$\mathbb{Z}^2$-actions on Kirchberg algebras

Masaki Izumi *
Graduate School of Science
Kyoto University
Sakyo-ku, Kyoto 606-8502, Japan

Hiroki Matui †
Graduate School of Science
Chiba University
Inage-ku, Chiba 263-8522, Japan

Dedicated to Professor Akitaka Kishimoto for his sixtieth birthday.

Abstract

We classify a large class of $\mathbb{Z}^2$-actions on the Kirchberg algebras employing the Kasparov group KK^1 as the space of classification invariants.

1 Introduction

Separable purely infinite simple nuclear C^*-algebras are said to be Kirchberg algebras. They form one of the most prominent classes of C^*-algebras from the viewpoint of classification, and they are completely classified by KK-theory (see [12], [22], and [25]). In this respect, they are compared to AFD factors, whose classification result is already classics in operator algebras. For AFD factors, their symmetries are also well-understood, namely, a complete classification is known for actions of countable amenable groups on AFD factors. However, classification of group actions on C^*-algebras is still a far less developed subject, partly because of K-theoretical difficulties.

For Kirchberg algebras, H. Nakamura [20] showed that aperiodic automorphisms are completely classified by their KK-classes up to, what we call, KK-trivial cocycle conjugacy. He followed a strategy developed by Kishimoto [15], [16] in the case of AT-algebras, and one of the main ingredients of the proof is the Rohlin property (see the review paper [7] for the outline of the strategy). While Nakamura's result can be considered as classification of outer actions of the integer group $\mathbb{Z}$, the Rohlin property is also formulated for finite group actions. In [8] and [9], the first-named author completely classified finite group actions with the Rohlin property on Kirchberg algebras. However, unlike the $\mathbb{Z}$ case where the Rohlin property is automatic, there are several outer finite group actions without the Rohlin property.

One of the purposes of this paper is to develop classification theory of discrete amenable group actions on the Kirchberg algebras. However, we should admit that this is too ambitious a goal now. The difficulties in the finite group case, in contrast to the $\mathbb{Z}$ case, are rather common in topology. For example, it is well-known that the classifying space of a non-trivial finite group is never finite dimensional while that of $\mathbb{Z}$ is a nice space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. To avoid this kind of difficulties, in this paper we work on the $\mathbb{Z}^2$ case as a first

*Supported in part by the Grant-in-Aid for Scientific Research (C) 19540214, JSPS.
†Supported in part by the Grant-in-Aid for Young Scientists (B) 18740085, JSPS.

step beyond the $\mathbb{Z}$ case. Indeed, the second-named author has already obtained successful results on classification of $\mathbb{Z}^2$-actions on the UHF algebras [11] and $\mathbb{Z}^N$-actions on the Cuntz algebra $\mathcal{O}_2$ [18].

General outer actions of $\mathbb{Z}^2$ on the Kirchberg algebras are still out of reach, and we concentrate on locally KK-trivial actions in this paper; we assume that each automorphism appearing in the actions has the trivial KK-class. Then the only remaining classification invariant should be a global one. It turns out that such an invariant is identified with an element of the Kasparov group KK^1, and it is indeed a complete invariant (Theorem 8.6). For example, our main theorem says that there are exactly $n-1$ cocycle conjugacy classes of outer $\mathbb{Z}^2$-actions on the Cuntz algebra $\mathcal{O}_n$ for finite n.

We briefly describe the basic idea of our approach now. For an outer $\mathbb{Z}^2$-action α on a Kirchberg algebra A, Nakamura's theorem says that the automorphism $\alpha_{(1,0)}$ is completely characterized by its KK-class. Therefore our task is to classify the other automorphism $\alpha_{(0,1)}$ commuting with the given fixed one $\alpha_{(1,0)}$. This problem is more or less equivalent to classifying the automorphisms of the crossed product $A \rtimes_{\alpha_{(1,0)}} \mathbb{Z}$ commuting with the dual $\mathbb{T}$-action. Roughly speaking, this means that for classification of $\mathbb{Z}^2$-actions it suffices to develop a $\mathbb{T}$-equivariant version of Kirchberg and Phillips' characterization of the KK-theory of Kirchberg algebras, and a $\mathbb{T}$-equivariant version of Nakamura's classification theorem of aperiodic automorphisms. Of course, this is not possible for a general $\mathbb{T}$-action. However, for an asymptotically representable action of a discrete amenable group Γ, we can prove the $\hat{\Gamma}$-equivariant versions for the crossed product by Γ equipped with the dual coaction. As a byproduct, we can show a uniqueness result for outer asymptotically representable actions of $\mathbb{Z}^N$ on Kirchberg algebras (Theorem 6.6). For algebras with sufficiently simple K-theory such as the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, every outer $\mathbb{Z}^N$-action turns out to be asymptotically representable, which implies the uniqueness of the cocycle conjugacy classes of outer $\mathbb{Z}^N$-actions on these algebras. This is a generalization of the main result in [18], and our proof is new even in the case of $\mathcal{O}_2$.

The price we have to pay for working on the crossed product $A \rtimes_{\alpha_{(1,0)}} \mathbb{Z}$ is that we need to show a second cohomology vanishing theorem for cocycle $\mathbb{Z}^2$-actions (Theorem 7.11). In the case of von Neumann algebras, this is known to be one of the standard steps toward classification results for group actions. To overcome the problem, we follow Ocneanu's idea in the von Neumann algebra case though we need a special care of K-theory in our setting.

The authors would like to thank Akitaka Kishimoto, Sergey Neshveyev, and Toshihiko Masuda for stimulating discussions.

2 Preliminaries

We denote by $\mathbb{K}$ the C^*-algebra of all compact operators on $\ell^2(\mathbb{Z})$. For a C^*-algebra A, we write the multiplier algebra of A by $M(A)$. We let $U(A)$ denote the set of all unitaries of $M(A)$. For $u \in U(A)$, the inner automorphism induced by u is written by $\operatorname{Ad} u$. An automorphism $\alpha \in \operatorname{Aut}(A)$ is called outer, when it is not inner. An automorphism $\alpha \in \operatorname{Aut}(A)$ is called aperiodic, when α^n is outer for all $n \in \mathbb{N}$. A single automorphism α is often identified with the $\mathbb{Z}$-action induced by α. The quotient group of $\operatorname{Aut}(A)$ by the inner automorphism group is denoted by $\operatorname{Out}(A)$. For $a, b \in A$, we mean by $[a, b]$ the

commutator $ab - ba$. For a Lipschitz continuous map f between metric spaces, $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f.

Let A, B and C be C^*-algebras. For a homomorphism $\rho : A \to B$, $K_0(\rho)$ and $K_1(\rho)$ mean the induced homomorphisms on K-groups, and $KK(\rho)$ means the induced element in $KK(A, B)$. We write $KK(\mathrm{id}_A) = 1_A$. For $x \in KK(A, B)$ and $i = 0, 1$, we let $K_i(x)$ denote the homomorphism from $K_i(A)$ to $K_i(B)$ induced by x. For $x \in KK(A, B)$ and $y \in KK(B, C)$, we denote the Kasparov product by $x \cdot y \in KK(A, C)$. When both A and B are unital, we denote by $\mathrm{Hom}(A, B)$ the set of all unital homomorphisms from A to B. Two unital homomorphisms $\rho, \sigma \in \mathrm{Hom}(A, B)$ are said to be asymptotically unitarily equivalent, if there exists a continuous family of unitaries $\{u_t\}_{t\in[0,\infty)}$ in B such that

$$\rho(a) = \lim_{t\to\infty} \mathrm{Ad}\, u_t(\sigma(a))$$

for all $a \in A$. When there exists a sequence of unitaries $\{u_n\}_{n\in\mathbb{N}}$ in B such that

$$\rho(a) = \lim_{n\to\infty} \mathrm{Ad}\, u_n(\sigma(a))$$

for all $a \in A$, ρ and σ are said to be approximately unitarily equivalent.

Let G be a countable discrete amenable group. The canonical generators in $C^*(G)$ is denoted by $\{\lambda_g\}_{g\in G}$. The homomorphism $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ sending λ_g to $\lambda_g \otimes \lambda_g$ is called the coproduct. Let $\alpha : G \curvearrowright A$ be an action of G on a C^*-algebra A. When α_g is outer for all $g \in G$ except for the neutral element, the action α is called outer. We let A^α denote the fixed points subalgebra of A. The canonical implementing unitaries in the crossed product C^*-algebra $A \rtimes_\alpha G$ are written by $\{\lambda_g^\alpha\}_{g\in G}$. The dual coaction $\hat{\alpha}$ of α is the homomorphism from $A \rtimes_\alpha G$ to $(A \rtimes_\alpha G) \otimes C^*(G)$ defined by

$$\hat{\alpha}(a) = a \otimes 1 \quad \text{and} \quad \hat{\alpha}(\lambda_g^\alpha) = \lambda_g^\alpha \otimes \lambda_g$$

for $a \in A$ and $g \in G$.

Definition 2.1. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions of a discrete group G on C^*-algebras A and B.

(1) We say that α is locally KK-trivial, if $KK(\alpha_g) = 1_A$ for all $g \in G$.

(2) The two actions α and β are said to be conjugate, when there exists an isomorphism $\mu : A \to B$ such that $\alpha_g = \mu^{-1} \circ \beta_g \circ \mu$ for all $g \in G$.

(3) The two actions α and β are said to be outer conjugate, when there exist an isomorphism $\mu : A \to B$ and a family of unitaries $\{u_g\}_{g\in G}$ in $M(A)$ such that $\mathrm{Ad}\, u_g \circ \alpha_g = \mu^{-1} \circ \beta_g \circ \mu$ for all $g \in G$.

(4) A family of unitaries $\{u_g\}_{g\in G}$ in $M(A)$ is called an α-cocycle, if one has $u_g \alpha_g(u_h) = u_{gh}$ for all $g, h \in G$. When $\{u_g\}_g$ is an α-cocycle, the perturbed action $\alpha^u : G \curvearrowright A$ is defined by $\alpha_g^u = \mathrm{Ad}\, u_g \circ \alpha_g$.

(5) The two actions α and β are said to be cocycle conjugate, if there exists an α-cocycle $\{u_g\}_{g\in G}$ in $M(A)$ such that α^u is conjugate to β.

(6) The two actions α and β are said to be strongly cocycle conjugate, if there exist an α-cocycle $\{u_g\}_{g\in G}$ in $M(A)$ and a sequence of unitaries $\{v_n\}_{n=1}^{\infty}$ in $M(A)$ such that α^u is conjugate to β and $\lim_{n\to\infty}\|u_g - v_n\alpha_g(v_n^*)\| = 0$ for all $g \in G$.

(7) Suppose A equals B. The two actions α and β are said to be KK-trivially cocycle conjugate, if there exist $\mu \in \mathrm{Aut}(A)$ with $KK(\mu) = 1$ and an α-cocycle $\{u_g\}_{g\in G}$ in $M(A)$ such that $\alpha_g^u = \mu^{-1} \circ \beta_g \circ \mu$ for all $g \in G$.

