NON-LINEAR MONOTONE POSITIVE MAPS

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Abstract. We study several classes of general non-linear positive maps between C^*-algebras, which are not necessary completely positive maps. We characterize the class of the compositions of *-multiplicative maps and positive linear maps as the class of non-linear maps of boundedly positive type abstractly. We consider three classes of non-linear positive maps defined only on the positive cones, which are the classes of being monotone, supercongruent or concave. Any concave maps are monotone. The intersection of the monotone maps and the supercongruent maps characterizes the class of monotone Borel functional calculus. We give many examples of non-linear positive maps, which show that there exist no other relations among these three classes in general.

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1. Introduction

We study several classes of general non-linear positive maps between C^*-algebras. Ando-Choi [1] and Arveson [3] investigated non-linear completely positive maps and extend the Stinespring dilation theorem. Ando-Choi showed that any non-linear completely positive map is decomposed as a doubly infinite sum of compressions of completely positive linear maps on certain C^*-tensor products. Arveson obtained the similar expression for bounded completely positive complex-valued functions on the open unit ball of a unital C^*-algebra. Hiai-Nakamura [17] studied a non-linear counterpart of Arveson's Hahn-Banach type extension theorem [2] for completely positive linear maps. Beltita-Neeb [7] studied non-linear completely positive maps and dilation theorems for real involutive algebras. Recently Dadkhah-Moslehian [10] studied some properties of non-linear positive maps like Lieb maps and the multiplicative domain for 3-positive maps.

We study general non-linear positive maps between C^*-algebras, which are not necessary completely positive maps. First we study a non-completely positive variation of Stinespring type dilation theorem. Let A and B be C^*-algebras. We consider non-linear positive maps $\varphi : A \to B$. For instance, *-multiplicative maps, positive linear maps and their compositions are typical examples of non-linear positive maps. We characterize the class of the compositions of these algebraically simple maps as non-linear maps of boundedly positive type abstractly. This class is different with the class of non-linear completely positive maps, because the transpose map of the n by n matrix algebra for $n \geq 2$ is contained in the class. They are not necessarily real analytic.

Another typical example of non-linear positive mas is given as the functional calculus by a continuous positive function. See, for example, [4] , [5] and [23]. In particular operator monotone functions are important to study operator means in Kubo-Ando theory in [21]. Osaka-Silvestrov-Tomiyama [22] studied monotone operator functions on C^*-algebras. Recently Hansen-Moslehian-Najafi [14] characterize the continuous functional calculus by a operator convex function by being of Jensen-type. Moreover a sufficient condition is given by Ehsan [12].

We consider three classes of non-linear positive maps defined only on the positive cones, which are the classes of being monotone, supercongruent or concave. Let A be a C^*-algebra. We denote by A^+ be the cone of all positive elements. A non-linear positive map $\varphi : A^+ \to B^+$ between C^*-algebras A and B is said to be *monotone* if for any $x, y \in A^+$, $x \leq y$ implies that $\varphi(x) \leq \varphi(y)$. We say $\varphi : A^+ \to A^+$ is *supercongruent* if $c\varphi(a)c \leq \varphi(cac)$ for any $a \in A^+$ and any contraction $c \in A^+$. A positive map $\varphi : A^+ \to B^+$ is said to be *concave* if $\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(x)$ for any $x, y \in A^+$ and $t \in [0, 1]$.

Let $f : [0, \infty) \to [0, \infty)$ be a operator monotone *continuous* function, H a Hilbert space and $\varphi_f : B(H)^+ \to B(H)^+$ be a continuous functional calculus by f denoted by $\varphi_f(a) = f(a)$ for $a \in B(H)^+$. Then φ_f is a monotone, supercongruent, concave and normal positive map.

Let M be a von Neumann algebra on a Hilbert space H and $\varphi : M^+ \to M^+$ be the non-linear positive map defined by the $\varphi(a) =$ (the range projection of a) for $a \in M^+$. Then φ is monotone, supercongruent and normal . In fact, this map is a functional calculus of a by a Borel function $\chi_{(0,\infty)}$ on $[0, \infty)$.

In this paper we shall show that any concave maps are monotone. The intersection of the monotone maps and the supercongruent maps characterizes the class of monotone Borel functional calculus. We give many examples of non-linear positive maps, which show that there exist no other relations among these three classes.

We also discuss the ambiguity of operator means for non-invertible positive operators related with our theorem. Based on the theory of Grassmann manifolds, Bonnabel-Sepulchre [8] and Batzies-H́uper-Machado-Leite [6] introduced the geometric mean for positive semidefinite matrices or projections of fixed rank. Fujii [13] extends it to a general theory of means of positive semideinite matrices of fixed rank.

Noncommutative function theory is important and related to our paper. But the domain of a noncommutative function is graded, which is different with our simple one domain setting. Therefore we do not disscuss a relation with them here. It will be discussed in the future.

Finally we show a matrix version of the Choquet integral [9], the Sugeno integral [25] or more generally the inclusion-exclusion integral by Honda-Okazaki [19] for non-addiive monotone measures as another type of examples of non-linear monotone positive maps.

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2. Non-linear maps of boundedly positive type

Let A and B be C^*-algebras. We consider a non-linear positive map $\varphi : A \to B$. For instance, *-multiplicative maps, positive linear maps and their compositions are typical examples of non-linear positive maps. In this section, we characterize the class of the compositions of these algebraically simple maps as non-linear maps of boundedly positive type abstractly. This class is different with the class of non-linear completely positive maps, because the transpose map of the n by n matrix algebra for $n \geq 2$ is contained in the class. They are not necessarily real analytic.

Definition 2.1. Let A and B be C^*-algebras. A map $\varphi : A \to B$ is said to be of positive type if for any finite subset $\{a_1, a_2, \ldots, a_n\} \subset A$ and any finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{C}$

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha_i} \alpha_j \varphi(a_i^* a_j).$$

A map $\varphi : A \to B$ is said to be of boundedly positive type if φ is of positive type and for any $a \in A$, there exists a constant $K = K_a \geq 0$ such that for any finite subset $\{a_1, a_2, \ldots, a_n\} \subset A$ and any finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{C}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha_i} \alpha_j \varphi(a_i^* a^* a a_j) \leq K \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha_i} \alpha_j \varphi(a_i^* a_j).$$

Recall that a map $\varphi : A \to B$ is said to be positive if for any $a \in A$ $0 \leq \varphi(a^* a)$. Assume that A is unital. Then it is clear that if $\varphi : A \to B$ is of boundedly positive type, then φ is of positive type. If φ is of positive type, then φ is positive.

Example 2.2. Let A and B be C^*-algebras. If a map $\varphi : A \to B$ is a positive linear map, then φ is of boundedly positive type. In fact, for any $a \in A$, put $K = \|a\|^2 \geq 0$. Then for any finite subset $\{a_1, a_2, \ldots, a_n\} \subset A$ and any finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{C}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha_i} \alpha_j \varphi(a_i^* a_j) = \varphi((\sum_{i=1}^{n} \alpha_i a_i)^* (\sum_{j=1}^{n} \alpha_j a_j)) \geq 0$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a^*aa_j) = \varphi((\sum_{i=1}^{n}\alpha_i a_i)^*a^*a(\sum_{j=1}^{n}\alpha_j a_j))$$
$$\leq \|a\|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a_j).$$

Example 2.3. Let A and B be C^*-algebras. If a map $\varphi : A \to B$ is *-multiplicative, that is, $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ for any $a, b \in A$, then φ is of boundedly positive type. In fact, for any $a \in A$, put $K = \|\varphi(a)\|^2$. Then for any finite subset $\{a_1, a_2, \dots, a_n\} \subset A$ and any finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathbb{C}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a_j) = (\sum_{i=1}^{n}\alpha_i\varphi(a_i))^*(\sum_{j=1}^{n}\alpha_j\varphi(a_j)) \geq 0$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a^*aa_j) = (\sum_{i=1}^{n}\alpha_i\varphi(a_i))^*\varphi(a)^*\varphi(a)(\sum_{j=1}^{n}\alpha_j\varphi(a_j))$$
$$\leq \|\varphi(a)\|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a_j).$$

For example the determinant $\det : M_n(\mathbb{C}) \to \mathbb{C}$ is of boundedly positive type. Let $B = A \otimes_{min} \cdots \otimes_{min} A$ and $\varphi : A \to B$ be defined by $\varphi(a) = a \otimes \cdots \otimes a$, then φ is of boundedly positive type.

We shall study the class of maps of boundedly positive type. Let A, B and C be unital C^*-algebras. If $\varphi_1 : A \to C$ is *-multiplicative and $\varphi_2 : C \to B$ is a positive linear map, then the composition $\varphi = \varphi_2 \circ \varphi_1$ is of boundedly positive type. Conversely any map of boundedly positive type is of this form.

