)\right)\psi=0. \tag{1}$$

Here η is a large parameter and Q is a rational function. The exact WKB analysis is very powerful in studying global problems of Equation (1). For example, if it is Fuchsian, we can calculate the monodromy group under some generic assumptions([SAKT], [AKT], [KT]). In order to carry out such calculations, first we need to know the topological configuration of turning points (zeros of Q), singular points (poles of Q) and Stokes curves (real one-dimensional curves defined by Q). These objects form a (multi)graph on the Riemann sphere, which is called the Stokes graph of Equation (1). According to [SAKT] and [KT], if Fuchsian, any Stokes graph has the following remarkable properties; it is connected and each face is quadrilateral (in a wide sense). When the number of turning points is small, it was also confirmed, though experimentally, that these properties, combined with some trivial ones, characterize the set of Stokes graphs, that is, for any sphere graph having such properties, there exists a potential Q which realizes the given graph as its Stokes graph (up to topological isomrphism)([SAKT], [KT], [AI]).

In this talk, reviewing the basic properties of Stokes graphs, we show mathematically that the properties mentioned above indeed characterize the set of Stokes graphs for second order Fuchsian equations even if there are many turning points.

References

[AI] T.Aoki and T.Iizuka. Classification of Stokes Graphs of Second Order Fuchsian Differential Equations of Genus Two. *Publ. RIMS, Kyoto Univ.*, **43** (2007), 241–276.

[AKT] T.Aoki, T.Kawai and Y.Takei. Algebraic analysis of singular perturbations—on exact WKB analysis. *Sugaku Expositions*, **8** (1995), no.2, 217–240. (Originally appeared in Japanese in *Sugaku*, **45** (1993).)

[KT] T.Kawai and Y.Takei. *Algebraic Analysis of Singular Perturbation Theory*, Translations of Mathematical Monographs, **227** (AMS, 2005). (Originally published in Japanese in 1998 from Iwanami.)

[SAKT] M.Sato, T.Aoki, T.Kawai and Y.Takei. Algebraic Analysis of Singular Perturbation. *RIMS Kokyuroku*, **750** (1991), 43–51. (Notes by A.Kaneko, in Japanese.)