(8) Suppose A equals B. The two actions α and β are said to be strongly KK-trivially cocycle conjugate, if there exist μ and $\{u_g\}_{g\in G}$ as in (7) such that there exists a sequence of unitaries $\{v_n\}_{n=1}^{\infty}$ in $M(A)$ satisfying $\lim_{n\to\infty}\|u_g - v_n\alpha_g(v_n^*)\| = 0$ for all $g \in G$.

Let α and β be actions of a discrete group G on unital C^*-algebras A and B, respectively. We let $\mathrm{Hom}_G(A, B)$ denote the set of all $\rho \in \mathrm{Hom}(A, B)$ such that $\rho \circ \alpha_g = \beta_g \circ \rho$ for every $g \in G$. Two homomorphisms $\rho, \sigma \in \mathrm{Hom}_G(A, B)$ are said to be G-asymptotically unitarily equivalent, if there exists a continuous family of unitaries $\{u_t\}_{t\in[0,\infty)}$ in B such that

$$\rho(a) = \lim_{t\to\infty} \mathrm{Ad}\, u_t(\sigma(a))$$

for all $a \in A$ and

$$\lim_{t\to\infty} \|u_t - \beta_g(u_t)\| = 0$$

for all $g \in G$. In an analogous way, one can define G-approximately unitarily equivalence.

We let $\mathrm{Hom}_{\hat{G}}(A \rtimes_\alpha G, B \rtimes_\beta G)$ denote the set of all $\rho \in \mathrm{Hom}(A \rtimes_\alpha G, B \rtimes_\beta G)$ such that

$$(\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha} = \hat{\beta} \circ \rho$$

and let $\mathrm{Aut}_{\hat{G}}(A \rtimes_\alpha G) = \mathrm{Hom}_{\hat{G}}(A \rtimes_\alpha G, A \rtimes_\alpha G) \cap \mathrm{Aut}(A \rtimes_\alpha G)$. Two homomorphisms $\rho, \sigma \in \mathrm{Hom}_{\hat{G}}(A \rtimes_\alpha G, B \rtimes_\beta G)$ are said to be $\hat{G}$-asymptotically unitarily equivalent, if there exists a continuous family of unitaries $\{u_t\}_{t\in[0,\infty)}$ in B such that

$$\rho(x) = \lim_{t\to\infty} \mathrm{Ad}\, u_t(\sigma(x))$$

for all $x \in A \rtimes_\alpha G$. In an analogous way, one can define $\hat{G}$-approximately unitarily equivalence.

Next, we give the definition of asymptotic representability of group actions.

Definition 2.2. Let G be a countable discrete group and let A be a unital C^*-algebra. An action $\alpha : G \curvearrowright A$ is said to be asymptotically representable, if there exists a continuous family of unitaries $\{v_g(t)\}_{t\in[0,\infty)}$ in $U(A)$ for each $g \in G$ such that

$$\lim_{t\to\infty} \|v_g(t)v_h(t) - v_{gh}(t)\| = 0,$$

$$\lim_{t\to\infty} \|\alpha_g(v_h(t)) - v_{ghg^{-1}}(t)\| = 0,$$

and

$$\lim_{t\to\infty} \|v_g(t)av_g(t)^* - \alpha_g(a)\| = 0$$

hold for all $g, h \in G$ and $a \in A$.

Approximate representability is defined in an analogous way (see [8, Definition 3.6]).

We now recall the definition of cocycle actions. Let A be a C^*-algebra and let G be a discrete group. A pair (α, u) of a map $\alpha : G \to \mathrm{Aut}(A)$ and a map $u : G \times G \to U(A)$ is called a cocycle action of G on A, if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$$

and

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk)$$

hold for any $g, h, k \in G$. A cocycle action (α, u) is said to be outer, if α_g is outer for every $g \in G$ except for the neutral element. Two cocycle actions (α, u) and (β, v) of G on a C^*-algebra A are said to be equivalent, if there exists a map $w : G \to U(A)$ such that

$$\alpha_g = \mathrm{Ad}\, w(g) \circ \beta_g$$

and

$$u(g,h) = \alpha_g(w(h))w(g)v(g,h)w(gh)^*$$

for every $g, h \in G$.

Let A be a separable C^*-algebra and let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. We set

$$c^\omega(A) = \{(a_n) \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\omega} \|a_n\| = 0\},$$

$$A^\omega = \ell^\infty(\mathbb{N}, A)/c^\omega(A).$$

We identify A with the C^*-subalgebra of A^ω consisting of equivalence classes of constant sequences. We let

$$A_\omega = A^\omega \cap A'.$$

When α is an automorphism on A or a (cocycle) action of a discrete group on A, we can consider its natural extension on A^ω and A_ω. We denote it by the same symbol α.

A simple C^*-algebra A is said to be purely infinite, if for every nonzero elements $x, y \in A$, there exist $a, b \in A$ such that $y = axb$. There are various remarkable properties fulfilled by purely infinite C^*-algebras [3], and we use them freely in the sequel as far as they are found in [25, Chapter 4]. Note that if A is purely infinite and simple, then so is A^ω. A purely infinite simple unital C^*-algebra is said to be in the Cuntz standard form, if $[1]$ equals zero in $K_0(A)$. The following fact is also used frequently.

Theorem 2.3 ([17, Lemma 10])**.** *Let $\alpha : G \curvearrowright A$ be an outer action of a countable discrete group G on a unital purely infinite simple C^*-algebra A. Then, the reduced crossed product C^*-algebra $A \rtimes_\alpha G$ is also purely infinite simple.*

A simple purely infinite nuclear separable C^*-algebra is called a Kirchberg algebra. We recall several facts from the classification theory of Kirchberg algebras mainly due to Kirchberg, Phillips and Rørdam.

Theorem 2.4 ([12],[13, Proposition 1.4])**.** *Let A be a unital purely infinite simple C^*-algebra and let $C \subset A$ be a unital separable subalgebra. For any nuclear unital completely positive map $\rho : C \to A$, there exists a sequence of nonunitary isometries $\{s_n\}_{n=1}^\infty$ in A such that $\rho(x) = \lim_{n\to\infty} s_n^* x s_n$ for all $x \in C$.*

Theorem 2.5 ([13, Proposition 3.4]). *When A is a unital Kirchberg algebra, A_ω is purely infinite and simple.*

Theorem 2.6 ([22, Theorem 4.1.1]). *Let A be a unital separable nuclear simple C^*-algebra and let B be a unital separable C^*-algebra that satisfies $B \cong B \otimes \mathcal{O}_\infty$.*

(1) *For every $x \in KK(A,B)$ satisfying $K_0(x)([1]) = [1]$, there exists $\rho \in \mathrm{Hom}(A,B)$ such that $KK(\rho) = x$.*

(2) *If $\rho, \sigma \in \mathrm{Hom}(A,B)$ satisfy $KK(\rho) = KK(\sigma)$, then ρ and σ are asymptotically unitarily equivalent.*

We next summarize a few results of [20].

Theorem 2.7. *Let A be a unital simple separable C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$ be an outer automorphism. Then, the extensions of α to A^ω and A_ω are both outer.*

Proof. This follows from [20, Lemma 2] and its proof. □

Theorem 2.8 ([20, Theorem 1]). *For an automorphism α of a unital Kirchberg algebra A, the following conditions are equivalent.*

(1) *α is aperiodic, that is, α^n is outer for every $n \in \mathbb{Z} \setminus \{0\}$.*

(2) *α has the Rohlin property.*

Theorem 2.9 ([20, Theorem 5]). *Let A be a unital Kirchberg algebra and let $\alpha, \beta \in \mathrm{Aut}(A)$ be aperiodic automorphisms. The following are equivalent.*

(1) *$KK(\alpha) = KK(\beta)$.*

(2) *α and β are KK-trivially cocycle conjugate.*

As an immediate consequence of the theorem above, we have the following.

Lemma 2.10. *Let A be a unital Kirchberg algebra and let α be an aperiodic automorphism. The following are equivalent.*

(1) *$KK(\alpha) = 1_A$.*

(2) *α is asymptotically representable.*

Proof. (2)⇒(1) is trivial, and so we show the other implication. If $KK(\alpha) = 1_A$, by Theorem 2.9, the automorphism α is cocycle conjugate to an automorphism of the form

$$(\mathrm{id}_A \otimes \bigotimes_{n=1}^{\infty} \mathrm{Ad}\, u_n, A \otimes \bigotimes_{n=1}^{\infty} \mathcal{O}_\infty),$$

where u_n is a unitary in $\mathcal{O}_\infty$ with finite spectrum, which shows the statement. □

3 Equivariant Kirchberg's theorem

Throughout this section, let A denote a unital Kirchberg algebra and let $\alpha : G \curvearrowright A$ be an approximately representable outer action of a discrete countable amenable group G. We show an equivariant version of Theorem 2.4 and 2.5. In what follows we often regard $A^\omega \rtimes_\alpha G$ as a subalgebra of $(A \rtimes_\alpha G)^\omega$ and identify $(A_\omega)^\alpha$ with $A^\omega \cap (A \rtimes_\alpha G)' \subset (A \rtimes_\alpha G)^\omega$.

Theorem 3.1. *Let $C \subset A^\omega$ be a unital separable nuclear globally α-invariant C^*-subalgebra. For any $\rho \in \mathrm{Hom}_G(C, A^\omega)$, there exists an isometry $s \in (A^\omega)^\alpha$ such that $\rho(x) = s^*xs$ for all $x \in C$.*

Proof. We regard $C \rtimes_\alpha G$ as a subalgebra of $A^\omega \rtimes_\alpha G$. We let $\tilde{\rho} : C \rtimes_\alpha G \to A^\omega \rtimes_\alpha G$ denote the extension of ρ determined by $\tilde{\rho}(\lambda_g^\alpha) = \lambda_g^\alpha$ for all $g \in G$.