Theorem 2.4. *Let A and B be unital C^*-algebras. Consider a map $\varphi : A \to B$. Then the following are equivalent:*

(1) *φ is of boundedly positive type.*
(2) *There exists a unital C^*-algebra C, a *-multiplicative map $\varphi_1 : A \to C$ and a positive linear map $\varphi_2 : C \to B$ such that φ is the composition $\varphi = \varphi_2 \circ \varphi_1$ of these maps.*

Proof. (2) $\Rightarrow$ (1): For any $a \in A$, put $K = \|\varphi_1(a)\|^2$. Then for any finite subset $\{a_1, a_2, \dots, a_n\} \subset A$ and any finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathbb{C}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a_j) = \varphi_2((\sum_{i=1}^{n}\alpha_i\varphi_1(a_i))^*(\sum_{j=1}^{n}\alpha_j\varphi_1(a_j))) \geq 0$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a^*aa_j)\\
=&\varphi_2((\sum_{i=1}^{n}\alpha_i\varphi_1(a_i))^*\varphi_1(a)^*\varphi_1(a)(\sum_{j=1}^{n}\alpha_j\varphi_1(a_j)))\\
\le&\|\varphi_1(a)\|^2\varphi_2((\sum_{i=1}^{n}\alpha_i\varphi_1(a_i))^*(\sum_{j=1}^{n}\alpha_j\varphi_1(a_j)))\\
\le&\|\varphi_1(a)\|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\alpha_i}\alpha_j\varphi(a_i^*a_j).
\end{aligned}$$

(1) $\Rightarrow$ (2): Let $\mathcal{S}_A$ be a *-semigroup defined by $\mathcal{S}_A = A$ as a set with the product ab and the involution a^* borrowed from A. Consider an algebraic *-semigroup algebra $\mathbb{C}[\mathcal{S}_A]$ of $\mathcal{S}_A$ with linear basis $\{u_a \mid a \in A\}$, that is,

$$\mathbb{C}[\mathcal{S}_A] = \{x = \sum_i x_i u_{a_i} \mid x_i \in \mathbb{C}, a_i \in A\}$$

with the product $u_a u_b = u_{ab}$ and the involution $(u_a)^* = u_{a^*}$ for $a, b \in A$. Thus $\mathbb{C}[\mathcal{S}_A]$ is an algebraic *-algebra. Define a linear map $\psi : \mathbb{C}[\mathcal{S}_A] \to B$ by

$$\psi(\sum_i x_i u_{a_i}) = \sum_i x_i \varphi(a_i).$$

For any state ω on B, define a linear functional ψ_ω on $\mathbb{C}[\mathcal{S}_A]$ by $\psi_\omega = \omega \circ \psi$. We introduce a pre-inner product $\langle\ ,\ \rangle_\omega$ on $\mathbb{C}[\mathcal{S}_A]$ by

$$\langle x, y\rangle_\omega := \psi_\omega(y^*x) = \omega(\sum_i\sum_j \overline{y_j}x_i\varphi(b_j^*a_i))$$

for $x = \sum_i x_i u_{a_i},\ y = \sum_i y_j u_{b_j} \in \mathbb{C}[\mathcal{S}_A]$, $(x_i, y_j \in \mathbb{C}, a_i, b_j \in A)$. Then

$$\langle x, x\rangle_\omega = \psi_\omega(x^*x) = \omega(\sum_i\sum_j \overline{x_j}x_i\varphi(a_j^*a_i)) \ge 0,$$

since φ is of positive type. Let $N_\omega := \{x \in \mathbb{C}[\mathcal{S}_A] \mid \langle x, x\rangle_\omega = 0\}$. Define a Hilbert space H_ω by the completion of $\mathbb{C}[\mathcal{S}_A]/N_\omega$. Let $\eta_\omega : \mathbb{C}[\mathcal{S}_A] \to H_\omega$ be the canonical map such that $\langle\eta_\omega(x), \eta_\omega(y)\rangle = \langle x, y\rangle_\omega$. Then we have a *-representation (π_ω, H_ω) of $\mathbb{C}(\mathcal{S}_A)$ on H_ω such that $\pi_\omega(u_a)\eta_\omega(x) = \eta_\omega(u_a x)$ and $\pi_\omega(u_a)^* = \pi_\omega(u_a^*) = \pi_\omega(u_{a^*})$ for $a \in A$. In fact

$$\begin{aligned}\|\eta_\omega(u_a x)\|^2 &= \langle \sum_i x_i u_{aa_i}, \sum_j x_j u_{aa_j}\rangle_\omega \\ &= \omega(\sum_i \sum_j \overline{x_j} x_i \varphi(a_j^* a^* a a_i)) \\ &\leq K\omega(\sum_i \sum_j \overline{x_j} x_i \varphi(a_j^* a_i)) = K\|\eta_\omega(x)\|^2\end{aligned}$$

since φ is of boundedly positive type, where K depends only on a. Therefore $\pi_\omega(u_a)$ is a well defined bounded operator with $\|\pi_\omega(u_a)\| \leq \sqrt{K}$. Since A has a unit I, we have that

$$\langle \pi_\omega(u_a)\eta_\omega(u_I), \eta_\omega(u_I)\rangle = \langle \eta_\omega(u_a u_I), \eta_\omega(u_I)\rangle = \omega(\psi(u_a)) = \omega(\varphi(a)).$$

Moreover for $x = \sum_i x_i u_{a_i} \in \mathbb{C}[\mathcal{S}_A]$, we have that

$$\omega(\psi(x)) = \langle \pi_\omega(x)\eta_\omega(u_I), \eta_\omega(u_I)\rangle.$$

Next we shall consider a φ-universal representation (π_U, H_U) of a *-algebra $\mathbb{C}[\mathcal{S}_A]$ as follows:

$$(\pi_U, H_U) = \bigoplus\{(\pi_\omega, H_\omega) \mid \omega \text{ is a state on } B\}.$$

For $x \in \mathbb{C}[\mathcal{S}_A]$ define the φ-universal seminorm

$$\|x\|_U := \sup\{\|\pi_\omega(x)\| \mid \omega \text{ is a state on } B\} \leq \sum_i |x_i|\sqrt{K_{a_i}} < \infty.$$

Let C be the completion of $\mathbb{C}[\mathcal{S}_A]/\mathrm{Ker}\pi_U$ by the induced norm $\|[x]\|_U$. Then C is a C^*-algebra and isomorphic to the closure of $\pi_U(\mathbb{C}[\mathcal{S}_A])$. We also have a *-representaion $(\tilde{\pi_U}, H_U)$ of C such that $\tilde{\pi_U}([x]) = \pi_U(x)$.

Next we shall show that for $x \in \mathbb{C}[\mathcal{S}_A]$

$$\|\psi(x)\| \leq 2\|\varphi(I)\|\|x\|_U.$$

In fact, since

$$\begin{aligned}&|\omega(\psi(x))| = |\langle \pi_\omega(x)\eta_\omega(u_I), \eta_\omega(u_I)\rangle| \\ \leq&\|\pi_\omega(x)\|\|\eta_\omega(u_I)\|^2 = \|\pi_\omega(x)\|\omega(\varphi(I)) \leq \|\varphi(I)\|\|\pi_u(x)\|,\end{aligned}$$

we have that

$$\begin{aligned}\|\psi(x)\| &\leq 2(\text{ the numerical radius of } \psi(x)) \\ &\leq 2\sup\{|\omega(\psi(x))| \mid \omega \text{ is a state on } B\} \\ &\leq 2\|\varphi(I)\|\|\pi_U(x)\| = 2\|\varphi(I)\|\|x\|_U.\end{aligned}$$

Therefore $\mathrm{Ker}\pi_U \subset \mathrm{Ker}\psi$. Hence there exists a linear map $\tilde{\psi} : \mathbb{C}[\mathcal{S}_A]/\mathrm{Ker}\pi_U \to B$ such that $\tilde{\psi}([x]) = \psi(x)$. Moreover $\tilde{\psi}$ extends to a linear map

$\varphi_2 : C \to B$ by the boundedness of $\tilde{\psi}$. Then φ_2 is a positive linear map. In fact, for $x \in \mathbb{C}[\mathcal{S}_A]$, since φ is of positive type,

$$\varphi_2([x^*x]) = \psi(x^*x) = \psi(\sum_i \sum_j \overline{x_j} x_i \varphi(a_j^* a_i)) \geq 0.$$

Any positive element in C can be approximated with these $[x^*x]$ for $x \in \mathbb{C}[\mathcal{S}_A]/\mathrm{Ker}\pi_U$. By the continuity of φ_2 , φ_2 is also positive.

We define $\varphi_1 : A \to C$ by $\varphi_1(a) = [u_a]$. Then φ_1 is *-multiplicative. Moreover

$$\varphi_2 \circ \varphi_1(a) = \varphi_2([u_a]) = \psi(u_a) = \varphi(a)$$

for $a \in A$. □

Definition 2.5. Let A and B be C^*-algebras. For a map $\varphi : A \to B$ and a natural number n, $\varphi_n : M_n(A) \to M_n(B)$ is defined by $\varphi_n((a_{ij})_{ij}) = (\varphi(a_{ij}))_{ij}$. Then φ is completely positive if φ_n is positive for any n. φ is said to be positive definite if, for any n and for any $\{a_1, a_2, \dots, a_n\} \subset A$, $(\varphi(a_i^* a_j))_{ij} \in M_n(B)$ is positive as in [7, Definition 2.8] .

Remark 2.6. (1) Let A and B be C^*-algebras. If a map $\varphi : A \to B$ is a non-linear map. If φ is completely positive, then φ is positive definite. If φ is positive definite,then φ is of positive type. But the converses do not hold. In fact, the transpose map of the n by n matrix algebra for $n \geq 2$ is of positive type but is not positive definite.

(2) We should note that the class of completely positive maps and the class of maps of boundedly positive type are different. For example, let $A = B = \mathbb{C}$ and $\varphi(z) = e^z$. Then φ is completely positive but is not of boundedly positive type. In fact, there exist no constant $K > 0$ such that for any $z \in \mathbb{C}$, $\varphi(z^* 3^2 z) \leq K\varphi(z^* z)$. The transpose map of the $n \times n$ matrix algebra for $n \geq 2$ is of boundedly positive type but is not completely positive.

3. Some classes of non-linear positive maps defined only on the positive cones

Let A be a C^*-algebra. We denote by A^+ be the cone of all positive elements. In this section we consider non-linear positive maps defined only on the positive cones.

Definition 3.1. Let A and B be C^*-algebras. A non-linear positive map $\varphi : A^+ \to B^+$ is said to be *monotone* if for any $x, y \in A^+$, $x \leq y$ implies that $\varphi(x) \leq \varphi(y)$. We say $\varphi : A^+ \to A^+$ is *supercongruent* if $c\varphi(a)c \leq \varphi(cac)$ for any $a \in A^+$ and any contraction $c \in A^+$. A positive map $\varphi : A^+ \to B^+$ is said to be *concave* if $\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(x)$ for any $x, y \in A^+$ and $t \in [0, 1]$.

