WKB analysis and small denominators for vector fields

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1. Introduction

In this talk we shall study the exact asymptotic analysis of a so-called homology equation and the normal form theory of a singular vector field. A homology equation is a system of partial differential equations which appear in linearizing a singular vector field by the change of independent variables. We shall introduce a (large) parameter in the homology equation by taking the eigenvalue(s) of the linearized vector field as a new parameter. This new parameter agrees with the one introduced by Aoki-Kawai-Takei for the Painlevé equation if we restrict the variable in the homology equation so that the homology equation reduces to the symmetric form of a Painlevé equation. (cf. [7]) By constructing the WKB solution of a homology equation, we will see that the classical Poincaré series solution of the homology equation coincides with the WKB solution via resummation procedure (and analytic continuation). Next we discuss the small denominator problem via resummed WKB solution and the appearence of a natural boundary of WKB solution.

2. Homology equation

Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $n \geq 2$ be the variable in $\mathbb{C}^n$. We consider a singular vector field near the origin of $\mathbb{C}^n$

$$\mathcal{X} = \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}, \quad a_j(0) = 0, \quad j = 1, \ldots, n,$$

where $a_j(x)\,(j = 1, 2, \ldots, n)$ are holomorphic in some neighborhood of the origin. We set

$$X(x) = (a_1(x), \ldots, a_n(x)), \quad \frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}),$$

and write

$$\mathcal{X} = X(x) \cdot \frac{\partial}{\partial x}, \quad X(x) = \Lambda x + R(x),$$

$$R(x) = (R_1(x), \ldots, R_n(x)), \quad R(x) = O(|x|^2),$$

where Λ is an n-square constant matrix.

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We want to linearize $\mathcal{X}$ by the change of variables,

$$(T) \qquad x = u(y), \quad u = (u_1, \ldots, u_n),$$

namely,

$$X(u(y))\frac{\partial y}{\partial x}\frac{\partial}{\partial y} = X(u(y))\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial}{\partial y} = \Lambda y\frac{\partial}{\partial y}.$$

It follows that u satisfies the equation

$$X(u(y))\left(\frac{\partial u}{\partial y}\right)^{-1} = \Lambda y,$$

that is

$$\Lambda u + R(u) = \Lambda y\frac{\partial u}{\partial y} \equiv \mathcal{L}u.$$

Now we introduce a parameter η^{-1} in the homology equation by regarding the eigenvalues of Λ as a new parameter. It follows that we obtain

$$\eta^{-1}\mathcal{L}u_j = \lambda_j u_j + R_j(u), \quad j = 1, \ldots, n.$$

In the following we write x instead of y for the sake of simplicity.

If $u = x + v(x)$, then by the same argument we obtain

$$(*)_\eta \qquad \eta^{-1}\mathcal{L}v_j = \lambda_j v_j + R_j(x + v(x)), \quad j = 1, \ldots, n.$$

Remark. If we restrict the variable of the homology equation to one variable appropriately, then the homology equation is reduced to the symmetric form of the Painlevé equation. Clearly, one can introduce a parameter η in the Schorödinger equation as the inverse of a Planck constant. Then by the monodromy preserving deformation, Aoki-Kawai-Takei introduced a parameter in the (symmetric form of) Painlevé equation.

$$\begin{aligned}
\eta^{-1}U_1' &= \lambda_1 + U_1(U_2 - U_3),\\
\eta^{-1}U_2' &= \lambda_2 + U_2(U_3 - U_1),\\
\eta^{-1}U_3' &= \lambda_3 + U_3(U_1 - U_2).
\end{aligned}$$

The above introduction of a large parameter in the homology equation agrees with that of the Painlevé equation, if the homology equation is reduced to the symmetric form of a Painlevé equation.

3. WKB solution

A WKB solution (0 - instanton solution)

A WKB solution (0 - instanton solution) $v(x, \eta)$ of $(*)_\eta$ is a formal power series solution of the form

$$v(x, \eta) = \sum_{\nu=0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1}v_1(x) + \cdots ,$$

where the series is a formal power series in η with coefficients $v_\nu(x)$ holomorphic vector functions in x in some open set independent of ν.