It suffices to show the following: for any finite subsets $F_1 \subset C$, $F_2 \subset G$ and $\varepsilon > 0$, there exists an isometry $s \in A^\omega$ such that

$$\|\rho(x) - s^*xs\| < \varepsilon \text{ and } \|\lambda_g^\alpha - s^*\lambda_g^\alpha s\| < \varepsilon$$

for all $x \in F_1$ and $g \in F_2$. By Theorem 2.3 and Theorem 2.7, $A^\omega \rtimes_\alpha G$ is purely infinite and simple. Then Kirchberg's theorem ([12],[13, Proposition 1.4]) shows that there exists an isometry $t \in A^\omega \rtimes_\alpha G$ such that

$$\|\tilde{\rho}(x) - t^*xt\| < \varepsilon$$

for all $x \in F_1 \cup \{\lambda_g^\alpha \mid g \in F_2\}$. Choose a unital separable α-invariant subalgebra D of A^ω so that $C, \rho(C) \subset D$ and $t \in D \rtimes_\alpha G$. Since $\alpha : G \curvearrowright A$ is approximately representable, we can find a family of unitaries $\{w_g\}_{g\in G}$ in A^ω such that

$$w_g w_h = w_{gh},\ \alpha_g(w_h) = w_{ghg^{-1}} \text{ and } \alpha_g(x) = w_g x w_g^*$$

for all $g, h \in G$ and $x \in D$. Define a homomorphism

$$\varphi : (D \rtimes_\alpha G) \otimes C^*(G) \to A^\omega \rtimes_\alpha G$$

by

$$\varphi(x \otimes 1) = x,\ \varphi(\lambda_g^\alpha \otimes 1) = w_g \text{ and } \varphi(1 \otimes \lambda_g) = w_g^*\lambda_g^\alpha$$

for every $x \in D$ and $g \in G$. From

$$\|\rho(x) \otimes 1 - (t^* \otimes 1)(x \otimes 1)(t \otimes 1)\| < \varepsilon, \quad \text{for all } x \in F_1$$

and

$$\|\lambda_g^\alpha \otimes \lambda_g - (t^* \otimes 1)(\lambda_g^\alpha \otimes \lambda_g)(t \otimes 1)\| < \varepsilon, \quad \text{for all } g \in F_2,$$

one has

$$\|\tilde{\rho}(x) - \varphi(t \otimes 1)^* x \varphi(t \otimes 1)\| < \varepsilon$$

for all $x \in F_1 \cup \{\lambda_g^\alpha \mid g \in F_2\}$. Hence $s = \varphi(t \otimes 1) \in A^\omega$ meets the requirement. □

Corollary 3.2. (1) *The C^*-algebra $(A^\omega)^\alpha$ is purely infinite and simple.*

(2) *The C^*-algebra $(A_\omega)^\alpha$ is purely infinite and simple.*

Proof. We show only (2). (1) can be shown in a similar way. It is easy to verify $(A_\omega)^\alpha \neq \mathbb{C}$ (see the proof of Proposition 3.5). Let $a \in (A_\omega)^\alpha$ be a positive element of norm one and let $C \subset A^\omega$ be the C^*-algebra generated by A and a. Since A is simple, C is isomorphic to $A \otimes C^*(a, 1)$. Hence there exists a homomorphism $\rho : C \to A^\omega$ such that $\rho(x) = x$ for $x \in A$ and $\rho(a) = 1$. From Theorem 3.1, one obtains an isometry $s \in (A^\omega)^\alpha$ such that $s^*xs = x$ for all $x \in A$ and $s^*as = 1$. For any $x \in A$,

$$\|[s, x]\|^2 = \|(sx - xs)^*(sx - xs)\| = \|x^*x - x^*s^*xs - s^*x^*sx + s^*x^*xs\| = 0,$$

and so s belongs to $(A_\omega)^\alpha$. Therefore $(A_\omega)^\alpha$ is purely infinite and simple. □

We will need the following lemma in Section 4.

Lemma 3.3. *Let $\varepsilon > 0$ be a positive real number and let $F \subset A \rtimes_\alpha G$ be a finite subset. If $u : [0, 1] \times [0, 1] \to U(A)$ is a continuous map satisfying*

$$\|[u(s, t), x]\| < \frac{\varepsilon}{27}$$

for all $x \in F$ and $s, t \in [0, 1]$, then there exists a continuous map $v : [0, 1] \times [0, 1] \to U(A)$ such that

$$v(s, 0) = u(s, 0), \quad v(s, 1) = u(s, 1), \quad \mathrm{Lip}(v(s, \cdot)) < 6\pi$$

and

$$\|[v(s, t), x]\| < \varepsilon$$

for all $x \in F$ and $s, t \in [0, 1]$.

Proof. By Corollary 3.2 (2), the α-fixed point subalgebra $(A_\omega)^\alpha$ of A_ω contains a unital copy of $\mathcal{O}_\infty$. Therefore the assertion follows from [20, Theorem 7] and its proof. □

The next lemma follows from [14, Lemma 1.1] and [20, Lemma 3].

Lemma 3.4. *Let $\{\beta_i\}$ be a countable family of outer automorphisms of A. Then there exists a non-zero projection $p \in A_\omega$ such that p is orthogonal to $\beta_i(p)$ for all i.*

The following proposition is an analogue of [10, Theorem 4.8]. Note that we do not need amenability of G and approximate representability of α for this proposition.

Proposition 3.5. *Let β be an automorphism of A such that $\beta\alpha_g$ is not inner for all $g \in G$. Then β is not the identity on $(A_\omega)^\alpha$.*

Proof. Let $F \subset G$ be a finite subset and let $n \in \mathbb{N}$. It suffices to construct $a \in A_\omega$ such that

$$\|a\| = 1, \ \|\alpha_g(a) - a\| \leq 1/n \text{ and } \|\beta(a) - a\| = 1$$

for every $g \in F$.

Let $G_0 \subset G$ be the subgroup generated by F and let $l : G_0 \to \mathbb{Z}$ be the length function with respect to $F \cup F^{-1}$. Applying the lemma above to $\{\alpha_g \mid g \in G_0 \setminus \{e\}\} \cup \{\alpha_h\beta\alpha_g \mid g, h \in G_0\}$, we get a non-zero projection $p \in A_\omega$ such that $\{\alpha_g(p), \beta(\alpha_h(p))\}_{g,h \in G_0}$ is a set of mutually orthogonal projections. Then

$$a = \sum_{g \in G_0} \frac{n - \min\{n, l(g)\}}{n} \alpha_g(p) \in A_\omega$$

is the desired element. □

The following theorem is an equivariant version of [20, Theorem 1].

Theorem 3.6. *Let β be an automorphism of A such that the map $(n, g) \mapsto \beta^n \alpha_g$ induces an injective homomorphism from $\mathbb{Z} \times G$ to $\mathrm{Out}(A)$. Then for any $N \in \mathbb{N}$, there exist projections $e_0, e_1, \dots, e_{N-1}$, $f_0, f_1, \dots, f_N$ in $(A_\omega)^\alpha$ such that*

$$\sum_{i=0}^{N-1} e_i + \sum_{j=0}^{N} f_j = 1, \quad \beta(e_i) = e_{i+1} \quad \text{and} \quad \beta(f_j) = f_{j+1}$$

for all $i = 0, 1, \dots, N-1$ and $j = 0, 1, \dots, N$, where e_N and f_{N+1} mean e_0 and f_0, respectively.

Proof. The proposition above shows that the restriction of β^n to $(A_\omega)^\alpha$ is a non-trivial automorphism for every $n \neq 0$, and hence it is outer thanks to [20, Lemma 2]. This means that we can choose a Rohlin tower for β in $(A_\omega)^\alpha$. We omit the detail, because the argument is exactly the same as [20, Theorem 1]. □

The following corollary is an immediate consequence of the theorem above and [19, Remark 2]. See [19, Section 2] (or [18, Section 2]) for the definition of the Rohlin property of $\mathbb{Z}^N$-actions.

Corollary 3.7. *When G is $\mathbb{Z}^N$, the action α has the Rohlin property.*

In Corollary 6.4, we will show that asymptotic representability of α is not necessary for the statement above.

4 Asymptotically representable actions

Throughout this section, we assume that A is a unital separable nuclear C^*-algebra and that α is an asymptotically representable action of a countable amenable group G on A. We fix $v_g = (v_g(t))_{t \geq 0} \in U(C^b([0, \infty), A))$ as in Definition 2.2. We denote the crossed product $A \rtimes_\alpha G$ by B and the implementing unitary representation of G in B by $\{\lambda_g^\alpha\} \subset B$. Let $C^*(G)$ be the group C^*-algebra of G, which is generated by the left regular representation $\{\lambda_g\}_{g \in G}$. Let $\hat{\alpha}$ be the dual coaction of α, which is a homomorphism from B to $B \otimes C^*(G)$ determined by

$$\hat{\alpha}(x) = x \otimes 1 \quad \text{and} \quad \hat{\alpha}(\lambda_g^\alpha) = \lambda_g^\alpha \otimes \lambda_g$$

for $x \in A$ and $g \in G$. Then $\hat{\alpha}$ satisfies coassociativity

$$(\hat{\alpha} \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha} = (\mathrm{id}_B \otimes \delta_G) \circ \hat{\alpha},$$

where $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ is the coproduct of $C^*(G)$ determined by $\delta_G(\lambda_g) = \lambda_g \otimes \lambda_g$.

Let $C_0 = C_0([0, \infty), B) = C_0([0, \infty)) \otimes B$. We regard B as a subalgebra of $C^b([0, \infty), B)$ and set C^b to be the C^*-subalgebra of $C^b([0, \infty), B)$ generated by $C^b([0, \infty), A)$ and $\{\lambda_g^\alpha\}_g$. The action α of G on A gives rise to an action of G on $C^b([0, \infty), A)/C_0([0, \infty), A)$ and the crossed product by G is isomorphic to C^b/C_0 in a canonical way. We denote the coaction of $C^*(G)$ on C^b/C_0 by $\hat{\alpha}^\infty$. We let π denote the quotient map $\pi : C^b \to C^b/C_0$.

Define a unital homomorphism φ from $B \otimes C^*(G)$ to C^b/C_0 by

$$\varphi(x \otimes 1) = \pi(x),\ \varphi(\lambda_g^\alpha \otimes 1) = \pi(v_g) \ \text{ and } \ \varphi(1 \otimes \lambda_g) = \pi(v_g^* \lambda_g^\alpha)$$

for $x \in A$ and $g \in G$. It is easy to show the following.

Lemma 4.1. *The homomorphism φ satisfies*

$$\varphi \circ \hat{\alpha} = \pi | B$$

and

$$\hat{\alpha}^\infty \circ \varphi = (\varphi \otimes \mathrm{id}_{C^*(G)}) \circ (\mathrm{id}_B \otimes \delta_G).$$

For simplicity, we often omit π and identify $x \in B$ with $\pi(x)$ in what follows. In particular, one has $\hat{\alpha} = \hat{\alpha}^\infty | B$.

Since $B \otimes C^*(G)$ is nuclear, there exists a unital completely positive map $\tilde{\varphi}$ from $B \otimes C^*(G)$ to C^b satisfying $\pi \circ \tilde{\varphi} = \varphi$. We fix such $\tilde{\varphi}$. For $t \geq 0$, we set $\tilde{\varphi}_t = \chi_t \circ \tilde{\varphi}$, where $\chi_t : C^b \to B$ is the point evaluation map at t. The family $\{\tilde{\varphi}_t\}_{t \geq 0}$ is an asymptotic morphism from $B \otimes C^*(G)$ to B.