When A and B are von Neumann algebras, $\varphi : A^+ \to B^+$ is said to be *normal* if , for any bounded increasing net $a_\nu \in A^+$,

$$\varphi(\sup_\nu a_\nu) = \sup_\nu \varphi(a_\nu).$$

It is clear that, the property for φ,

$$c^*\varphi(a)c \leq \varphi(c^*ac)$$

for any $a \in A^+$ and any contraction $c \in A$, is stronger than the supercongruence of φ. We will sometime show this strong property instead of the supercongruence of φ.

Example 3.2. Let $f : [0, \infty) \to [0, \infty)$ be a operator monotone continuous function, H a Hilbert space and $\varphi_f : B(H)^+ \to B(H)^+$ be a continuous functional calculus by f denoted by $\varphi_f(a) = f(a)$ for $a \in B(H)^+$. Then φ_f is a monotone, supercongruent, concave and normal positive map.

There exists a non-linear positive map $\varphi : B(H)^+ \to B(H)^+$ which is monotone, supercongruent and normal but is not a continuous functional calculus. For example, let $\varphi(a)$ be the projection onto the closure of the range of $a \in B(H)^+$, then $\varphi(a)$ is called the range projection of a or the supprot projection of a and is equal to the projection onto the orthogonal complement of the kernel of a ([24, 2.22]).

Proposition 3.3. *Let M be a von Neumann algebra on a Hilbert space H and $\varphi : M^+ \to M^+$ be the non-linear positive map defined by the $\varphi(a) =$ (the range projection of a) for $a \in M^+$. Then φ is is monotone, supercongruent and normal.*

Proof. For $a, b \in M^+$, we remark the following facts:

- $\varphi(a) \leq \varphi(b)$ is equivalent to $\mathrm{Ker}(a) \supset \mathrm{Ker}(b)$.
- $\mathrm{Ker}(a) = \mathrm{Ker}(\varphi(a))$.
- $\varphi(a) \geq a$ if $\|a\| \leq 1$.
- $\varphi(a) = a$ if a is a projection.

For $0 \leq a \leq b$, it is clear that $\mathrm{Ker}(a) \supset \mathrm{Ker}(b)$. So φ is monotone.

For any contraction c, we have

$$c^*ac\xi = 0 \Rightarrow a^{1/2}c\xi = 0 \Rightarrow \varphi(a)c\xi = 0 \Rightarrow c^*\varphi(a)c\xi = 0.$$

Since $\mathrm{Ker}(c^*ac) \subset \mathrm{Ker}(c^*\varphi(a)c)$, $\varphi(c^*\varphi(a)c) \leq \varphi(c^*ac)$. Because $c^*\varphi(a)c$ is a contraction,

$$c^*\varphi(a)c \leq \varphi(c^*\varphi(a)c) \leq \varphi(c^*ac).$$

So φ is supercongruent.

Let $\{a_\nu\}$ be a bounded increasing net in M^+. Since

$$\mathrm{Ker}(\sup_\nu \varphi(a_\nu)) = \bigcap_\nu \mathrm{Ker}(\varphi(a_\nu)) = \bigcap_\nu \mathrm{Ker}(a_\nu) = \mathrm{Ker}(\sup_\nu a_\nu)$$

and $\sup_\nu \varphi(a_\nu)$ is a projection, we have

$$\varphi(\sup_\nu a_\nu) = \varphi(\sup_\nu \varphi(a_\nu)) = \sup_\nu \varphi(a_\nu).$$

So φ is normal on M^+. $\square$

We shall study and compare these properties of being monotone, supercongruent and concave for general non-linear positive maps $\varphi : A^+ \to A^+$ on the whole positive cone A^+ of a C^*-algebra A. If φ is concave, then φ is monotone. But there exist no other relations between them in general as follows:

Proposition 3.4. *Let A and B be C^*-algebras and $\varphi : A^+ \to B^+$ be a non-linear positive map. If φ is concave, then φ is monotone.*

Proof. We assume that $0 \le a \le b$ and $0 < t < 1$. Then $\frac{1}{1-t}(b - ta) \ge 0$ and

$$b = ta + (1-t)\frac{1}{1-t}(b - ta).$$

By the concavity of φ, we have

$$\varphi(b) \ge t\varphi(a) + (1-t)\varphi(\frac{1}{1-t}(b - ta)) \ge t\varphi(a).$$

When t tends to 1, this implies $\varphi(a) \le \varphi(b)$. $\square$

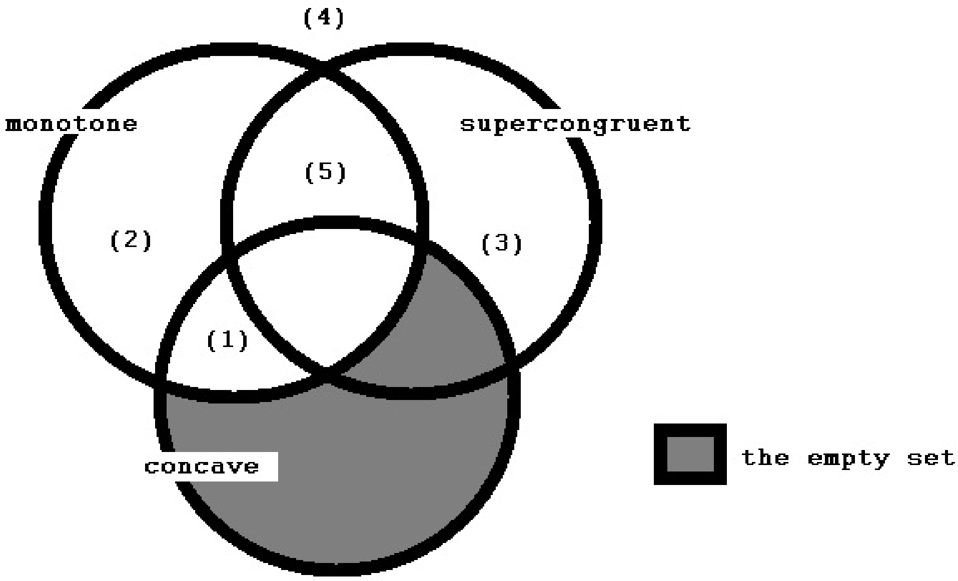
Proposition 3.5. *There exist many non-linear positive maps on the positive cones of some C^*-algebras which satisfy anyone of the following conditions:*

(1) *φ is concave and not supercongruent.*
(2) *φ is monotone, not concave and not supercongruent.*
(3) *φ is not monotone and supercongruent.*
(4) *φ is not monotone and not supercongruent.*
(5) *φ is monotone, supercongruent and not concave.*

Proof. In each case, this was verified by constructing many concrete examples in the below. $\square$

Remark 3.6. (1) When M is an infinite dimensional factor, we shall show that a map $\varphi : M \longrightarrow M$ is concave if φ is monotone and supercongruent in the next section.

(2) Related to the statement of Theorem 3.5, we can get the following figure:



Example (1-1). Let M be a II_1-factor and τ the trace on M. Define $\varphi : M^+ \to M^+$ by $\varphi(a) = \tau(a)\mathbf{1}$ for $a \in M^+$. Since φ is the restriction of a linear map, φ is concave. Let p be a projention in M with $\tau(p) = \frac{1}{2}$. Since

$$p\tau(\mathbf{1})p = p \not\leq \tau(p\mathbf{1}p)\mathbf{1} = \tau(p)\mathbf{1} = \frac{1}{2}\mathbf{1},$$

φ is not supercongruent.

Example (1-2). Let H be a Hilbert space and $M = B(H)$. Condsider a projection $p(\neq \mathbf{1})$ of $B(H)$ and take a vector $\xi \in pH$ with $\|\xi\| = 1$. Define $\varphi : M^+ \to M^+$ by $\varphi(a) = \langle a\xi, \xi\rangle\mathbf{1}$ for $a \in B(H)^+$. Since φ is the restriction of a linear map, φ is concave. By the fact

$$(\mathbf{1}-p)\varphi(p)(\mathbf{1}-p) = \mathbf{1} - p \not\leq \varphi((\mathbf{1}-p)p(\mathbf{1}-p) = 0,$$

φ is not supercongruent.

Example (2-1). Let $H = \ell^2(\mathbb{N})$ and $M = B(H)$. Consider a maximal abelian *-subalgebra $A \cong \ell^\infty(\mathbb{N})$ of $B(H)$ and a conditional expectation E of $B(H)$ onto A. We define $\varphi(a) = E(a)^2$ for $a \in B(H)^+$. Since E is positive linear map and the mapping $\mathcal{A}^+ \ni a \mapsto a^2 \in A^+$ is monotone, φ is monotone. By the fact

$$\frac{\varphi(0\mathbf{1}) + \varphi(2\mathbf{1})}{2} = 2\mathbf{1} \not\leq \mathbf{1} = \varphi(\mathbf{1}) = \varphi(\frac{0\mathbf{1} + 2\mathbf{1}}{2}),$$

$$\frac{\mathbf{1}}{2}\varphi(\mathbf{1})\frac{\mathbf{1}}{2} = \frac{\mathbf{1}}{4} \not\leq \frac{\mathbf{1}}{16} = \varphi(\frac{\mathbf{1}}{4}) = \varphi(\frac{\mathbf{1}}{2} \cdot \mathbf{1} \cdot \frac{\mathbf{1}}{2}),$$

φ is not concave and not supercongruent.