Definition (turning point). The point x such that

$$\det\left(\Lambda + (\partial R/\partial z)(x + v_0)\right) = 0 \tag{3.1}$$

is called a *turning point* of the equation $(*)_\eta$.

Remark. If

$$\lambda_j \neq 0, \qquad j = 1, \dots, n, \tag{A.1}$$

then the origin $x = 0$ is not a turning point of $(*)_\eta$ for any v_0, because $\det \Lambda \neq 0$. Then, we have

Proposition *Assume that* $\det \Lambda \neq 0$. *Then every coefficients of a WKB solution is uniquely determined as a holomorphic function of x in a neihborhood of the origin $x = 0$ independent of ν.*

Definition (Resonance condition). We say that η is *resonant*, if

$$\sum_{i=1}^{n} \lambda_i \alpha_i - \eta \lambda_j = 0, \tag{3.2}$$

for some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| \geq 2$ and $j, 1 \leq j \leq n$. If η is not resonant, we say that η is *nonresonant*.

Definition (Poincaré condition) *We say that a homolgy equation satisfies a Poincaré condition, if the convex hull of $\lambda_j, (j = 1, \dots, n)$ in the complex plane does not contain the origin.*

4. Summability of a WKB solution in a Poincaré region

For the direction ξ, $(0 \leq \xi < 2\pi)$ and the opening $\theta > 0$ we define the sector $S_{\xi,\theta}$ by

$$S_{\xi,\theta} = \left\{ \eta \in \mathbb{C};\ |\mathrm{Arg}\eta - \xi| < \frac{\theta}{2} \right\}, \tag{4.1}$$

where the branch of the argument is the principal value. Then we have

Theorem 1. (Resummation) *Suppose that*

$$|\ \mathit{Arg}\ \lambda_j| < \frac{\pi}{4}, \quad j = 1, \dots, n. \tag{C}$$

Then, there exist a direction ξ, an opening $\theta > \pi$, a neighborhood U of the origin $x = 0$ and $V(x, \eta)$ such that $V(x, \eta)$ is holomorphic in $(x, \eta) \in U \times S_{\xi,\theta}$ and satisfies $(*)_\eta$. The function $V(x, \eta)$ is a Borel sum of the WKB solution $v(x, \eta)$ in $U \times S_{\xi,\theta}$ when $\eta \to \infty$ in the following sense. Namely, for every $N \geq 1$ and $R > 0$, there exist $C > 0$ and

$K > 0$ such that

$$\left| V(x,\eta) - \sum_{\nu=0}^{N} \eta^{-\nu} v_\nu(x) \right| \le CK^N N! |\eta|^{-N-1}, \tag{4.2}$$
$$\forall (x,\eta) \in U \times S_{\xi,\theta}, \ |\eta| \ge R.$$

5. A Poincaré solution via analytic continuation of a WKB solution

We shall make an analytic continuation (with respect to η) of a resummed WKB solution to the right half plane. We note that there exist an infinte number of resonaces on the right-half plane $\Re\eta > 0$ which accumulate only at infinity. The solution may be singular with respect to η at the resonances. We have

Theorem 2. *Suppose that (C) is verified. Then the resummed WKB solution is analytically continued to the right half plane as a single-valued function except for resonances. If the nonresonance condition holds, then the analytic continuation of a resummed WKB solution to $\eta = 1$ coincides with a classical Poincaré solution of a homology equation.*

6. WKB solution in a Siegel domain- small denominators

In this section we assume that we are in a Siegel domain. Moreover, we assume, for the sake of simplicity

$\lambda_j \in \mathbb{R}$ $(j = 1, 2, \dots, n)$ *are linearly independent over* $\mathbb{Q}$.

Then the set of all resonances is dense on $\mathbb{R}$. This implies that the resummed WKB solution has a natural boundary on the real axis. We will study the small denominator problem from the viewpoint of a WKB analysis.

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