Lemma 4.2. *For any compact subsets $K \subset B \otimes C^*(G)$, $K' \subset B$ and a positive number ε, there exists $t_0 \geq 0$ such that the following hold for all $t \geq t_0$.*

(1) *For all $x, y \in K$, $\|\tilde{\varphi}_t(x)\tilde{\varphi}_t(y) - \tilde{\varphi}_t(xy)\| < \varepsilon$.*

(2) *For all $x \in K$, $\|\hat{\alpha}(\tilde{\varphi}_t(x)) - (\tilde{\varphi}_t \otimes \mathrm{id}_{C^*(G)})(\mathrm{id}_B \otimes \delta_G)(x)\| < \varepsilon$.*

(3) *For all $x \in K'$, $\|\tilde{\varphi}_t(\hat{\alpha}(x)) - x\| < \varepsilon$.*

Proof. This immediately follows from Lemma 4.1. □

Lemma 4.3. *Let ρ be a unital endomorphism of B. Assume*

$$\|(\mathrm{Ad}\, w \circ \hat{\alpha} \circ \rho)(x) - ((\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha})(x)\| < \varepsilon, \tag{4.1}$$

$$\|((\mathrm{Ad}\, w \circ \hat{\alpha} \circ \rho) \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x)) - (((\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha}) \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < \varepsilon, \tag{4.2}$$

for some $x \in B$ and $w \in U(B \otimes C^*(G))$. Then the following inequalities hold:

$$\|[(\mathrm{id}_B \otimes \delta_G)(w^*)(w \otimes 1)(\hat{\alpha} \otimes \mathrm{id}_{C^*(G)})(w), ((\hat{\alpha} \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha} \circ \rho)(x)]\| < 3\varepsilon, \tag{4.3}$$

$$\|(\hat{\alpha}^\infty \circ \mathrm{Ad}\, \varphi(w) \circ \rho)(x) - ((\mathrm{Ad}\, \varphi(w) \circ \rho) \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < 4\varepsilon. \tag{4.4}$$

Proof. From (4.1) and (4.2), we get

$$\begin{aligned} &\|(\mathrm{Ad}(w \otimes 1) \circ \mathrm{Ad}(\hat{\alpha} \otimes \mathrm{id}_{C^*(G)}(w)) \circ (\hat{\alpha} \otimes \mathrm{id}_{C^*(G)}))(\hat{\alpha}(\rho(x))) \\ &\quad - ((\rho \otimes \mathrm{id}_{C^*(G)} \otimes \mathrm{id}_{C^*(G)}) \circ (\hat{\alpha} \otimes \mathrm{id}_{C^*(G)}))(\hat{\alpha}(x))\| < 2\varepsilon. \end{aligned} \tag{4.5}$$

On the other hand, applying $\mathrm{id}_B \otimes \delta_G$ to the inside of the norm of (4.1) and using coassociativity, we get

$$\begin{aligned} &\|(\mathrm{Ad}((\mathrm{id}_B \otimes \delta_G)(w)) \circ (\hat{\alpha} \otimes \mathrm{id}_{C^*(G)}))(\hat{\alpha}(\rho(x))) \\ &\quad - ((\rho \otimes \mathrm{id}_{C^*(G)} \otimes \mathrm{id}_{C^*(G)}) \circ (\hat{\alpha} \otimes \mathrm{id}_{C^*(G)}))(\hat{\alpha}(x))\| < \varepsilon. \end{aligned} \tag{4.6}$$

Then, (4.5) and (4.6) imply (4.3).

Direct computation implies

$$\begin{aligned}
&\|(\hat{\alpha}^\infty \circ \mathrm{Ad}\,\varphi(w) \circ \rho)(x) - ((\mathrm{Ad}\,\varphi(w) \circ \rho) \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| \\
&= \|\mathrm{Ad}((\varphi(w^*) \otimes 1)\hat{\alpha}^\infty(\varphi(w)))(\hat{\alpha}^\infty(\rho(x)) - (\rho \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x)))\| \\
&\leq \varepsilon + \|\mathrm{Ad}((\varphi(w^*) \otimes 1)\hat{\alpha}^\infty(\varphi(w)))(\hat{\alpha}^\infty(\rho(x)) - \mathrm{Ad}\,w(\hat{\alpha}^\infty(\rho(x))))\| \\
&= \varepsilon + \|\mathrm{Ad}(w^*(\varphi(w^*) \otimes 1)\hat{\alpha}^\infty(\varphi(w)))(\hat{\alpha}^\infty(\rho(x))) - \hat{\alpha}^\infty(\rho(x))\|,
\end{aligned}$$

where we used (4.1). Thanks to Lemma 4.1, we have

$$\begin{aligned}
&w^*(\varphi(w^*) \otimes 1)\hat{\alpha}^\infty(\varphi(w)) \\
&= ((\varphi\hat{\alpha}) \otimes \mathrm{id}_{C^*(G)})(w^*)(\varphi(w^*) \otimes 1)((\varphi \otimes \mathrm{id}_{C^*(G)}) \circ (\mathrm{id}_B \otimes \delta_G))(w) \\
&= (\varphi \otimes \mathrm{id}_{C^*(G)})((\hat{\alpha} \otimes \mathrm{id}_{C^*(G)})(w^*)(w^* \otimes 1)(\mathrm{id}_B \otimes \delta_G)(w)).
\end{aligned}$$

Now Lemma 4.1 and (4.3) imply (4.4). □

Lemma 4.4. *Let ρ be a unital endomorphism of B and let $\{w_0(s)\}_{s\geq 0}$ be a continuous family of unitaries in $B \otimes C^*(G)$ such that*

$$\lim_{s\to\infty} \|\mathrm{Ad}\,w_0(s)(\hat{\alpha}(\rho(x))) - (\rho \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| = 0$$

for all $x \in B$. Then we have the following.

(1) *There exists a continuous family of unitaries $\{w(t)\}_{t\geq 0}$ in B such that*

$$\lim_{t\to\infty} \|\hat{\alpha}((\mathrm{Ad}\,w(t) \circ \rho)(x)) - ((\mathrm{Ad}\,w(t) \circ \rho) \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| = 0.$$

(2) *Assume further that $w_0(0) = 1$ and that for a finite subset F of B and a positive number ε, the inequality*

$$\|[w_0(s), \hat{\alpha}(\rho(x))]\| < \varepsilon$$

holds for all $s \geq 0$ and $x \in F$. Then $w(t)$ can be chosen so that $w(0) = 1$ and

$$\|[w(t), \rho(x)]\| < \varepsilon$$

for all $t \geq 0$ and $x \in F$.

Proof. (1) We choose an increasing sequence $\{F_n\}_{n=0}^\infty$ of finite subsets of B whose union is dense in B. We may assume that the following hold for any non-negative integer n, $x \in F_n$ and $s \geq n$:

$$\|\mathrm{Ad}\,w_0(s)(\hat{\alpha}(\rho(x))) - (\rho \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < 2^{-n},$$

$$\|((\mathrm{Ad}\,w_0(s) \circ \hat{\alpha} \circ \rho) \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x)) - (((\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha}) \otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < 2^{-n}.$$

We set $w_1(s,t) = \tilde{\varphi}_t(w_0(s))$. Thanks to Lemma 4.2 and Lemma 4.3, for each non-negative integer n, there exists $t_n \geq 0$ such that the following hold: for all $t \geq t_n$, $s \in [n, n+1]$ and $x \in F_n$, we have

$$\|w_1(s,t)^* w_1(s,t) - 1\| < 2^{-n}, \qquad \|w_1(s,t) w_1(s,t)^* - 1\| < 2^{-n}$$

and

$$\|(\hat{\alpha}\circ \mathrm{Ad}\, w_1(s,t)\circ\rho)(x) - ((\mathrm{Ad}\, w_1(s,t)\circ\rho)\otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < 2^{-n+2}.$$

We may assume that the sequence $\{t_n\}_{n=0}^{\infty}$ is increasing. Let L be the piecewise linear path in $[0,\infty)^2$ starting from $(0,t_0)$ and connecting the following points by line segments in order:

$$(0,t_0),(1,t_0),(1,t_1),(2,t_1),\dots,(n,t_n),(n+1,t_n),(n+1,t_{n+1}),\dots.$$

Then the function $w_1(s,t)$ is continuous on L and $w_1(s,t)$ is invertible for all $(s,t)\in L$. Let f be a homeomorphism from $[0,\infty)$ onto L with $f(0)=(0,t_0)$. Then $w(t)=w_1(f(t))|w_1(f(t))|^{-1}$ gives the desired family of unitaries.

(2) Since $\varphi\circ\hat{\alpha}=\pi|B$, we have

$$\|[\varphi(w_0(s)),\rho(x)]\| \le \|[w_0(s),\hat{\alpha}(\rho(x))]\|.$$

Thus the statement follows from the above construction with a slight modification. □

By using this lemma, we can prove the following theorem.

Theorem 4.5. *Let ρ be a unital endomorphism of B such that $\hat{\alpha}\circ\rho$ and $(\rho\otimes\mathrm{id}_{C^*(G)})\circ\hat{\alpha}$ are asymptotically unitarily equivalent. Then there exists a unital endomorphism ρ_1 of B such that ρ_1 is asymptotically unitarily equivalent to ρ and*

$$\hat{\alpha}\circ\rho_1 = (\rho_1\otimes\mathrm{id}_{C^*(G)})\circ\hat{\alpha}.$$

Proof. We choose an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of B whose union is dense in B. Thanks to Lemma 4.4, there exists a continuous family of unitaries $\{w(t)\}_{t\ge 0}$ in B satisfying

$$\lim_{t\to\infty}\|\hat{\alpha}((\mathrm{Ad}\, w(t)\circ\rho)(x)) - ((\mathrm{Ad}\, w(t)\circ\rho)\otimes\mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| = 0$$

for all $x\in B$. We set $\rho^{(0)}=\rho$, $w^{(0)}(t)=w(t)$ and $t_0=0$.

We choose $t_1\ge 0$ such that

$$\|\hat{\alpha}((\mathrm{Ad}\, w(t)\circ\rho)(x)) - ((\mathrm{Ad}\, w(t)\circ\rho)\otimes\mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < 2^{-1}$$

holds for all $t\ge t_1$ and $x\in F_1$. Let $\rho^{(1)}=\mathrm{Ad}\, w^{(0)}(t_1)\circ\rho^{(0)}$ and $u^{(1)}(t)=w^{(0)}(t)w^{(0)}(t_1)^*$ for $t\ge t_1$. Then we have $u^{(1)}(t_1)=1$ and

$$\|\hat{\alpha}((\mathrm{Ad}\, u^{(1)}(t)\circ\rho^{(1)})(x)) - ((\mathrm{Ad}\, u^{(1)}(t)\circ\rho^{(1)})\otimes\mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < 2^{-1}$$

for all $t\ge t_1$ and $x\in F_1$. Besides,

$$\lim_{t\to\infty}\|\hat{\alpha}((\mathrm{Ad}\, u^{(1)}(t)\circ\rho^{(1)})(x)) - ((\mathrm{Ad}\, u^{(1)}(t)\circ\rho^{(1)})\otimes\mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| = 0$$

holds for all $x\in B$.