Example (2-2). Let $f:[0,\infty)\longrightarrow[0,\infty)$ as follows:

$$f(t)=\begin{cases}\frac{-4}{t+1}+4 & 0\le t\le 1\\ t+1 & t\ge 1\end{cases}.$$

Define $\varphi: M_2(\mathbb{C})\longrightarrow M_2(\mathbb{C})$ by the functional calculus by f. Since

$$f'(t)=\begin{cases}\frac{4}{(t+1)^2} & 0\le t\le 1\\ 1 & t\ge 1\end{cases},$$

is monotone and convex on $[0,\infty)$ and the matrix

$$\begin{pmatrix} f'(t) & \frac{f''(t)}{2!}\\ \frac{f''(t)}{2!} & \frac{f'''(t)}{3!}\end{pmatrix}$$

is positive on $[0,\infty)\setminus\{1\}$. By [11, p.82 Theorem IV], f is 2-matrix monotone on $[0,\infty)$, that is, φ is monotone. It is clear that $f(t)$ is not continuously twice differentiable. By [20], [16, Theorem 2.4.4], $f(t)$ is not 2-matrix concave, that is, φ is not concave.

For

$$a=\begin{pmatrix}3/2 & 1/2\\ 1/2 & 3/2\end{pmatrix}\in M_2(\mathbb{C})^+ \text{ and } 0\le c=\begin{pmatrix}1 & 0\\ 0 & \frac{4}{5}\end{pmatrix}\le \mathbf{1}_2,$$

we have that

$$c\varphi(a)c=\begin{pmatrix}5/2 & 2/5\\ 2/5 & 8/5\end{pmatrix},\quad cac=\begin{pmatrix}3/2 & 2/5\\ 2/5 & 24/25\end{pmatrix}$$

$$\text{and } \varphi(cac)=\begin{pmatrix}\frac{2297}{948}+\frac{3155}{948\sqrt{2329}} & \frac{437}{1185}+\frac{2651}{1185\sqrt{2329}}\\ \frac{437}{1185}+\frac{2651}{1185\sqrt{2329}} & \frac{22813}{11850}+\frac{3649}{11850\sqrt{2329}}\end{pmatrix}.$$

Since the $(1,1)$-compornent of the matrix $\varphi(cac)-c\varphi(a)c$ is negative, it does not hold $c\varphi(a)c\le\varphi(cac)$. So φ is not supercongruent.

Example (3-1). Let H be a Hilbert space and $M=B(H)$. For $a\in B(H)^+$, define $\varphi(a)=\begin{cases}\mathbf{1} & \|a\|\le 1\\ a & \|a\|>1\end{cases}$. Let $p(\neq\mathbf{1})$ be a projection. Then we have

$$\varphi(\frac{1}{2}p)=\mathbf{1}\not\le\varphi(2p)=2p.$$

So φ is not monotone.

Let $c\in B(H)$ be a contraction. If $\|a\|\le 1$, then $c^*\varphi(a)c=c^*c\le\mathbf{1}=\varphi(c^*ac)$. If $\|a\|>1$, then $\varphi(a)=a$ and

$$\begin{aligned}\varphi(c^*ac)&=\begin{cases}c^*ac, & \|c^*ac\|>1\\ \mathbf{1}, & \|c^*ac\|\le 1\end{cases}\\ &\ge c^*\varphi(a)c.\end{aligned}$$

So φ is supercongruent.

Example (3-2). Let M be a II_1-factor. Define $\varphi : M^+ \to M^+$ by $\varphi(a) = \begin{cases} \mathbf{1} & a \text{ is invertible} \\ 2\mathbf{1} & a \text{ is not invertible} \end{cases}$, for $a \in M^+$. It is clear that φ is not monotone, since, for any non invertible positive contraction a,

$$\varphi(a) = 2\mathbf{1} \geq \mathbf{1} = \varphi(\mathbf{1}), \text{ and } a \leq \mathbf{1}.$$

If a is invertible, then

$$c^*\varphi(a)c = c^*c \leq \mathbf{1} \leq \varphi(c^*ac).$$

If a is not invertible,

$$c^*\varphi(a)c = 2c^*c \leq 2\mathbf{1} = \varphi(c^*ac),$$

where we use the fact that a left invertible element in a factor of type II_1 is invertible. So we have that φ is supercongruent.

Example (3-3). Let M be a II_1-factor and τ the normalized trace on M. Let $\alpha : [0,1] \longrightarrow [0,\infty)$ be a decreasing and non-constant function. For $x \in M$, we denote $r(x)$ (resp. $s(x)$) the range projection of x (resp. the support projection of x). Define $\varphi : M^+ \to M^+$ by

$$\varphi(a) = \alpha(\tau(r(a)))\mathbf{1}.$$

By definition, there exist t_0, t_1 with $0 \leq t_0 < t_1$ and $\alpha(t_0) > \alpha(t_1)$. We can choose projections p, q with $\tau(p) = t_0$, $\tau_(q) = t_1$, and $p \leq q$. Then we have $\varphi(p) > \varphi(q)$. So φ is not monotone.

Let $c, x \in M^+$ with $\|c\| \leq 1$. We set $p = r(x)$ and consider the polar decomposition of pc as follows:

$$pc = hv,$$

where $h \geq 0$ and $r(v) = s(h) \leq p$, $v^*v = s(pc)$, and $vv^* = s(h)$. Since M is a factor of type II_1, there exists a unitary $u \in M$ satisfying $u^*s(h) = v^*$. Then we have

$$\begin{aligned} s(c^*xc) &= s(c^*pxpc) = s(v^*hxhv) = s(u^*hxhu) \\ &= u^*s(hxh)u \leq u^*s(p)u = u^*s(x)u. \end{aligned}$$

Since

$$\tau(s(c^*xc)) \leq \tau(u^*s(x)u) = \tau(s(x)),$$

we can prove the supercongruence of φ as follows:

$$\varphi(c^*xc) = \alpha(\tau(c^*xc))\mathbf{1} \geq \alpha(\tau(s(x)))\mathbf{1} \geq c^*\alpha(\tau(s(x)))c = c^*\varphi(x)c.$$

Example (3-4). Let H be a Hilbert space and $M = B(H)$. For $a \in B(H)^+$, define

$$\varphi(a) = \begin{cases} \mathbf{1}, & \operatorname{rank}(a) = \infty \\ 2\mathbf{1} & \operatorname{rank}(a) < \infty \end{cases},$$

where $\operatorname{rank}(a)$ means the dimension of the closure of the subspace $\{a\xi \mid \xi \in H\}$ of H. Let p be a finite rank projection. By the fact

$\varphi(p) = 2\mathbf{1} > \mathbf{1} = \varphi(\mathbf{1})$, φ is not monotone. If $\mathrm{rank}(a) < \infty$, then $\mathrm{rank}(c^*ac) < \infty$ and

$$c^*\varphi(a)c = 2c^*c \leq 2I = \varphi(c^*ac).$$

If $\mathrm{rank}(a) = \infty$, then

$$c^*\varphi(a)c = c^*c \leq I \leq \varphi(c^*ac).$$

So φ is supercongruent.

Example (4-1). Let H be a Hilbert space and $M = B(H)$. For $a \in B(H)^+$, define $\varphi(a) = a^2$. Because $f(x) = x^2$ is not an operator monotone function, φ is not monotone.

$$\frac{1}{2}\mathbf{1} \cdot \varphi(\mathbf{1}) \cdot \frac{1}{2}\mathbf{1} = \frac{1}{4}\mathbf{1} \not\leq \varphi(\frac{1}{2}\mathbf{1} \cdot \mathbf{1} \cdot \frac{1}{2}\mathbf{1}) = \varphi(\frac{1}{4}\mathbf{1}) = \frac{1}{16}\mathbf{1}.$$

This implies that φ is not supercongruent.

Example (4-2). Let f be a real function as follows: $f(x) = 1 \vee x$ on $[0, \infty)$. Let H be a Hilbert space and $M = B(H)$. For $a \in B(H)^+$, define $\varphi(a) = \mathbf{1} \vee a = f(a)$ by a functional calculus. Consider

$$a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \leq b = \begin{pmatrix} 3 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

$$\varphi(a) = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix} \not\leq \varphi(b) = \begin{pmatrix} 3 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

Thus φ is not monotone.

Consider

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

then

$$c\varphi(a)c = c\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}c = \begin{pmatrix} 3/4 & 3/4 \\ 3/4 & 3/4 \end{pmatrix} \not\leq \varphi(cac) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence φ is not supercongruent.

Example (5-1). Let f be a real function as follows: $f(x) = x \vee (\frac{x+1}{2})$ on $[0, \infty)$. For a one-dimensional C*-algebra $\mathbb{C}$, $f : \mathbb{C} \longrightarrow \mathbb{C}$ is monotone, supercongruent and not concave.

4. Characterization of monotone maps given by Borel functional calculus

Let M be a von Neumann algebra on a Hilbert space H and $\varphi : M^+ \to M^+$ be the non-linear positive map defined by the range projection $\varphi(a)$ of $a \in M^+$. Then we showed that φ is monotone, supercongruent and normal. This is a typical example of non-linear positive map which is monotone, supercongruent and normal but is not a form of continuous functional calculus. We should remark that this map is given by a Borel functional culculus of the Borel function $\chi_{(0,\infty)}$ on $[0,\infty)$ as follows:

$$\varphi(a) = \chi_{(0,\infty)}(a),$$

where

$$\chi_{(0,\infty)}(t) = \begin{cases} 0 & t = 0 \\ 1 & t > 0 \end{cases}.$$

In this section, we shall characterize monotone maps given by Borel functional calculus.

