Instanton-type formal solutions for the first Painlevé hierarchy

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The first Painlevé hierarchy $(P_{\mathrm{I}})_m$ $(m = 1, 2, \ldots)$ with a large parameter η is the following system of first order nonlinear ordinary differential equations:

$$\begin{cases} \dfrac{du_j}{dt} = 2\eta v_j \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) \end{cases} \qquad (j = 1, \ldots, m)$$

where w_j is a polynomial of unknown functions $\{u_k, v_k\}_{k \le j}$ and constants $\{c_k\}_{k \le j}$ that is determined by the following recursive relations:

$$w_j = \frac{1}{2}\sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm} t.$$

In [1] and [2] we have clarified the structure of the Stokes geometry of $(P_{\mathrm{I}})_m$ (especially its relationship with that of its underlying Lax pair). We have also shown in [3] and [4] that a 0-parameter solution of $(P_{\mathrm{I}})_m$ can be locally reduced to that of the traditional first Painlevé equation (P_{I}) near its turning point of the first kind.

In this talk, as the next step toward establishment of the exact WKB analysis for the first Painlevé hierarchy $(P_{\mathrm{I}})_m$, we discuss the construction of instanton-type formal solutions of $(P_{\mathrm{I}})_m$ which are expected to play a central role in the description of the connection formula for $(P_{\mathrm{I}})_m$. To construct instanton-type formal solutions, we make use of the Hamiltonian structure of $(P_{\mathrm{I}})_m$; after rewriting $(P_{\mathrm{I}})_m$ in the Hamiltonian form and localizing it at a 0-parameter solution, we consider its transformation into the Birkhoff normal form. Using this approach, we may obtain formal solutions of $(P_{\mathrm{I}})_m$ with $2m$ free parameters. This construction is a generalization of the construction of 2-parameter solutions of traditional Painlevé equations discussed in [5].

References

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