For $t\ge t_1$, let $v^{(1)}(t)=(u^{(1)}(t)^*\otimes 1_{C^*(G)})\hat{\alpha}(u^{(1)}(t))$. Then $\{v^{(1)}(t)\}_{t\ge t_1}$ is a continuous family of unitaries in $B\otimes C^*(G)$ with $v^{(1)}(t_1)=1$ satisfying

$$\|[v^{(1)}(t),\hat{\alpha}(\rho^{(1)}(x))]\| < 1 \quad \text{for all } t\ge t_1 \text{ and } x\in F_1$$

and

$$\lim_{t\to\infty}\|\mathrm{Ad}\, v^{(1)}(t)(\hat{\alpha}(\rho^{(1)}(x))) - (\rho^{(1)}\otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| = 0 \quad \text{for all } x \in B.$$

Thus thanks to Lemma 4.4, there exists a continuous family of unitaries $\{w^{(1)}(t)\}_{t\geq t_1}$ in B with $w^{(1)}(t_1) = 1$ satisfying

$$\|[w^{(1)}(t), \rho^{(1)}(x)]\| < 1 \quad \text{for all } t \geq t_1 \text{ and } x \in F_1$$

and

$$\lim_{t\to\infty}\|\hat{\alpha}((\mathrm{Ad}\, w^{(1)}(t)\circ\rho^{(1)})(x)) - ((\mathrm{Ad}\, w^{(1)}(t)\circ\rho^{(1)})\otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| = 0 \quad \text{for all } x \in B.$$

Repeating this argument, we can construct a sequence of unital endomorphisms $\{\rho^{(n)}\}_{n=0}^{\infty}$, an increasing sequence of positive numbers $\{t_n\}_{n=0}^{\infty}$ and a sequence of continuous families of unitaries $\{\{w^{(n)}(t)\}_{t\geq t_n}\}_{n=0}^{\infty}$ with $w^{(n)}(t_n) = 1$ for $n \geq 0$ satisfying

$$\rho^{(n+1)} = \mathrm{Ad}\, w^{(n)}(t_{n+1})\circ\rho^{(n)},$$

$$\|\hat{\alpha}(\rho^{(n)}(x)) - (\rho^{(n)}\otimes \mathrm{id}_{C^*(G)})(\hat{\alpha}(x))\| < 2^{-n} \quad \text{for all } x \in F_n$$

and

$$\|[w^{(n)}(t), \rho^{(n)}(x)]\| < 2^{-n+1} \quad \text{for all } t \geq t_n \text{ and } x \in F_n.$$

We construct a continuous family of unitaries $\{w'(t)\}_{t\geq 0}$ in B as follows. For $t \in [t_0, t_1]$, we set $w'(t) = w^{(0)}(t)$. For $t \in [t_n, t_{n+1}]$ with $n \geq 1$, we set $w'(t) = w^{(n)}(t)w'(t_n)$. Note that $\rho^{(n)} = \mathrm{Ad}\, w'(t_n)\circ\rho$ holds.

Let $x \in F_n$ and $n < m$. For $t \geq t_m$, we choose an integer $l \geq m$ such that $t \in [t_l, t_{l+1}]$. Then one has

$$\begin{aligned}
&\|\mathrm{Ad}\, w'(t)(\rho(x)) - \mathrm{Ad}\, w'(t_m)(\rho(x))\| \\
&\leq \|\mathrm{Ad}\, w'(t)(\rho(x)) - \mathrm{Ad}\, w'(t_l)(\rho(x))\| + \sum_{k=m}^{l-1}\|\mathrm{Ad}\, w'(t_{k+1})(\rho(x)) - \mathrm{Ad}\, w'(t_k)(\rho(x))\| \\
&= \|\mathrm{Ad}\, w^{(l)}(t)(\rho^{(l)}(x)) - \rho^{(l)}(x)\| + \sum_{k=m}^{l-1}\|\mathrm{Ad}\, w^{(k)}(t_{k+1})(\rho^{(k)}(x)) - \rho^{(k)}(x)\| \\
&\leq 2^{-l+1} + \sum_{k=m}^{l-1} 2^{-k+1} < 2^{-m+2},
\end{aligned}$$

which implies that $\lim_{t\to\infty}\mathrm{Ad}\, w'(t)(\rho(x))$ exists for any x in the dense subset $\bigcup_{n=1}^{\infty} F_n$ of B. This shows that there exists a unital endomorphism ρ_1 of B satisfying $\hat{\alpha}\circ\rho_1 = (\rho_1\otimes \mathrm{id}_{C^*(G)})\circ\hat{\alpha}$ and

$$\lim_{t\to\infty}\mathrm{Ad}\, w'(t)(\rho(x)) = \rho_1(x)$$

for all $x \in B$, which completes the proof. □

Remark 4.6. Let $\gamma : G \curvearrowright C$ be an asymptotically representable action of G on a unital separable nuclear C^*-algebra C and let $D = C \rtimes_\gamma G$. In a similar fashion to the theorem above, we can show the following: if $\rho : D \to B$ is an isomorphism such that $\hat{\alpha}\circ\rho$ and $(\rho\otimes \mathrm{id}_{C^*(G)})\circ\hat{\gamma}$ are asymptotically unitarily equivalent, then there exists an isomorphism $\rho_1 : D \to B$ such that ρ_1 is asymptotically unitarily equivalent to ρ and $\hat{\alpha}\circ\rho_1 = (\rho_1\otimes \mathrm{id}_{C^*(G)})\circ\hat{\gamma}$.

Remark 4.7. When $\alpha : G \curvearrowright A$ is an approximately representable action of G, one can prove the following in an analogous fashion to the theorem above: if ρ is a unital endomorphism of B such that $\hat{\alpha} \circ \rho$ and $(\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha}$ are approximately unitarily equivalent, then there exists a unital endomorphism ρ_1 of B such that ρ_1 is approximately unitarily equivalent to ρ and $\hat{\alpha} \circ \rho_1 = (\rho_1 \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha}$.

Moreover, when $\gamma : G \curvearrowright C$ is an approximately representable action of G on a unital separable nuclear C^*-algebra C and $D = C \rtimes_\gamma G$, one can prove the following in a similar way: if $\rho : D \to B$ is an isomorphism such that $\hat{\alpha} \circ \rho$ and $(\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\gamma}$ are approximately unitarily equivalent, then there exists an isomorphism $\rho_1 : D \to B$ such that ρ_1 is approximately unitarily equivalent to ρ and $\hat{\alpha} \circ \rho_1 = (\rho_1 \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha}$. This is a generalization of [8, Corollary 3.9].

Theorem 4.8. *Let $\gamma : G \curvearrowright C$ be an action of G on a unital separable nuclear C^*-algebra C and let $D = C \rtimes_\gamma G$. For any two homomorphisms $\rho, \sigma \in \mathrm{Hom}_{\hat{G}}(D, B)$, the following conditions are equivalent.*

(1) *The two homomorphisms ρ and σ are $\hat{G}$-asymptotically unitarily equivalent.*

(2) *The two homomorphisms ρ and σ are asymptotically unitarily equivalent.*

Proof. The implication (1)⇒(2) is trivial. We would like to show that (2) implies (1), assuming that α is asymptotically representable. Suppose that $\{w_0(s)\}_{s \geq 0}$ is a continuous family of unitaries in $B = A \rtimes_\alpha G$ satisfying

$$\lim_{s \to \infty} \mathrm{Ad}\, w_0(s)(\rho(x)) = \sigma(x)$$

for all $x \in D$. We choose an increasing sequence $\{F_n\}_{n=0}^{\infty}$ of finite subsets of D whose union is dense in D. We may assume

$$\|((\mathrm{Ad}\, w_0(s) \circ \rho) \otimes \mathrm{id}_{C^*(G)})(\hat{\gamma}(x)) - (\sigma \otimes \mathrm{id}_{C^*(G)})(\hat{\gamma}(x))\| < 2^{-n}$$

holds for all $s \geq n$ and $x \in F_n$. From

$$\hat{\alpha} \circ \rho = (\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\gamma}$$

and

$$\hat{\alpha} \circ \sigma = (\sigma \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\gamma},$$

we have

$$\|(w_0(s) \otimes 1)\hat{\alpha}(\rho(x)) - \hat{\alpha}(\sigma(x))(w_0(s) \otimes 1)\| < 2^{-n}$$

for all $s \geq n$ and $x \in F_n$. Then, Lemma 4.1 implies

$$\|\varphi(w_0(s) \otimes 1)\rho(x) - \sigma(x)\varphi(w_0(s) \otimes 1)\| < 2^{-n}$$

for all $s \geq n$ and $x \in F_n$. Let $E : B \to A$ be the canonical conditional expectation determined by $E(u_g^\alpha) = 0$ for $g \in G \setminus \{e\}$. Note that since $\varphi(B \otimes \mathbb{C}) \subset C^*(A \cup \{v_g\})/C_0$, the two asymptotic morphisms $\{\tilde{\varphi}_t\}_{t \geq 0}$ and $\{E \circ \tilde{\varphi}_t\}_{t \geq 0}$ are equivalent on $B \otimes \mathbb{C}$, that is,

$$\lim_{t \to \infty} \|\tilde{\varphi}_t(x \otimes 1) - E(\tilde{\varphi}_t(x \otimes 1))\| = 0$$

for all $x \in B$. We set $w_1(s,t) = E(\tilde{\varphi}_t(w_0(s) \otimes 1)) \in A$.

Thanks to Lemma 4.2, for each non-negative integer n, there exists $t_n \geq 0$ such that the following hold: for any $t \geq t_n$, $s \in [n, n+1]$ and $x \in F_n$, one has

$$\|w_1(s,t)^* w_1(s,t) - 1\| < 2^{-n}, \quad \|w_1(s,t) w_1(s,t)^* - 1\| < 2^{-n}$$

and

$$\|w_1(s,t)\rho(x) - \sigma(x) w_1(s,t)\| < 2^{-n}.$$

The rest of the proof is the same as that of Lemma 4.4. □

Remark 4.9. When $\alpha : G \curvearrowright A$ is an approximately representable action of G, one can prove the following in an analogous fashion to the theorem above: for any two homomorphisms $\rho, \sigma \in \mathrm{Hom}_{\hat{G}}(D, B)$, the following conditions are equivalent.

(1) The two homomorphisms ρ and σ are $\hat{G}$-approximately unitarily equivalent.

(2) The two homomorphisms ρ and σ are approximately unitarily equivalent.

When $\{u_g\}_{g \in G}$ is an α-cocycle in A, one can define a homomorphism $\iota_u : C^*(G) \to B$ by $\iota_u(\lambda_g) = u_g \lambda_g^\alpha$ for $g \in G$. Clearly ι_u belongs to $\mathrm{Hom}_{\hat{G}}(C^*(G), B)$, where $C^*(G)$ is regarded as the crossed product of $\mathbb{C}$ by the trivial action of G. The perturbed action $\alpha^u : G \curvearrowright A$ is the action defined by $\alpha_g^u = \mathrm{Ad}\, u_g \circ \alpha_g$.

Corollary 4.10. *Suppose that $\alpha : G \curvearrowright A$ is approximately representable and $\{u_g\}_g, \{v_g\}_g$ are α-cocycles in A.*

(1) *There exists a sequence of unitaries $\{s_n\}_{n=1}^\infty$ in A satisfying*

$$\lim_{n \to \infty} s_n u_g \alpha_g(s_n^*) = v_g$$

for all $g \in G$ if and only if ι_u and ι_v are approximately unitarily equivalent.

(2) *If there exists $\mu \in \mathrm{Aut}_{\hat{G}}(B)$ such that $\mu \circ \iota_u$ and ι_v are approximately unitarily equivalent, then the two actions α^u and α^v are strongly cocycle conjugate. When $G = \mathbb{Z}^N$ and B is purely infinite simple, the converse also holds.*

Proof. (1) Suppose that there exists a sequence of unitaries $\{s_n\}_{n=1}^\infty$ in A such that $\lim_{n\to\infty} s_n u_g \alpha_g(s_n^*) = v_g$ for all $g \in G$. It follows that

$$\lim_{n \to \infty} s_n \iota_u(\lambda_g) s_n^* = \lim_{n \to \infty} s_n u_g \lambda_g^\alpha s_n^* = \lim_{n \to \infty} s_n u_g \alpha_g(s_n^*) \lambda_g^\alpha = v_g \lambda_g^\alpha = \iota_v(\lambda_g),$$

which means that ι_u and ι_v are approximately unitarily equivalent.