At first we recall Borel functional calculus. Let Ω be a metrizable topological space and $C(\Omega)$ a set of all complex valued continuous functions on Ω. We denote by $\mathcal{B}(\Omega)$ the set of all bounded complex Borel functions on Ω. For a bound self-adjoint linear operator $a \in M$ there exists a correspondence

$$\mathcal{B}(\sigma(a)) \ni f \mapsto f(a) \in M$$

satisfying

(1) $f(a) = \alpha_0 \mathbf{1} + \alpha_1 a + \cdots + \alpha_n a^n$ for any polynomial $f(\lambda) = \alpha_0 + \alpha_1 \lambda + \cdots + \alpha_n \lambda^n$.
(2) $(f_n)_n$ is a bounded sequence in $\mathcal{B}(\sigma(a))$. If $(f_n)_n$ tends to $f \in \mathcal{B}(\sigma(a))$ with respect to the point-wise convergent topology, then the sequence $(f_n(a))_n$ of operators tends to the operator $f(a)$ in the strong operator topology.
(3) If f is continuous on $\sigma(a)$, then the Borel functional calculus coincides with the continuous functional calculus.

Moreover, this correspondence is a *-homomorphism of $\mathcal{B}(\sigma(a))$ onto the von Neumann algebra generated by a (see, [24, 2.20]). We call $f(a)$ the Borel functional calculus of a by $f \in \mathcal{B}(\sigma(a))$.

The following fact is well-known (see, [4, Theorem V.2.3, V2.5]).

Lemma 4.1. *Let f be a continuous mapping on $[0,\infty)$ into itself. Then the following are equivalent:*

(1) *f is operator monotone.*
(2) *f is operator concave.*
(3) *For any $a = a^* \in B(H)$ with $\sigma(a) \subset [0,\infty)$ and any $c \in B(H)$ with $\|c\| \leq 1$, it holds*

$$c^* f(a) c \leq f(c^* a c),$$

Theorem 4.2. *Let M be an infinite-dimensional factor on a Hilbert space H and $\varphi : M^+ \longrightarrow M^+$ be a non-linear positive map. Then the following are equivalent:*

(1) *φ is monotone and supercongruent.*
(2) *There exists a Borel function $f : [0,\infty) \longrightarrow [0,\infty)$ such that f is continuous on $(0,\infty)$, operator monotone on $(0,\infty)$ with*

$$f(0) \leq \lim_{t \to +0} f(t),$$

and $\varphi(a)$ is equal to the Borel functional calculus $f(a)$ of a by f for any $a \in M^+$.

Proof. (1) $\Rightarrow$ (2): Assume that φ is monotone and supercongruent. Firstly, we shall show that, for any $a \in M^+$ and any projection $p \in M$, if $ap = pa$, then $p\varphi(a) = \varphi(a)p = p\varphi(pap)p$.

In fact, suppose that $ap = pa$. Then $pap = a^{1/2}pa^{1/2} \leq a$. Since φ is supercongruent and monotone, we have

$$p\varphi(a)p \leq \varphi(pap) \leq \varphi(a) \text{ and } p\varphi(a)p = p\varphi(pap)p.$$

The positivity of $\varphi(a) - p\varphi(a)p$ implies $p\varphi(a)(\mathbf{1} - p) = 0$ and $(\mathbf{1} - p)\varphi(a)p = 0$. So $p\varphi(a) = \varphi(a)p = p\varphi(pap)p$.

Take $t\mathbf{1}$ for any $t \in [0, \infty)$. Because $p(t\mathbf{1}) = (t\mathbf{1})p$, we have that $p\varphi(t\mathbf{1}) = \varphi(t\mathbf{1})p$. Since M is a factor, $\varphi(t\mathbf{1})$ is a scalar operator $f(t)\mathbf{1}$. Thus f turns out to be a (not necessarily continuous) function $f : [0, \infty) \longrightarrow [0, \infty)$ such that

$$\varphi(t\mathbf{1}) = f(t)\mathbf{1} \quad for\ any\ t \in [0, \infty).$$

By the monotonicity of φ, f is increasing on $[0, \infty)$. In particular, we have

$$f(0) \leq \lim_{t \to +0} f(t).$$

Moreover for any $a \in M^+$ and any $\xi \in \mathrm{Ker}(a)$, we have

$$\varphi(a)\xi = f(0)\xi$$

In fact, let $r(a)$ (resp. $s(a)$) be the range projection (resp. the support projection) of a. Then $q = \mathbf{1} - r(a) = \mathbf{1} - s(a)$ is the projection onto the kernel of a. Since $aq = 0 = qa$, we have that $\varphi(a)q = q\varphi(a) = q\varphi(qaq)q = q\varphi(0\mathbf{1})q = f(0)q$. Hence $\varphi(a)\xi = \varphi(a)q\xi = f(0)q\xi = f(0)\xi$.

We shall show that for any $n \in \mathbb{N}$, $t_i \in [0, \infty)$ and projections $p_i \in M$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n p_i = \mathbf{1}$,

$$\varphi(\sum_{i=1}^n t_i p_i) = \sum_{i=1}^n f(t_i)p_i.$$

In fact, put $a = \sum_{i=1}^n t_i p_i$ and take any $k = 1, 2, \ldots, n$ and fix it. Put $b = t_k\mathbf{1}$. Then $ap_k = p_k a$ and $bp_k = p_k b$. Therefore

$$p_k\varphi(a) = \varphi(a)p_k = p_k\varphi(p_k a p_k)p_k = p_k\varphi(t_k p_k)p_k$$

and

$$f(t_k)p_k = \varphi(b)p_k = p_k\varphi(p_k b p_k)p_k = p_k\varphi(t_k p_k)p_k.$$

Hence

$$\varphi(\sum_{i=1}^n t_i p_i) = \varphi(\sum_{i=1}^n t_i p_i)(\sum_{k=1}^n p_k) = \sum_{k=1}^n f(t_k)p_k.$$

Next we shall show that, for any *invertible* $a \in M^+$ and any sequence $(a_n)_n$ in M^+ with $a_n \le a$, if $\|a_n - a\| \to 0$, then $\|\varphi(a_n) - \varphi(a)\| \to 0$. In fact, let

$$c_n = a^{-1} \# a_n,$$

where $X \# Y$ means the geometric operator mean for $X, Y \in M^+$ and is defined by

$$X \# Y = X^{1/2}(X^{-1/2} Y X^{-1/2})^{1/2} X^{1/2}$$

if X is invertible (see, [21]). Then $a_n = c_n a c_n$ and

$$0 \le c_n \le a^{-1} \# a = \mathbf{1}.$$

Because $\|a_n - a\| \to 0$, we have that $\|c_n - \mathbf{1}\| \to 0$. Since φ is monotone and supercongruent,

$$0 \le c_n \varphi(a) c_n \le \varphi(c_n a c_n) = \varphi(a_n) \le \varphi(a).$$

Then the relation $0 \le \varphi(a) - \varphi(a_n) \le \varphi(a) - c_n \varphi(a) c_n$ implies

$$\|\varphi(a) - \varphi(a_n)\| \le \|\varphi(a) - c_n \varphi(a) c_n\| \to 0.$$

We shall also show that, for any *invertible* $a \in M^+$ and any sequence $(b_n)_n$ in M^+ with $a \le b_n$, if $\|b_n - a\| \to 0$, then $\|\varphi(b_n) - \varphi(a)\| \to 0$. In fact, let $d_n = a \# b_n^{-1}$. Then $a = d_n b_n d_n$ and

$$0 \le d_n \le a \# a^{-1} = \mathbf{1}.$$

Because $\|b_n - a\| \to 0$, we have that $\|d_n - \mathbf{1}\| \to 0$. Since φ is monotone and supercongruent,

$$0 \le d_n \varphi(b_n) d_n \le \varphi(d_n b_n d_n) = \varphi(a),$$

and $0 \le \varphi(b_n) \le d_n^{-1} \varphi(a) d_n^{-1}$. Then the relation

$$\varphi(b_n) - \varphi(a) \le d_n^{-1} \varphi(a) d_n^{-1} - \varphi(a)$$

implies $\|\varphi(b_n) - \varphi(a)\| \le \|d_n^{-1} \varphi(a) d_n^{-1} - \varphi(a)\| \to 0$.

In particular, Since $\varphi(t\mathbf{1}) = f(t)\mathbf{1}$, the function f is continuous on $(0, \infty)$. Moreover f is a Borel function on $[0, \infty)$.

For any *invertible* elememt $a \in M^+$. we shall show that $\varphi(a)$ is equal to the continuous functional calculus of a by $f|_{(0,\infty)}$ on $(0, \infty)$, that is, $\varphi(a) = f(a)$. We may assume that $\sigma(a) \subset [\alpha, \beta]$ for some $0 < \alpha \le \beta$ in $(0, \infty)$. For any positive integer n, we define a function g_n on $[\alpha, \beta]$ as follows:

$$g_n(t) = \begin{cases} \alpha, & \alpha \le t \le \alpha + \dfrac{\beta - \alpha}{2^n} \\ \alpha + (k-1)\dfrac{\beta - \alpha}{2^n}, & \alpha + (k-1)\dfrac{\beta - \alpha}{2^n} < t \le \alpha + k\dfrac{\beta - \alpha}{2^n} \end{cases},$$

where $k = 2, 3, \ldots, 2^n$. Put $a_n = g_n(a)$. Then we have $0 \le a_n \le a$, $\sigma(a_n) \subset [\alpha, \beta]$ and $\varphi(a_n) = f(a_n)$, because a_n has a finite spectra. Since $\|a_n - a\| \to 0$, $\|\varphi(a_n) - \varphi(a)\| \to 0$. On the otherhand, since f is continuous on $(0, \infty)$ and the continuous functional calculus by f on $[\alpha, \beta]$ is norm continuous, $\|f(a_n) - f(a)\| \to 0$. Therefore $\varphi(a) = f(a)$.

Because M is an infinite-dimensional factor, M contains any finite matrix algebra $M_n(\mathbb{C})$. Hence f is an operator monotone continuous function on $(0, \infty)$.

For possiblly non-invertible element $a \in M^+$ in general, we shall show that $\varphi(a)$ is equal to the Borel functional calculus of a by f, that is $\varphi(a) = f(a)$. This case is a little bit subtle. We may assume that $\sigma(a) \subset [0, \beta]$ fo some $\beta \geq 0$.

