Bounded Trajectories in the Sixth Painlevé Dynamics

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The sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is a second-order nonlinear ordinary differential equation with an independent variable $x \in \mathbb{P}^1 - \{0, 1, \infty\}$ and an unknown function $q = q(x)$,

$$\begin{aligned} q_{xx} &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x}\right) q_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x}\right) q_x \\ &+ \frac{q(q-1)(q-x)}{2x^2(x-1)^2}\left\{\kappa_4^2 - \kappa_1^2 \frac{x}{q^2} + \kappa_2^2 \frac{x-1}{(q-1)^2} + (1-\kappa_3^2)\frac{x(x-1)}{(q-x)^2}\right\}, \end{aligned} \tag{1}$$

depending on complex parameters $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in a 4-dimensional affine space

$$\mathcal{K} = \{\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \,:\, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1\}. \tag{2}$$

Equation (1) is only a fragmentary appearance of a more substantial entity, called the sixth Painlevé dynamics and still denoted by the same symbol $\mathrm{P_{VI}}(\kappa)$, which is a time-dependent Hamiltonian dynamical system on the fibration

$$\pi_\kappa : \mathcal{M}(\kappa) \to T. \tag{3}$$

Here the *phase space* $\mathcal{M}(\kappa)$ is the total space of a family of moduli spaces of certain stable parabolic connections on $\mathbb{P}^1$ with four regular singular points; the base space T is the configuration space of ordered four points in $\mathbb{P}^1$, with $t \in T$ playing the role of *time variables*; and π_κ is the projection associating to each stable parabolic connection its ordered regular singular points. The fiber $\mathcal{M}_t(\kappa) = \pi_\kappa^{-1}(t)$ over $t \in T$, which is referred to as the *space of initial values* at time t, is the moduli space of stable parabolic connections with singularities fixed at t. The parameters $\kappa \in \mathcal{K}$ represent data on the local exponents of stable parabolic connections.

The space of initial values $\mathcal{M}_t(\kappa)$ has the structure of a generalized Halphen surface in Sakai's terminology, or that of an Okamoto-Painlevé pair in Saito-Takebe-Terajima's terminology. Namely, the moduli space $\mathcal{M}_t(\kappa)$ admits a natural compactification $\overline{\mathcal{M}}_t(\kappa)$ with a unique effective anti-canonical divisor $Y_t(\kappa) \in |-K_{\overline{\mathcal{M}}_t(\kappa)}|$ such that

$$\mathcal{M}_t(\kappa) = \overline{\mathcal{M}}_t(\kappa) - Y_t(\kappa)_{\mathrm{red}}, \tag{4}$$

where $\overline{\mathcal{M}}_t(\kappa)$ is known to be isomorphic to an 8-point blow-up of the Hirzebruch surface of degree 2. The divisor $Y_t(\kappa)$ is called the vertical leaf at time t, each point of which may be thought of as *a point at infinity* of the space $\mathcal{M}_t(\kappa)$.

The dynamical system $\mathrm{P_{VI}}(\kappa)$ enjoys the Painlevé property, that is, any local trajectory can be continued endlessly along any path on the time-variable space T to yield a global trajectory. Thus one can speak of the Poincaré return map homomorphism, or the nonlinear monodromy,

$$\mathrm{PRM}_\kappa : \pi_1(T, t) \to \mathrm{Aut}(\mathcal{M}_t(\kappa)), \tag{5}$$

which faithfully describes the multivaluedness of global trajectories. Then one can say that a $\pi_1(T, t)$-orbit in $\mathcal{M}_t(\kappa)$ just represents a global trajectory of $\mathrm{P_{VI}}(\kappa)$, while a point of the orbit signifies an initial value of the corresponding global trajectory.

Now we make a definition: A global trajectory is said to be *bounded* if the corresponding $\pi_1(T,t)$-orbit in $\mathcal{M}_t(\kappa)$ is bounded away from the vertical leaf $Y_t(\kappa)$, namely, a bounded trajectory is such a trajectory that remains bounded, accumulating no points at infinity, under the analytic continuation along any path in T. Then the following problem naturally occurs to us.

Problem. Determine all those parameters $\kappa \in \mathcal{K}$ for which $\mathrm{P}_{\mathrm{VI}}(\kappa)$ admits bounded trajectories. Moreover, given any such parameter $\kappa \in \mathcal{K}$, classify all the bounded trajectories of $\mathrm{P}_{\mathrm{VI}}(\kappa)$.

Being so difficult, at least to the speaker, this problem is far remote from its complete resolution. In this talk, we make some preliminary discussions about this problem and some related issues.