Conversely, if ι_u and ι_v are approximately unitarily equivalent, then by Remark 4.9, ι_u and ι_v are $\hat{G}$-approximately unitarily equivalent. Thus there exists a sequence of unitaries $\{s_n\}_{n=1}^\infty$ in A such that $\lim_{n\to\infty} s_n \iota_u(\lambda_g) s_n^* = \iota_v(\lambda_g)$ for all $g \in G$. Hence $s_n u_g \alpha_g(s_n^*) = s_n u_g \lambda_g^\alpha s_n^* \lambda_g^{\alpha *}$ goes to $v_g \lambda_g^\alpha \lambda_g^{\alpha *} = v_g$ as n goes to infinity.

(2) Take a finite subset $F \subset G$ and $\varepsilon > 0$ arbitrarily. By Remark 4.9, $\mu \circ \iota_u$ and ι_v are $\hat{G}$-approximately unitarily equivalent. Therefore there exists a sequence of unitaries $\{s_n\}_{n=1}^\infty$ in A such that

$$\lim_{n \to \infty} \|s_n \mu(u_g \lambda_g^\alpha) s_n^* - v_g \lambda_g^\alpha\| = 0$$

for all $g \in G$. Put $w_g = \mu(u_g \lambda_g^\alpha)\lambda_g^{\alpha *} v_g^*$. Then $\{w_g\}_{g\in G}$ is an α^v-cocycle in A and $\operatorname{Ad} w_g \circ \alpha_g^v = \mu \circ \alpha_g^u \circ \mu^{-1}$ on A. In addition,

$$\lim_{n\to\infty} \|w_g - s_n^* \alpha_g^v(s_n)\| = 0$$

which completes the proof.

We turn to the case $G = \mathbb{Z}^N$. Suppose that the two actions α^u and α^v are strongly cocycle conjugate. Let $F \subset \mathbb{Z}^N$ be a finite generating set and let $\varepsilon > 0$. There exists an isomorphism $\mu : A \to A$ and an α^v-cocycle $\{w_g\}_{g\in\mathbb{Z}^N}$ such that

$$\operatorname{Ad} w_g \circ \alpha_g^v = \mu \circ \alpha_g^u \circ \mu^{-1} \quad \forall g \in \mathbb{Z}^N \quad \text{and} \quad \|w_g - 1\| < \varepsilon \quad \forall g \in F.$$

Define $\tilde{\mu} \in \operatorname{Aut}_{\mathbb{T}^N}(B)$ by

$$\tilde{\mu}(a) = a \quad \forall a \in A \quad \text{and} \quad \tilde{\mu}(\lambda_g^\alpha) = \mu(u_g^*) w_g v_g \lambda_g^\alpha \quad \forall g \in \mathbb{Z}^N.$$

It is easy to see $\|\tilde{\mu}(\iota_u(\lambda_g)) - \iota_v(\lambda_g)\| < \varepsilon$ for every $g \in F$. Since $C^*(\mathbb{Z}^N)$ satisfies the universal coefficient theorem and its K-groups are free abelian groups of finite rank, if $\varepsilon > 0$ is sufficiently small, we can conclude that $KK(\tilde{\mu} \circ \iota_u) = KK(\iota_v)$. It follows from [4, Theorem 1.7] that $\tilde{\mu} \circ \iota_u$ and ι_v are approximately unitarily equivalent. □

Finally in this section, we show a G-equivariant version of Nakamura's theorem [20, Theorem 5]. Suppose that A is a unital Kirchberg algebra and that $\alpha : G \curvearrowright A$ is outer. Let β_1 and β_2 be automorphisms on A satisfying

$$\beta_i \circ \alpha_g = \operatorname{Ad} u_{i,g} \circ \alpha_g \circ \beta_i$$

for $i = 1, 2$ and $g \in G$, where $\{u_{1,g}\}_{g\in G}$ and $\{u_{2,g}\}_{g\in G}$ are α-cocycles in A. Then, for each $i = 1, 2$, there exists an extension $\tilde{\beta}_i \in \operatorname{Aut}(B)$ determined by $\tilde{\beta}_i(\lambda_g^\alpha) = u_{i,g}\lambda_g^\alpha$. Note that $\tilde{\beta}_i$ belongs to $\operatorname{Aut}_{\hat{G}}(B)$. We further assume that, for each $i = 1, 2$, the map $(n, g) \mapsto \beta_i^n \alpha_g$ induces an injection from $\mathbb{Z} \times G$ to $\operatorname{Out}(A)$.

Theorem 4.11. *In the setting above, if $KK(\tilde{\beta}_1) = KK(\tilde{\beta}_2)$, then there exists an automorphism $\mu \in \operatorname{Aut}_{\hat{G}}(B)$ and a unitary $v \in A$ such that $KK(\mu) = 1_B$, $KK(\mu|A) = 1_A$ and*

$$\mu \circ \tilde{\beta}_1 \circ \mu^{-1} = \operatorname{Ad} v \circ \tilde{\beta}_2.$$

Proof. We apply the argument of [20, Theorem 5] to $\tilde{\beta}_1$ and $\tilde{\beta}_2$. By Theorem 2.3, B is a unital Kirchberg algebra. Since $KK(\tilde{\beta}_1) = KK(\tilde{\beta}_2)$, by Theorem 2.6 (2), $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are asymptotically unitarily equivalent. Then Theorem 4.8 implies that $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are $\hat{G}$-asymptotically unitarily equivalent. Moreover, by Theorem 3.6, we can find Rohlin projections for $\tilde{\beta}_i$ in $(A_\omega)^\alpha \subset B_\omega$. Hence, by using Lemma 3.3 instead of [20, Theorem 7], the usual intertwining argument shows the statement. □

5 Uniqueness of $\mathbb{Z}^N$-actions on $\mathcal{O}_\infty$

Throughout this section, we let $\mathbb{T}$ denote the set of all complex numbers with absolute value one. For a discrete group G, let $Z^2(G,\mathbb{T})$ denote the abelian group of all 2-cocycles from $G\times G$ to $\mathbb{T}$, that is,

$$Z^2(G,\mathbb{T}) = \{\omega : G\times G\to\mathbb{T} \mid \omega(g,h)\omega(gh,k) = \omega(h,k)\omega(g,hk) \quad \forall g,h,k\in G\}$$

and let $B^2(G,\mathbb{T})$ denote the subgroup of 2-coboundaries in $Z^2(G,\mathbb{T})$, i.e. elements of the form $\omega(g,h) = f(g)f(h)f(gh)^{-1}$ for some map $f: G\to\mathbb{T}$. Two 2-cocycles $\omega,\omega'\in Z^2(G,\mathbb{T})$ are said to be cohomologous, if $\omega\omega'^{-1}$ is in $B^2(G,\mathbb{T})$.

Let $\alpha : G\curvearrowright A$ be an action of a discrete group G on a unital C^*-algebra A. We say that α absorbs $\omega\in Z^2(G,\mathbb{T})$, if there exists an automorphism μ of A and a family of unitaries $\{u_g\}_{g\in G}$ in A satisfying

$$\mu\circ\alpha_g\circ\mu^{-1} = \operatorname{Ad} u_g\circ\alpha_g$$

and

$$u_g\alpha_g(u_h) = \omega(g,h)u_{g,h}$$

for every $g,h\in G$. If $\omega,\omega'\in Z^2(G,\mathbb{T})$ are cohomologous and α absorbs ω, then clearly α absorbs ω', too.

The following is an easy observation and we omit the proof.

Lemma 5.1. *Suppose that two actions α,β of a discrete group G on a unital simple C^*-algebra A are outer conjugate.*

(1) *If α absorbs $\omega\in Z^2(G,\mathbb{T})$, then so does β.*

(2) *If α absorbs all elements in $Z^2(G,\mathbb{T})$, then α and β are cocycle conjugate.*

When G is abelian, as described in [21], the second cohomology group $H^2(G,\mathbb{T})\cong Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ is isomorphic to the subgroup

$$\{\chi\in\operatorname{Hom}(G,\hat{G}) \mid \langle\chi(g),g\rangle = 1 \text{ for every } g\in G\}$$

of $\operatorname{Hom}(G,\hat{G})$, where $\langle\cdot,\cdot\rangle$ is the paring of $\hat{G}$ and G. The isomorphism between them is given by sending $\omega\in Z^2(G,\mathbb{T})$ to

$$\chi(g)(h) = \omega(g,h)\omega(h,g)^{-1} \quad g,h\in G.$$

In particular, when G is $\mathbb{Z}^N$, $H^2(G,\mathbb{T}) = H^2(\mathbb{Z}^N,\mathbb{T})$ is isomorphic to

$$\{(\theta_{i,j})_{1\le i\le j\le N}\in\mathbb{T}^{N^2} \mid \theta_{i,i} = 1 \text{ and } \theta_{i,j}\theta_{j,i} = 1 \text{ for all } i,j\}\cong\mathbb{T}^{N(N-1)/2}.$$

Lemma 5.2. *For any $\omega\in Z^2(\mathbb{Z}^N,\mathbb{T})$, there exists an outer $\mathbb{Z}^N$-action α on the Cuntz algebra $\mathcal{O}_\infty$ which absorbs ω.*

Proof. From the fact mentioned above, ω corresponds to some $(\theta_{i,j})_{i,j} \in \mathbb{T}^{N^2}$ satisfying $\theta_{i,i} = 1$ and $\theta_{i,j}\theta_{j,i} = 1$ for all $i, j = 1, 2, \ldots, N$. Therefore, it suffices to construct an outer $\mathbb{Z}^N$-action α on $\mathcal{O}_\infty$, a family of unitaries $\{u_i\}_{i=1}^N$ in $\mathcal{O}_\infty$ and an automorphism μ of $\mathcal{O}_\infty$ such that

$$\mu \circ \alpha_i \circ \mu^{-1} = \operatorname{Ad} u_i \circ \alpha_i$$

and

$$u_i \alpha_i(u_j) = \theta_{i,j} u_j \alpha_j(u_i)$$

for every $i, j = 1, 2, \ldots, N$, where α_i denotes the i-th generator of the $\mathbb{Z}^N$-action α.

First, we claim that $\mathcal{O}_\infty$ contains a family of unitaries $\{v_i\}_{i=1}^N$ satisfying $v_i v_j = \theta_{i,j} v_j v_i$. For each $\theta_{i,j}$, it is easy to find two unitaries $x_{i,j}$ and $y_{i,j}$ such that $x_{i,j} y_{i,j} = \theta_{i,j} y_{i,j} x_{i,j}$ in a C^*-algebra $B_{i,j}$ isomorphic to $\mathcal{O}_\infty$ (see [6] for example). Let $B = \bigotimes_{1 \le i < j \le N} B_{i,j}$. We regard $B_{i,j}$ as a subalgebra of B. For $i = 1, 2, \ldots, N$, we set

$$v_i = \left(\prod_{1 \le k < i} y_{k,i} \right) \left(\prod_{i < l \le N} x_{i,l} \right).$$

It is straightforward to see the commutation relation $v_i v_j = \theta_{i,j} v_j v_i$.