For any positive integer n, we define a function $\tilde{g_n}$ on $[0, \beta]$ as follows:

$$\tilde{g_n}(t) = \begin{cases} 0, & 0 \leq t \leq \dfrac{\beta}{2^n} \\ \dfrac{(k-1)\beta}{2^n}, & \dfrac{(k-1)\beta}{2^n} < t \leq \dfrac{k\beta}{2^n} \end{cases},$$

where $k = 2, 3, \ldots, 2^n$. Put $\tilde{a_n} = \tilde{g_n}(a)$. Then, for $m \leq n$, we have

$$\tilde{a_m} \leq \tilde{a_n} \leq a \text{ and } \varphi(\tilde{a_m}) \leq \varphi(\tilde{a_n}) \leq \varphi(a)$$

by the monotonicity of φ, and $\varphi(\tilde{a_n}) = f(\tilde{a_n})$ because $\tilde{a_n}$ has a finite spectra. Since the sequence $\{\tilde{g}_n\}$ converges the identity map on $[0, \beta]$ with respect to the pointwise convergent topology, the increasing sequence $(\tilde{a_n})_n = (\tilde{g}_n(a))_n$ in M^+ converges to a in the strong operator topology.

We do not know that φ is normal in this moment. But, only for this particular sequence $(\tilde{a_n})_n$, we can show that $\varphi(\tilde{a_n})$ converges $\varphi(a)$ in the weak operator topology. In fact, let $\tilde{h_n}$ be a bounded Borel function on $[0, \beta]$ as follows:

$$\tilde{h_n}(t) = \begin{cases} 0, & 0 \leq t \leq \dfrac{\beta}{2^n} \\ \sqrt{\dfrac{\tilde{g_n}(t)}{t}} & \dfrac{\beta}{2^n} < t \leq \beta \end{cases}.$$

Then $0 \leq \tilde{h}_n \leq 1$ and $\{\tilde{h}_n\}$ poitwise converges to $\chi_{(0,\beta]}$. We set $\tilde{c_n} = \tilde{h_n}(a)$. Then the sequence $(\tilde{c_n})_n$ of positive contractions strongly converges to the range projection $r = \chi_{(0,\beta]}(a)$ of a and $\tilde{c_n} a \tilde{c_n} = \tilde{a_n}$.

For any $\xi \in H$, put $\xi_1 := r\xi$ and $\xi_2 := (\mathbf{1} - r)\xi \in \mathrm{Ker}(a)$. Since $\tilde{a_n} \leq a$, $\mathrm{Ker}(a) \subset \mathrm{Ker}(\tilde{a_n})$ and $\xi_2 \in \mathrm{Ker}(\tilde{a_n})$. Because $ar = ra$ and $\tilde{a_n} r = r \tilde{a_n}$, $\varphi(a) r = r \varphi(a)$ and $\varphi(\tilde{a_n}) r = r \varphi(\tilde{a_n})$. Since φ is monotone and supercongruent, $\tilde{c_n} \varphi(a) \tilde{c_n} \leq \varphi(\tilde{c_n} a \tilde{c_n}) = \varphi(\tilde{a_n}) \leq \varphi(a)$. Thus we have

$$0 \leq \varphi(a) - \varphi(\tilde{a_n}) \leq \varphi(a) - \tilde{c_n} \varphi(a) \tilde{c_n}.$$

Then we have

$$\begin{aligned} 0 \leq & \langle (\varphi(a) - \varphi(\tilde{a_n}))\xi, \xi \rangle \\ = & \langle (\varphi(a) - \varphi(\tilde{a_n}))\xi_1, \xi_1 \rangle + \langle (\varphi(a) - \varphi(\tilde{a_n}))\xi_2, \xi_2 \rangle \\ \leq & \langle (\varphi(a) - \tilde{c_n} \varphi(a) \tilde{c_n})\xi_1, \xi_1 \rangle + \langle f(0)\xi_2, \xi_2 \rangle - \langle f(0)\xi_2, \xi_2 \rangle \\ = & \langle \varphi(a)\xi_1, \xi_1 \rangle - \langle \varphi(a) \tilde{c_n} \xi_1, \tilde{c_n} \xi_1 \rangle. \end{aligned}$$

Since $\tilde{c_n}\xi_1$ converges to $r\xi_1 = \xi_1$, we conclude that $\varphi(\tilde{a_n})$ converges to $\varphi(a)$ in the weak operator topology.

We should note that a Borel functional calculus is not normal in general. But we shall show that the Borel functional calculus φ_f on M^+ by the particular function f is normal. In fact, define a continuous function $F : [0,\infty) \longrightarrow [0,\infty)$ by

$$F(t) = \begin{cases} f(0) + f(t) - \lim_{t\to+0} f(t) & \text{if } t > 0 \\ f(0) & \text{if } t = 0 \end{cases}.$$

Then F is operator monotone on $[0,\infty)$. In fact, for $0 \le a \le b$ and any $\epsilon > 0$, $F(a+\epsilon\mathbf{1}) \le F(b+\epsilon\mathbf{1})$ because f is operator monotone on $(0,\infty)$. By the continuity of F, we can get $F(a) \le F(b)$ by making ϵ tend to 0. Thus F is operator monotone function on $[0,\infty)$ with $F(0) = f(0)$. The functional calculus φ_F by the continuous function F is normal. The function f is decomposed into

$$f(t) = F(t) + k\chi_{(0,\infty)}(t), \qquad k = \lim_{t\to 0+} f(t) - f(0) \ge 0.$$

Then the Borel functional calculus φ_f of $a \in \mathcal{M}^+$ by f has the form:

$$\varphi_f(a) = \varphi_F(a) + k\varphi_{(0,\infty)}(a),$$

where $\varphi_{(0,\infty)}(a)$ is the Borel functional calculus of a by $\chi_{(0,\infty)}$ and in fact the range projection of a. Hence $\varphi_{(0,\infty)}$ is normal by Propositon 3.3. Therefore the Borel functional calculus φ_f by f is normal.

Finally, since $\varphi(\tilde{a_n})$ converges $\varphi(a)$ and $f(\tilde{a_n})$ converges $f(a)$ in the weak operator topology and $\varphi(\tilde{a_n}) = f(\tilde{a_n})$, we conclude that $\varphi(a) = f(a) = \varphi_f(a)$, the Borel functional calculus of a by f.

(2) $\Rightarrow$ (1): Suppose that there exists a Borel function $f : [0,\infty) \longrightarrow [0,\infty)$ such that f is continuous on $(0,\infty)$, operator monotone on $(0,\infty)$ with

$$f(0) \le \lim_{t\to+0} f(t),$$

and $\varphi(a)$ is equal to the Borel functional calculus $f(a) = \varphi_f(a)$ of a by f for any $a \in M^+$. We define a continuous function $F : [0,\infty) \longrightarrow [0,\infty)$ by

$$F(t) = \begin{cases} f(0) + f(t) - \lim_{t\to+0} f(t) & \text{if } t > 0 \\ f(0) & \text{if } t = 0 \end{cases}.$$

Then as in the preceding discussion, F is operator monotone on $[0,\infty)$ with $F(0) = f(0)$. Hence the continuous functional calculus φ_F is supercongruent as in Example 3.2. The function f is decomposed into

$$f(t) = F(t) + k\chi_{(0,\infty)}(t), \qquad k = \lim_{t\to 0+} f(t) - f(0) \ge 0,$$

and

$$\varphi_f(a) = \varphi_F(a) + k\varphi_{(0,\infty)}(a),$$

where $\varphi_{(0,\infty)}(a)$ is the range projection of a and $\varphi_{(0,\infty)}$ is supercongruent. Hence φ is monotone and supercongruent. $\square$

By the above theorem, the restricted norm continuity and the normality of the non-linear positive map are satisfied automatically without assuming them a priori.

Corollary 4.3. *Let M be an infinite-dimensional factor and $\varphi : M^+ \longrightarrow M^+$ be a non-linear positive map. If φ is monotone and supercongruent, then φ is normal on M^+ and φ is norm continuous on the set of positive invertible elements $(M^+)^{-1}$. Moreover φ is concave.*

Proof. The almost all except concavity are proved in the discussion of the proof in the theorem above. Since $f_n(t) = t^{1/n}$ is a operator concave function on $[0,\infty)$ and $\chi_{(0,\infty)}(t) = \lim_{n\to\infty} f_n(t)$, $\chi_{(0,\infty)}$ is also operator concave function. Since F is operator monotone on $[0,\infty)$, F is also operator concave. Therfore φ is concave. $\square$

Related to the theorem, we also state the following fact as a remark:

Corollary 4.4. *Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a Borel function and $\varphi_f : B(H)^+ \longrightarrow B(H))^+$ defined as a Borel functional calculus $\varphi_f(a) = f(a)$ of $a \in B(H)^+$ by f. Then the following are equivalent:*

(1) *φ_f is monotone.*
(2) *φ_f is supercongruent.*
(3) *φ_f is concave.*

Proof. (3) $\Rightarrow$ (1) It follows from Proposition 3.4.

(1) $\Rightarrow$ (2) Since φ_f is positive, we have $f(0) \le f(t)$ for any $t \in (0,\infty)$ and that f is n-matrix monotone on $(0,\infty)$ for any positive integer n, where n-matrix monotone means $\varphi_f(a) \le \varphi_f(b)$ for any $a, b \in M_n(\mathbb{C})^+$ with $0 \le a \le b$. This means f is operator monotone on $(0,\infty)$ and $f(0) \le \lim_{t\to+0} f(t)$. So φ_f is supercongruence by the above theorem.

