Virtual Turning Points
— A gift of microlocal analysis to the exact WKB analysis

by

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A virtual turning point plays a crucially important role in the WKB analysis of higher order ordinary differential equations. As the notion of a virtual turning point does not find any similar precedents in the traditional asymptotic analysis, we begin our discussion by recalling its definition, which makes use of a bicharacteristic strip associated with the Borel transform P_B of an ordinary differential operator P with a large parameter.

We then discuss how it is relevant to the bifurcation of a Stokes curve — a phenomenon peculiar to higher order operators. Making use of the relevance we explain how and why concentrating our attention only upon ordinary turning points results in a contradiction.

It is of some interest to see how our attitude toward the notion has been changed: at first we were quite timid in introducing this novel notion and used a modest terminology "a new turning point" in

[1] T. Aoki, T. Kawai and Y. Takei: New turning points in the exact WKB analysis for higher-order ordinary differential equations, Analyse algébrique des perturbations singulières, I, ed. par L. Boutet de Monvel, Hermann, 1994, pp.69-84.

1

Later we became much more confident of its importance, but we were still somewhat nervous about the transcendental character of its definition. Thus, around the turn of the century, we changed the wording "a new turning point" to "a virtual turning point" as is used in

[2] T. Aoki, T. Kawai and Y. Takei: Exact WKB analysis of non-adiabatic transition probabilities for three-levels, J. Phys., **A35** (2002), 2401-2430.

The same attitude is also observed in our recent paper

[3] T. Aoki, T. Kawai, T. Koike and Y. Takei: On the global aspects of exact WKB analysis of operators admitting infinitely many phases, Contemporary Math., **373** (2005), 11-47,

but it is just because this is a report of a conference held in 2002.

We are now rather tempted to call a "virtual turning point" simply a "turning point", particularly after the publication of the paper

[4] T. Aoki, T. Kawai, S. Sasaki, A. Shudo and Y. Takei: Virtual turning points and bifurcation of Stokes curves for higher order ordinary differential equations, J. Phys., **A38** (2005), 3317-3336;

actually we are claiming:

Virtual is real, real is virtual

in

[5] T. Aoki, T. Kawai, T. Koike, S. Sasaki, A. Shudo and Y. Takei: A background story and some know-how of virtual turning points, RIMS Kôkyûroku, No. 1424 (ISSN 1880-2818), pp.53-63 (2005).