We define a $\mathbb{Z}^N$-action α on $A = \bigotimes_{k=0}^\infty B$ by

$$\alpha_i = \mathrm{id} \otimes \mathrm{id} \otimes \bigotimes_{k=2}^{\infty} \operatorname{Ad} v_i$$

for $i = 1, 2, \ldots, N$. Let

$$u_i = 1 \otimes v_i \otimes 1 \otimes \cdots.$$

Since $\mathcal{O}_\infty$ is isomorphic to $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$, α is conjugate to the $\mathbb{Z}^N$-action determined by $\operatorname{Ad} u_i \circ \alpha_i$. In addition, $u_i \alpha_i(u_j) = \theta_{i,j} u_j \alpha_j(u_i)$ holds. By tensoring another outer $\mathbb{Z}^N$-action if necessary, we may assume that α is outer, thereby completing the proof. □

Lemma 5.3. *Let α be an action of $\mathbb{Z}^N$ on the Cuntz algebra $\mathcal{O}_\infty$. Then, the canonical inclusion $\iota_N : C^*(\mathbb{Z}^N) \to \mathcal{O}_\infty \rtimes_\alpha \mathbb{Z}^N$ induces a KK-equivalence.*

Proof. We use the induction on N. When $N = 0$, the assertion is clear. Assume that the claim has been shown for $N - 1$.

Let α be an action of $\mathbb{Z}^N$ on $\mathcal{O}_\infty$. We regard $\mathbb{Z}^{N-1}$ as a subgroup of $\mathbb{Z}^N$ via the map $(n_1, n_2, \ldots, n_{N-1}) \mapsto (n_1, n_2, \ldots, n_{N-1}, 0)$. Let A be the crossed product of $\mathcal{O}_\infty$ by $\mathbb{Z}^{N-1}$ and let α_N denote the N-th generator of α. Then, α_N extends to an automorphism on A and $\mathcal{O}_\infty \rtimes_\alpha \mathbb{Z}^N$ is canonically identified with $A \rtimes_{\alpha_N} \mathbb{Z}$. From the induction hypothesis, the inclusion $\iota_{N-1} : C^*(\mathbb{Z}^{N-1}) \to A$ induces a KK-equivalence. In addition, ι_{N-1} is a covariant homomorphism from $(C^*(\mathbb{Z}^{N-1}), \mathrm{id})$ to (A, α_N). By the naturality of the Pimsner-Voiculescu exact sequence, we obtain the following commutative diagram, in

which the vertical sequences are exact.

$$\begin{array}{ccc}
K_*(C^*(\mathbb{Z}^{N-1})) & \xrightarrow[\cong]{K_*(\iota_{N-1})} & K_*(A) \\
\downarrow & & \downarrow \\
K_*(C^*(\mathbb{Z}^{N-1})) & \xrightarrow[\cong]{K_*(\iota_{N-1})} & K_*(A) \\
\downarrow & & \downarrow \\
K_*(C^*(\mathbb{Z}^{N})) & \xrightarrow{K_*(\iota_{N})} & K_*(A \rtimes_{\alpha_N} \mathbb{Z}) \\
\downarrow & & \downarrow \\
K_{1-*}(C^*(\mathbb{Z}^{N-1})) & \xrightarrow[\cong]{K_{1-*}(\iota_{N-1})} & K_{1-*}(A) \\
\downarrow & & \downarrow \\
K_{1-*}(C^*(\mathbb{Z}^{N-1})) & \xrightarrow[\cong]{K_{1-*}(\iota_{N-1})} & K_{1-*}(A)
\end{array}$$

From an easy diagram chase, we can conclude that $K_*(\iota_N)$ is an isomorphism for $* = 0, 1$. The universal coefficient theorem [26] implies that $KK(\iota_N)$ is invertible. □

Now we can prove the uniqueness of outer $\mathbb{Z}^N$-actions on $\mathcal{O}_\infty$ up to cocycle conjugacy.

Theorem 5.4. *Any outer $\mathbb{Z}^N$-actions on $\mathcal{O}_\infty$ are cocycle conjugate to each other. In consequence, they are asymptotically representable.*

Proof. The proof is by induction on N. When $N = 1$, we get the conclusion from Theorem 2.9 and Lemma 2.10. Assume that the claim has been shown for $N - 1$.

Let α and β be outer $\mathbb{Z}^N$-actions on $A \cong \mathcal{O}_\infty$. Let α' and β' be the $\mathbb{Z}^{N-1}$-actions generated by the first $N - 1$ generators of α and β, respectively. We denote the N-th generator of α and β by α_N and β_N, respectively. From the induction hypothesis, by conjugating β if necessary, we may assume that there exists an α'-cocycle $\{u_g\}_{g\in\mathbb{Z}^{N-1}}$ in A such that $\beta_g = \operatorname{Ad} u_g \circ \alpha_g$. Moreover, α' is asymptotically representable. It is easy to check

$$\beta_N \circ \alpha'_g = (\operatorname{Ad} \beta_N(u_g^*)u_g) \circ \alpha'_g \circ \beta_N$$

for all $g \in \mathbb{Z}^{N-1}$ and $\{\beta_N(u_g^*)u_g\}_g$ is an α'-cocycle.

Let B_α and B_β be the crossed product of A by the $\mathbb{Z}^{N-1}$-actions α' and β', respectively. There exists an isomorphism $\pi : B_\beta \to B_\alpha$ such that $\pi(a) = a$ for all $a \in A$ and $\pi(\lambda_g^\beta) = u_g\lambda_g^\alpha$ for all $g \in \mathbb{Z}^{N-1}$. The automorphisms α_N and β_N of A extend to automorphisms $\tilde{\alpha}_N$ and $\tilde{\beta}_N$ of B_α and B_β, respectively. By Lemma 5.3, we have $KK(\tilde{\alpha}_N) = 1$ and $KK(\tilde{\beta}_N) = 1$, and hence

$$KK(\pi \circ \tilde{\beta}_N \circ \pi^{-1}) = 1 = KK(\tilde{\alpha}_N)$$

in $KK(B_\alpha, B_\alpha)$. By applying Theorem 4.11 to α', $\tilde{\alpha}_N$ and $\pi \circ \tilde{\beta}_N \circ \pi^{-1}$, we obtain $\mu \in \operatorname{Aut}_{\mathbb{T}^{N-1}}(B_\alpha)$ and $v \in U(A)$ satisfying the conditions stated there. For each $g \in \mathbb{Z}^{N-1}$,

$\mu|A$ commutes with α_g up to an inner automorphism of A. Furthermore,

$$\begin{aligned}\operatorname{Ad} v \circ \alpha_N &= (\operatorname{Ad} v \circ \tilde{\alpha}_N)|A \\ &= (\mu \circ \pi \circ \tilde{\beta}_N \circ \pi^{-1} \circ \mu^{-1})|A \\ &= (\mu|A) \circ \beta_N \circ (\mu|A)^{-1}.\end{aligned}$$

It follows that the $\mathbb{Z}^N$-actions α and β are outer conjugate. Thus, any outer $\mathbb{Z}^N$-actions on $\mathcal{O}_\infty$ are outer conjugate to each other. Now the conclusion follows from Lemma 5.1 and Lemma 5.2. □

6 Uniqueness of asymptotically representable $\mathbb{Z}^N$-actions

Let G be a countable infinite discrete amenable group and let $\{S_g\}_{g\in G}$ be the generator of the Cuntz algebra $\mathcal{O}_\infty$. Define an action γ^G of G on $\mathcal{O}_\infty$ by $\gamma^G_g(S_h) = S_{gh}$. We consider the diagonal action $\gamma^G \otimes \gamma^G$ on $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$. Clearly γ^G and $\gamma^G \otimes \gamma^G$ are both outer. Let

$$\rho_l : \mathcal{O}_\infty \ni x \mapsto x \otimes 1 \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty,$$

$$\rho_r : \mathcal{O}_\infty \ni x \mapsto 1 \otimes x \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty.$$

Then clearly ρ_l, ρ_r are in $\operatorname{Hom}_G(\mathcal{O}_\infty, \mathcal{O}_\infty \otimes \mathcal{O}_\infty)$. We let

$$\tilde{\rho}_l, \tilde{\rho}_r \in \operatorname{Hom}_{\hat{G}}(\mathcal{O}_\infty \rtimes_{\gamma^G} G,\ (\mathcal{O}_\infty \otimes \mathcal{O}_\infty) \rtimes_{\gamma^G \otimes \gamma^G} G)$$

denote the natural extensions of ρ_l and ρ_r, respectively.

Lemma 6.1. *If the diagonal action $\gamma^G \otimes \gamma^G$ is asymptotically (resp. approximately) representable, then ρ_l and ρ_r are G-asymptotically (resp. G-approximately) unitarily equivalent.*

Proof. Note that γ^G is a quasi-free action. Let $\iota : C^*(G) \to \mathcal{O}_\infty \rtimes_{\gamma^G} G$ be the inclusion map. It is well-known that $KK(\iota) \in KK(C^*(G), \mathcal{O}_\infty \rtimes_{\gamma^G} G)$ is invertible (see [24, Section 4]). From $\tilde{\rho}_l \circ \iota = \tilde{\rho}_r \circ \iota$, we can conclude $KK(\tilde{\rho}_l) = KK(\tilde{\rho}_r)$. It follows from Theorem 2.6 (2) that $\tilde{\rho}_l$ and $\tilde{\rho}_r$ are asymptotically unitarily equivalent. Hence, by Theorem 4.8 and Remark 4.9, $\tilde{\rho}_l$ and $\tilde{\rho}_r$ are $\hat{G}$-asymptotically unitarily equivalent (resp. $\hat{G}$-approximately unitarily equivalent). Therefore, ρ_l and ρ_r are G-asymptotically unitarily equivalent (resp. G-approximately unitarily equivalent). □

Lemma 6.2. *Let A be a unital Kirchberg algebra and let (α, u) be an outer cocycle action of G on A. Then, there exists $\sigma \in \operatorname{Hom}_G(\mathcal{O}_\infty, A_\omega)$.*

Proof. By Lemma 3.4, there exists a non-zero projection $p \in A_\omega$ such that $p\alpha_g(p) = 0$ for every $g \in G \setminus \{e\}$. Let $T \in A_\omega$ be an isometry satisfying $TT^* \le p$. Define a unital homomorphism $\sigma : \mathcal{O}_\infty \to A_\omega$ by $\sigma(S_g) = \alpha_g(T)$. Evidently σ belongs to $\operatorname{Hom}_G(\mathcal{O}_\infty, A_\omega)$. □

Theorem 6.3. *Let A be a unital Kirchberg algebra and let (α, u) be an outer cocycle action of G on A. If the diagonal action $\gamma^G \otimes \gamma^G$ of G on $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$ is approximately representable, then there exists an isomorphism $\mu : A \otimes \mathcal{O}_\infty \to A$ such that (α, u) is equivalent to $(\mu \circ (\alpha \otimes \gamma^G) \circ \mu^{-1}, \mu(u \otimes 1))$.*

Proof. By using Lemma 6.1 and Lemma 6.2, we can show the statement in a similar fashion to the proofs of Theorem 7.2.2, Theorem 7.2.1 and Proposition 2.3.5 of [25]. We leave the details to the reader. □

We obtain the following two corollaries.