(2) $\Rightarrow$ (3) For any positive integer n and $a, b \in M_n(\mathbb{C})^+$, we put

$$X = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \ K = \begin{pmatrix} \sqrt{1-t}\mathbf{1}_n & 0 \\ \sqrt{t}\mathbf{1}_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

Then there exists a unitary $U \in M_{2n}(\mathbb{C})$ such that $K = |K^*|U$, where $|K| = (KK^*)^{1/2}$. Since φ_f is supercongruence, we have

$$\begin{aligned} K^*\varphi_f(X)K &= U^*|K^*|\varphi_f(X)|K^*|U \le U^*\varphi_f(|K^*|X|K^*|)U \\ &= \varphi_f(U^*|K^*|X|K^*|U) = \varphi_f(K^*XK). \end{aligned}$$

By the simple calculation,

$$K^*\varphi_f(X)K = K^* \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix} K = \begin{pmatrix} (1-t)f(a) + tf(b) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\varphi_f(K^*XK) = \varphi_f(\begin{pmatrix} (1-t)a + tb & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f((1-t)a + tb) & 0 \\ 0 & f(0)\mathbf{1}_n \end{pmatrix}.$$

So we have $f(0) \geq 0$ and

$$(1-t)f(a) + tf(b) \leq f((1-t)a + tb).$$

This means that f is concave on $[0, \infty)$ and n-matrix concave on $[0, \infty)$ for any n, that is, operator concave on $(0, \infty)$. Because f is also operator monotone on $(0, \infty)$, φ_f is concave by Corollary 4.3. $\square$

In the above theorem, if we weaken the supercongruent condition as only for positive *invertible* contraction $c \in M^+$ and $a \in M^+$

$$c\varphi(a)c \leq \varphi(cac),$$

then the conclusion of the theorem above does not hold in general. In fact, let the function f_α $(\alpha \geq 0)$ be operator monotone on $[0, \infty)$ and increasing for α, that is

$$\begin{aligned} &a, b \in M^+ \text{ with } a \leq b \Rightarrow f_\alpha(a) \leq f_\alpha(b) \\ (*) \qquad &\text{and } \alpha \leq \beta \Rightarrow f_\alpha(t) \leq f_\beta(a) \quad (t \in [0, \infty)), \end{aligned}$$

for any factor M. For an example, it is well-known

$$f(t) = \alpha - \frac{1}{t+1} \quad (\alpha \geq 0)$$

is operator monotone for $[0, \infty)$ ([4], [5], [18]). So the function

$$f_\alpha(t) = \frac{\alpha}{\alpha+1} - \frac{1}{t+1} \quad (\alpha \geq 0)$$

satisfies the condition (*).

Proposition 4.5. *We assume that $M = B(H)$ for a separable Hilbert space H and the operator monotone continuous function f_α on $[0, \infty)$ with the property (*) and*

$$f_\infty(t) = \lim_{\alpha \to \infty} f_\alpha(t) < \infty$$

exists for all $t \in [0, \infty)$. We define the map $\varphi : M^+ \longrightarrow M^+$ as follows:

$$\varphi(a) = f_{\mathrm{rank}(a)}(a) \qquad a \in M^+,$$

where $\mathrm{rank}(a) = \dim($*the closure of* $a\mathcal{H})$. *Then we have the following.*

1. *$a, b \in M^+ \Rightarrow \varphi(a) \leq \varphi(b)$.*
2. *For any invertible $c \in M$, $c^*\varphi(a)c \leq \varphi(c^*ac)$ $(a \in M^+)$.*
3. *If $f_m \neq f_n$ for some $m, n \in \mathbb{N}$, then φ is not given as the continuous function calculus.*

Proof. (1) Since $a \leq b$, $\mathrm{rank}(a) \leq \mathrm{rank}(b)$. So we have

$$\varphi(a) = f_{\mathrm{rank}(a)}(a) \leq f_{\mathrm{rank}(a)}(b) \leq f_{\mathrm{rank}(b)}(b) = \varphi(b).$$

(2) Since the mapping $f_{\mathrm{rank}(a)}$ is operator monotone on $[0, \infty)$, we have

$$c^* f_{\mathrm{rank}(a)}(a)c \leq f_{\mathrm{rank}(a)}(c^*ac),$$

using the approximation of polynomials for $f_{\mathrm{rank}(a)}$. By the invertibility of c, we have $\mathrm{rank}(c^*ac) = \mathrm{rank}(a)$ and

$$c^*\varphi(a)c \leq \varphi(c^*ac).$$

(3) By definiton, we have $\varphi(t\mathbf{1}) = f_\infty(t)\mathbf{1}$ for any $t \in [0, \infty)$. We assume $m < n$ and $f_m(t_0) < f_n(t_0)$ for some $t_0 \in (0, \infty)$. For a projection $p \in M$ with $\mathrm{rank}(p) = m$, we have

$$\varphi(t_0 p) = f_m(t_0 p) < f_\infty(t_0 p).$$

□

Finally we shall discuss the ambiguity of operator means for non-invertible positive operators related with our theorem , if we do *not* assume the upper semi-continuity for operator means. We follow the original paper of Kubo-Ando [21], see also [5], [18].

Corollary 4.6. *Let M be an infinite-dimensional factor. If the mapping*

$$\sigma : M^+ \times M^+ \ni (a, b) \mapsto a\sigma b \in M^+$$

satisfies the following conditions:

(1) *$a \leq c$ and $b \leq d$ imply $a\sigma b \leq c\sigma d$.*
(2) *For any $c \in M^+$, $c(a\sigma b)c \leq (cac)\sigma(cbc)$.*

then there exist non-negative real valued, increasing, continuous functions f and g on $(0, \infty)$ such that

$$\begin{aligned} a\sigma b &= b^{1/2} f(b^{-1/2} a b^{-1/2}) b^{1/2} \\ &= a^{1/2} g(A^{-1/2} b a^{-1/2}) a^{1/2} \end{aligned}$$

for any positive invertible operators $a, b \in M^+$. But we do not know how to represent $a\sigma b$ for positive non-invertible operators $a, b \in M^+$.

Proof. We define the mapping $\varphi : M^+ \longrightarrow M^+$ as follows:

$$\varphi(a) = a\sigma\mathbf{1} \qquad (a \in M^+).$$

It is clear that

$$a \leq b \ \Rightarrow \varphi(a) = a\sigma\mathbf{1} \leq b\sigma\mathbf{1} = \varphi(b)$$

and for any contraction $c \in M^+$,

$$cf(a)c = c(a\sigma\mathbf{1})c \leq (cac)\sigma(c^2) \leq (cac)\sigma\mathbf{1} = f(cac).$$

By Theorem 4.2, we can get the desired function f and the relation

$$f(a) = a\sigma\mathbf{1} \qquad (a \in (M^+)^{-1}).$$

For any positive invertible operators $a, b \in M^+$, we have

$$a\sigma b = b^{1/2} f(b^{-1/2} a b^{-1/2}) b^{1/2}$$

as usual way:

$$\begin{aligned}a\sigma b &= b^{1/2}b^{-1/2}(a\sigma b)b^{-1/2}b^{1/2}\\ &\le b^{1/2}((b^{-1/2}ab^{-1/2})\sigma \mathbf{1})b^{1/2} = b^{1/2}f(b^{-1/2}ab^{-1/2})b^{1/2}\\ &\le a\sigma b.\end{aligned}$$

We also define $\psi : M^+ \longrightarrow M^*$ as follows:

$$\psi(a) = \mathbf{1}\sigma a \qquad (a \in M^+).$$

Then we can get the function g satisfying

$$a\sigma b = a^{1/2}g(a^{-1/2}ba^{-1/2})a^{1/2} \quad \text{ for any } a, b \in (M^+)^{-1}.$$

□

Remark 4.7. We do not know how to represent $a\sigma b$ for positive non-invertible operators $a, b \in M^+$. Based on the theory of Grassmann manifolds, Bonnabel-Sepulchre [8] and Batzies-H́uper-Machado-Leite [6] introduced the geometric mean for positive semidefinite matrices or projections of fixed rank. Fujii [13] extends it to a general theory of means of positive semideinite matrices of fixed rank.

5. Non-additive measures and non-linear monotone positive maps

In this section we begin to study non-linear monotone positive maps related with non-additive measures. A non-additive measure is also called capacity, fuzzy measure, submeasure, monotone measure, etc. in different fields. Non-additive measures were firstly studied by Choquet [9] and Sugeno [25]. They proposed Choquet integral and Sugeno integral with respect to monotone measures.

Definition 5.1. Let Ω be a set and $\mathcal{B}$ a σ-field on Ω. A function $\mu : \mathcal{B} \to [0, \infty]$ is called a monotone measure if μ satisfies

(1) $\mu(\emptyset) = 0$, and
(2) For any $A, B \in \mathcal{B}$, if $A \subset B$, then $\mu(A) \le \mu(B)$.

We recall the discrete Choquet integral with respect to a monotone measure on a finite set $\Omega = \{1, 2, \ldots, n\}$. Let $\mathcal{B} = P(\Omega)$ be the set of all subsets of Ω and $\mu : \mathcal{B} \to [0, \infty)$ be a finite monotone measure .

Definition 5.2. The discrete Choquet integral of $f = (x_1, x_2, \ldots, x_n) \in [0, \infty)^n$ with respect to a monotone measure μ on a finite set $\Omega = \{1, 2, \ldots, n\}$ is defined as follows:

$$(C)\int f d\mu = \sum_{i=1}^{n-1}(x_{\sigma(i)} - x_{\sigma(i+1)})\mu(A_i) + x_{\sigma(n)}\mu(A_n),$$

where σ is a permutaion on Ω such that $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(n)}$, $A_i = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$. Here we should note that

$$f = \sum_{i=1}^{n-1} (x_{\sigma(i)} - x_{\sigma(i+1)}) \chi_{A_i} + x_{\sigma(n)} \chi_{A_n}$$

Let $A = \mathbb{C}^n$ and define $(C-\varphi)_\mu : (\mathbb{C}^n)^+ \to \mathbb{C}^+$ by the Choquet integral $(C-\varphi)_\mu(f) = (C)\int f d\mu$. Then $(C-\varphi)_\mu$ is a non-linear monotone positive map such that $(C-\varphi)_\mu(\alpha f) = \alpha (C-\varphi)_\mu(f)$ for a positive scalar α.

