Inverse Problem for Reconstructing Medium Singularity

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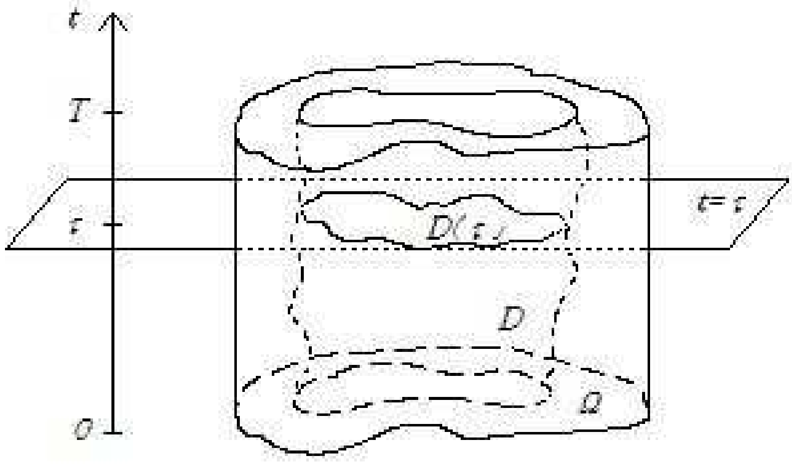
Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain. $\partial\Omega$ is C^2 if $n \geq 2$. Ω is considered as an isotropic heat conductive medium with heat conductivity:

$$\gamma(x,t) = \begin{cases} 1 & \text{in } \Omega \setminus \overline{D(t)} \\ k & \text{in } D(t) \end{cases} \qquad i.e. \qquad \gamma(x,t) = 1 + (k-1)\chi_{D(t)}$$

for each $0 \leq t \leq T$ with $0 < T < \infty$. Here $k > 0$ is a constant such that $k \neq 1$, $D(t)$ is a bounded domain with C^2 boundary $\partial D(t)$ such that $\overline{D(t)} \subset \Omega$, $\Omega \setminus \overline{D(t)}$ is connected, the dependency of $\partial D(t)$ on $t \in [0,T]$ is C^1 and $\chi_{D(t)}$ is the characteristic function of $D(t)$. The two dimensional figure of Ω and $D := \bigcup_{0 \leq t \leq T} D(t) \times \{t\}$ is given below.



We will use the following notations in this note. For any $E \subset \mathbb{R}^n$ and $a, b\,(a < b)$, $T > 0$, we denote $E_{(a,b)} := E \times (a,b)$ and $E_T := E \times (0,T)$.

We also use the standard notation $H^{p,q}(Q)$ for the Sobolev space when Q is either an open set in $\mathbb{R}^n_x \times \mathbb{R}_t$ or its lateral boundary $\partial_x Q := \partial Q \cap \mathbb{R}^n_x$, where $p \in \mathbb{R}$ and $q \in \mathbb{R}$ represent the L^2 regularity in x or $\sigma \in \partial_x Q$ and L^2 regularity in t, respectively. We also use the notation $H_0^{p,q}(Q) := \{u \in H^{p,q}(Q); u|_{\partial_x Q} = 0\}$ and $W(\Omega_T) := \{u \in H^{1,0}(\Omega_T); \partial_t u \in H^{-1,0}(\Omega)\}$ for the former case. Of course we have to be aware of some necessary conditions on Q, p, q.

Now, we consider the boundary value problem (MP):

$$\begin{cases} \big(P_D u\big)(x,t) := \partial_t u(x,t) - \operatorname{div}\big(\gamma(x,t)\nabla_x u(x,t)\big) = 0 \quad \text{in } \Omega_T \\ \partial_\nu u(x,t) = f(x,t) \quad \text{on } \partial\Omega_T, \quad u(x,0) = 0\,. \end{cases}$$

The physical meaning of u and f are the temperature and heat flux, respectively.

Theoremt(Unique Solvability)
For given $f \in H^{-\frac{1}{2},0}(\partial\Omega_T)$, there exists a unique solution u=u(f) $\in W(\Omega_T)$ to (MP).

Next, we define the Neumann to Dirichlet map Λ_D as follow.

Definition(Neumann-to-Dirichlet map)
Let $u(f)$ be the solution to (MP). Define $\Lambda_D : H^{-\frac{1}{2},0}(\partial\Omega_T) \to H^{\frac{1}{2},0}(\partial\Omega_T)$ by

$$\Lambda_D f := u(f) \quad \text{on } \partial\Omega_T.$$

The measurement Λ_D is to measure the temperature induced from inputting current or heat flux infinitely many times.

Now, we consider the inverse problems:

(IP) **Suppose k, D are unknown. Reconstruct D from Λ_D.**

Our main theorem is the following.

Theorem
If $n = 1$, there is a reconstruction procedure for the inverse problem for (IP). The details of the reconstruction procedure will be given in my talk.

The uniqueness and stability of the identification are known. See [1] and [2], respectively. However, the reconstruction has not been known. For the reconstruction, we tied to develop the analogue of probe method known for elliptic inverse problem. This is the first attempt to study the reconstruction for the inverse boundary value problem for non-stationary heat equation.

References

[1] A. Elayyan and V. Isakov, On uniqueness of recovery of the discontinuous conductivity coefficient of a parabolic equation, SIAM. J. Math. Anal. 28 No.1 49-59 (1997).

[2] M. Di Crist, L. Rondi and S. Vessella, oral communication.