We shall consider a matrix version of the discrete Choquet integral.

Proposition 5.3. *Let $\mu : \mathcal{B} \to [0, \infty)$ be a finite monotone measure on a finite set $\Omega = \{1, 2, \ldots, n\}$ with $\mathcal{B} = P(\Omega)$. Let $A = M_n(\mathbb{C})$ and define $(C-\varphi)_\mu : (M_n(\mathbb{C}))^+ \to \mathbb{C}^+$ as follows: For $a \in (M_n(\mathbb{C}))^+$, let $\lambda(a) = (\lambda_1(a), \lambda_2(a), \ldots, \lambda_n(a))$ be the list of the eigenvalues of a in decreasing order : $\lambda_1(a) \geq \lambda_2(a) \geq \cdots \geq \lambda_n(a)$ with counting multiplicities. Let*

$$(C-\varphi)_\mu(a) = \sum_{i=1}^{n-1} (\lambda_i(a) - \lambda_{i+1}(a)) \mu(A_i) + \lambda_n(a) \mu(A_n),$$

where $A_i = \{1, 2, \ldots, i\}$. Then $(C-\varphi)_\mu$ is a unitarily invariant non-linear monotone positive map such that $(C-\varphi)_\mu(\alpha a) = \alpha (C-\varphi)_\mu(a)$ for a positive scalar α.

Proof. For $a, b \in (M_n(\mathbb{C}))^+$, suppose that $0 \leq a \leq b$. By the mini-max principle for eigenvalues, we have that $\lambda_i(a) \leq \lambda_i(b)$ for $i = 1, 2, \ldots, n$.

$$\begin{aligned}
(C-\varphi)_\mu(a) &= \sum_{i=1}^{n-1} (\lambda_i(a) - \lambda_{i+1}(a)) \mu(A_i) + \lambda_n(a) \mu(A_n) \\
&= \sum_{i=2}^{n} \lambda_i(a) (\mu(A_i) - \mu(A_{i-1})) + \lambda_1(a) (\mu(A_1) \\
&\leq \sum_{i=2}^{n} \lambda_i(b) (\mu(A_i) - \mu(A_{i-1})) + \lambda_1(b) \mu(A_1) \\
&= (C-\varphi)_\mu(b)
\end{aligned}$$

since μ is a monotone measure. Thus φ_μ is monotone. It is clear that $\varphi_\mu(\alpha a) = \alpha \varphi_\mu(a)$ for a positive scalar α by the definiton and φ_μ is unitarily invariant. $\square$

Furthermore we can replace a monotone measure on a finite set $\Omega = \{1, 2, \ldots, n\}$ by a positive operator-valued monotone measure $\mu : \mathcal{B} \to B(H)^+$ for some Hilbert space H , that is,

(1) $\mu(\emptyset) = 0$, and
(2) For any $X, Y \in \mathcal{B} = P(\Omega)$, if $X \subset Y$, then $\mu(X) \leq \mu(Y)$.

We have a similar result as follows:

Proposition 5.4. *Let H be a Hilbert space, $\mu : \mathcal{B} \to B(H)^+$ be a positive operator-valued monotone measure on a finite set $\Omega = \{1, 2, \ldots, n\}$ with $\mathcal{B} = P(\Omega)$. Define $(C - \varphi)_\mu : (M_n(\mathbb{C}))^+ \to B(H)^+$ as follows: For $a \in (M_n(\mathbb{C}))^+$, let $\lambda(a) = (\lambda_1(a), \lambda_2(a), \ldots, \lambda_n(a))$ be the list of the eigenvalues of a in decreasing order with counting multiplicities. Let*

$$(C - \varphi)_\mu(a) = \sum_{i=1}^{n-1}(\lambda_i(a) - \lambda_{i+1}(a))\mu(A_i) + \lambda_n(a)\mu(A_n),$$

where $A_i = \{1, 2, \ldots, i\}$. Then $(C - \varphi)_\mu$ is a a unitarily invariant non-linear monotone positive map such that $(C - \varphi)_\mu(\alpha a) = \alpha(C - \varphi)_\mu(a)$ for a positive scalar α.

Proof. Use the similar argument as above. □

Honda and Okazaki [19] proposed the inclusion-exclusion integral with respect to a monotone measure, which is a generalization of the Lebesgue integral and the the Choquet integral. We can also consider a matrix version of the inclusion-exclusion integral.

Proposition 5.5. *Let H be a Hilbert space , $\mu : \mathcal{B} \to B(H)^+$ be a positive operator-valued monotone measure on a finite set $\Omega = \{1, 2, \ldots, n\}$ with $\mathcal{B} = P(\Omega)$. Fix a positive number K. Let $(\Omega, P(\Omega), \mu, I, K)$ be an interactive monotone measure space such that the interaction operator I is positive and monotone in the sense of* [19]. *Define*

$$(I - \varphi)_\mu : \{a \in (M_n(\mathbb{C}))^+ \mid \sigma(a) \subset [0, K]\} \to B(H)^+$$

as follows: For $a \in (M_n(\mathbb{C}))^+$ with the spectra $\sigma(a) \subset [0, K]$, let $\lambda(a) = (\lambda_1(a), \lambda_2(a), \ldots, \lambda_n(a))$ be the list of the eigenvalues of a in decreasing order with counting multiplicities. Let

$$(I - \varphi)_\mu(a) = \sum_{A \in P(\Omega)} (\sum_{B \supset A}(-1)^{|B \setminus A|} I(\lambda(a)|B))\mu(A).$$

Then $(I - \varphi)_\mu$ is a a unitarily invariant non-linear monotone positive map.

Proof. For $a, b \in (M_n(\mathbb{C}))^+$ with the spectra $\sigma(a) \subset [0, K]$ and $\sigma(b) \subset [0, K]$, suppose that $0 \leq a \leq b$. By the mini-max principle for eigenvalues, we have that $\lambda_i(a) \leq \lambda_i(b)$ for $i = 1, 2, \ldots, n$. Since the interaction operator I is monotone,

$$\sum_{B \supset A}(-1)^{|B \setminus A|} I(\lambda(a)|B) \leq \sum_{B \supset A}(-1)^{|B \setminus A|} I(\lambda(b)|B).$$

Therefore $(I - \varphi)_\mu(a) \leq (I - \varphi)_\mu(b)$. □

Next we recall the Sugeno integral with respect to a monotone measure on a finite set $\Omega = \{1, 2, \ldots, n\}$.

Definition 5.6. The discrete Sugeno integral of $f = (x_1, x_2, \ldots, x_n) \in [0, \infty)^n$ with respect to a monotone measure μ on a finite set $\Omega = \{1, 2, \ldots, n\}$ is defined as follows:

$$(S) \int f d\mu = \vee_{i=1}^{n} (x_{\sigma(i)} \wedge \mu(A_i)),$$

where σ is a permutaion on Ω such that $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(n)}$, $A_i = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ and $\vee = \max$, $\wedge = \min$. Here we should note that

$$f = \vee_{i=1}^{n} (x_{\sigma(i)} \chi_{A_i}).$$

Let $A = \mathbb{C}^n$ and define $(S-\varphi)_\mu : (\mathbb{C}^n)^+ \to \mathbb{C}^+$ by the Sugeno integral $(S - \varphi)_\mu(f) = (S) \int f d\mu$. Then $(S - \varphi)_\mu$ is a non-linear monotone positive map such that $(S - \varphi)_\mu(\alpha f) = \alpha(S - \varphi)_\mu(f)$ for a positive scalar α.

We shall consider a matrix version of the discrete Sugeno integral.

Proposition 5.7. *Let $\mu : \mathcal{B} \to [0, \infty)$ be a finite monotone measure on a finite set $\Omega = \{1, 2, \ldots, n\}$ with $\mathcal{B} = P(\Omega)$. Let $A = M_n(\mathbb{C})$ and define $(S - \varphi)_\mu : (M_n(\mathbb{C}))^+ \to \mathbb{C}^+$ as follows: For $a \in (M_n(\mathbb{C}))^+$, let $\lambda(a) = (\lambda_1(a), \lambda_2(a), \ldots, \lambda_n(a))$ be the list of the eigenvalues of a in decreasing order : $\lambda_1(a) \geq \lambda_2(a) \geq \cdots \geq \lambda_n(a)$ with counting multiplicities. Let*

$$(S - \varphi)_\mu(a) = \vee_{i=1}^{n} (\lambda_i(a) \wedge \mu(A_i))$$

where $A_i = \{1, 2, \ldots, i\}$. Then $(S - \varphi)_\mu$ is a unitarily invariant non-linear monotone positive map such that $(S - \varphi)_\mu(\alpha a) = \alpha(S - \varphi)_\mu(a)$ for a positive scalar α.

Proof. For $a, b \in (M_n(\mathbb{C}))^+$, suppose that $0 \leq a \leq b$. Since $\lambda_i(a) \leq \lambda_i(b)$ for $i = 1, 2, \ldots, n$,

$$\begin{aligned} (S - \varphi)_\mu(a) &= \vee_{i=1}^{n} (\lambda_i(a) \wedge \mu(A_i)) \\ &\leq \vee_{i=1}^{n} (\lambda_i(b) \wedge \mu(A_i)) = (S - \varphi)_\mu(b). \end{aligned}$$

Thus φ_μ is monotone. It is clear that $\varphi_\mu(\alpha a) = \alpha \varphi_\mu(a)$ for a positive scalar α by the definiton and φ_μ is unitarily invariant. □

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