Corollary 6.4. *Let A be a unital Kirchberg algebra and let (α, u) be an outer cocycle action of $\mathbb{Z}^N$ on A. Let γ be an outer action of $\mathbb{Z}^N$ on $\mathcal{O}_\infty$.*

(1) *There exists an isomorphism $\mu : A \otimes \mathcal{O}_\infty \to A$ such that (α, u) is equivalent to $(\mu \circ (\alpha \otimes \gamma) \circ \mu^{-1}, \mu(u \otimes 1))$.*

(2) *(α, u) has the Rohlin property.*

Proof. The first statement follows from Theorem 5.4 and Theorem 6.3. The second statement follows from Corollary 3.7, Theorem 5.4 and the first statement. □

Corollary 6.5. *Let A be a unital Kirchberg algebra and let α, β be outer $\mathbb{Z}^N$-actions on A. If α and β are outer conjugate, then they are cocycle conjugate.*

Proof. By Corollary 6.4 (1), (A, α) is cocycle conjugate to $(A \otimes \mathcal{O}_\infty, \alpha \otimes \gamma^{\mathbb{Z}^N})$. Then the assertion follows from Lemma 5.1, Lemma 5.2 and Theorem 5.4. □

Now we can prove the uniqueness of asymptotically representable outer $\mathbb{Z}^N$-actions on a unital Kirchberg algebra up to KK-trivial cocycle conjugacy in a similar fashion to Theorem 5.4.

Theorem 6.6. *Any asymptotically representable outer $\mathbb{Z}^N$-actions on a unital Kirchberg algebra A are KK-trivially cocycle conjugate to each other.*

Proof. The proof is by induction on N. When $N = 1$, we get the conclusion Theorem 2.9 and Lemma 2.10. Assume that the claim has been shown for $N - 1$.

Let α and β be asymptotically representable outer $\mathbb{Z}^N$-actions on A. We use the same notation as in the proof of Theorem 5.4. Since α' and β' are asymptotically representable outer $\mathbb{Z}^{N-1}$-actions on A, by the induction hypothesis, we may assume that there exists an α'-cocycle $\{u_g\}_{g \in \mathbb{Z}^{N-1}}$ in A such that $\beta_g = \mathrm{Ad}\, u_g \circ \alpha_g$. Moreover, as α and β are both asymptotically representable, one has $KK(\tilde{\alpha}_N) = 1$ in $KK(B_\alpha, B_\alpha)$ and $KK(\tilde{\beta}_N) = 1$ in $KK(B_\beta, B_\beta)$. In the same way as Theorem 5.4, we can find $\mu \in \mathrm{Aut}(A)$ with $KK(\mu) = 1_A$ that induces outer conjugacy between α and β. Then, from Corollary 6.5 and its proof, we can conclude that α and β are KK-trivially cocycle conjugate. □

Remark 6.7. A key step in the proof above is $KK(\tilde{\alpha}_N) = 1$ in $KK(B_\alpha, B_\alpha)$. When $A = \mathcal{O}_2$, the crossed product $B_\alpha = A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$ is again $\mathcal{O}_2$, and so $KK(B_\alpha, B_\alpha) = 0$. By using this observation, one can show the uniqueness of outer $\mathbb{Z}^N$-actions on $\mathcal{O}_2$ up to cocycle conjugacy, which is the main result of [18].

If there exists a path $\{\gamma_t\}_{t \in [0,1]}$ of automorphisms of A such that $\gamma_0 = \mathrm{id}$, $\gamma_1 = \alpha_N$ and $\gamma_t \circ \alpha'_g = \alpha'_g \circ \gamma_t$ for all $t \in [0, 1]$ and $g \in \mathbb{Z}^{N-1}$, then $KK(\tilde{\alpha}_N) = 1$ in $KK(B_\alpha, B_\alpha)$. Hence any outer $\mathbb{Z}^N$-action α on a unital Kirchberg algebra that extends to an $\mathbb{R}^N$-action is asymptotically representable. In particular, any outer quasi-free $\mathbb{Z}^N$-actions on the Cuntz algebra $\mathcal{O}_n$ are mutually cocycle conjugate.

In the rest of this section, we let $\alpha : \mathbb{Z}^N \curvearrowright A$ be an outer asymptotically representable action of $\mathbb{Z}^N$ on a unital Kirchberg algebra A and let $B = A \rtimes_\alpha \mathbb{Z}^N$. We would like to characterize elements in $\mathrm{Hom}_{\mathbb{T}^N}(B, B)$ in terms of KK-theory. As applications, we show Theorem 6.18 and Theorem 6.20. Let $H_{\mathbb{T}^N}(B, B)$ be the subset of $KK(B, B)$ consisting of x satisfying

$$K_0(x)([1]) = [1]$$

and

$$x \cdot KK(\hat{\alpha}) = KK(\hat{\alpha}) \cdot (x \otimes 1_{C^*(\mathbb{Z}^N)}),$$

and let $H_{\mathbb{T}^N}(B, B)^{-1}$ be the subset of $H_{\mathbb{T}^N}(B, B)$ consisting of invertible elements. Thanks to Theorem 4.5, Remark 4.6 and Theorem 4.8, we obtain the following.

Lemma 6.8. (1) *There exists a natural bijective correspondence between $H_{\mathbb{T}^N}(B, B)$ and the quotient space of $\mathrm{Hom}_{\mathbb{T}^N}(B, B)$ by $\mathbb{T}^N$-asymptotically unitary equivalence.*

(2) *There exists a natural bijective correspondence between $H_{\mathbb{T}^N}(B, B)^{-1}$ and the quotient space of $\mathrm{Aut}_{\mathbb{T}^N}(B)$ by $\mathbb{T}^N$-asymptotically unitary equivalence.*

Let $\delta_N : C^*(\mathbb{Z}^N) \to C^*(\mathbb{Z}^N) \otimes C^*(\mathbb{Z}^N)$ be the coproduct and let $j_\alpha : A \to B$ and $j : A \to A \otimes C^*(\mathbb{Z}^N)$ be the canonical embeddings. By the uniqueness of asymptotically representable outer actions of $\mathbb{Z}^N$ on a unital Kirchberg algebra, we get the following.

Lemma 6.9. *There exists a KK-equivalence $w \in KK(B, A \otimes C^*(\mathbb{Z}^N))$ such that*

$$KK(j_\alpha) \cdot w = KK(j)$$

and

$$w \cdot KK(\mathrm{id}_A \otimes \delta_N) = KK(\hat{\alpha}) \cdot (w \otimes 1_{C^*(\mathbb{Z}^N)}).$$

Proof. Let $\gamma : \mathbb{Z}^N \curvearrowright \mathcal{O}_\infty$ be an outer action and let $\iota : C^*(\mathbb{Z}^N) \to \mathcal{O}_\infty \rtimes_\gamma \mathbb{Z}^N$ be the canonical inclusion. We have the following commutative diagram.

$$\begin{array}{ccc} A \otimes C^*(\mathbb{Z}^N) & \xrightarrow{\mathrm{id}_A \otimes \iota} & (A \otimes \mathcal{O}_\infty) \rtimes_{\mathrm{id} \otimes \gamma} \mathbb{Z}^N \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes \delta_N} & & \downarrow{\scriptstyle \widehat{\mathrm{id} \otimes \gamma}} \\ A \otimes C^*(\mathbb{Z}^N) \otimes C^*(\mathbb{Z}^N) & \xrightarrow[\mathrm{id}_A \otimes \iota \otimes \mathrm{id}_{C^*(\mathbb{Z}^N)}]{} & ((A \otimes \mathcal{O}_\infty) \rtimes_{\mathrm{id} \otimes \gamma} \mathbb{Z}^N) \otimes C^*(\mathbb{Z}^N) \end{array}$$

By Lemma 5.3, $KK(\iota)$ is invertible, and so $KK(\mathrm{id}_A \otimes \iota)$ is also invertible.

Both α and $\mathrm{id} \otimes \gamma$ are asymptotically representable outer $\mathbb{Z}^N$-actions on $A \cong A \otimes \mathcal{O}_\infty$. It follows from Theorem 6.6 that there exists an isomorphism

$$\mu \in \mathrm{Hom}_{\mathbb{T}^N}(B, (A \otimes \mathcal{O}_\infty) \rtimes_{\mathrm{id} \otimes \gamma} \mathbb{Z}^N).$$

In other words, the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\mu} & (A \otimes \mathcal{O}_\infty) \rtimes_{\mathrm{id} \otimes \gamma} \mathbb{Z}^N \\ \downarrow{\scriptstyle \hat{\alpha}} & & \widehat{\mathrm{id} \otimes \gamma} \downarrow \\ B \otimes C^*(\mathbb{Z}^N) & \xrightarrow[\mu \otimes \mathrm{id}_{C^*(\mathbb{Z}^N)}]{} & ((A \otimes \mathcal{O}_\infty) \rtimes_{\mathrm{id} \otimes \gamma} \mathbb{Z}^N) \otimes C^*(\mathbb{Z}^N) \end{array}$$

is commutative. Then $w = KK(\mu) \cdot KK(\mathrm{id}_A \otimes \iota)^{-1}$ meets the requirement. □

In what follows we fix the KK-equivalence $w \in KK(B, A \otimes C^*(\mathbb{Z}^N))$ described in the lemma above.

Let $S = C_0((0,1))$ and let $\mathcal{T} = \mathbb{C} \oplus S$. Put $[N] = \{1, 2, \ldots, N\}$. The N-fold tensor product $\mathcal{T}^{\otimes N}$ has 2^N direct sum components and each of them is isomorphic to a tensor product of several copies of S. For $I \subset [N]$, we let $S_I \subset \mathcal{T}^{\otimes N}$ denote the tensor product of S's of the i-th tensor product component for all $i \in I$, so that

$$\mathcal{T}^{\otimes N} = \bigoplus_{I \subset [N]} S_I.$$

Note that $S_\emptyset$ is isomorphic to $\mathbb{C}$. When $I, J \subset [N]$ are disjoint, we may identify $S_I \otimes S_J$ with $S_{I \cup J}$. For $I \subset [N]$ and $J \subset I$, we let $\sigma(I, J)$ be the canonical invertible element in $KK(S_I, S_{I \setminus J} \otimes S_J)$, where $S_{I \setminus J} \otimes S_J$ is a subalgebra of $\mathcal{T}^{\otimes N} \otimes \mathcal{T}^{\otimes N}$. Define

$$\sigma_N \in KK(\mathcal{T}^{\otimes N}, \mathcal{T}^{\otimes N} \otimes \mathcal{T}^{\otimes N})$$

by

$$\sigma_N = \sum_{J \subset I \subset [N]} \sigma(I, J).$$

For $n \in \mathbb{N}$, we let $\delta_n : C^*(\mathbb{Z}^n) \to C^*(\mathbb{Z}^n) \otimes C^*(\mathbb{Z}^n)$ denote the coproduct. By a suitable permutation of tensor components of $C^*(\mathbb{Z}^n) \otimes C^*(\mathbb{Z}^n) \cong C^*(\mathbb{Z})^{\otimes 2n}$, δ_n can be viewed as $(\delta_1)^{\otimes n}$. Let $z \in KK(C^*(\mathbb{Z}), \mathcal{T})$ be an invertible element. We denote the N-fold tensor product of z by $z_N \in KK(C^*(\mathbb{Z}^N), \mathcal{T}^{\otimes N})$.

Lemma 6.10. *In the setting above, one has* $z_N \cdot \sigma_N \cdot (z_N \otimes z_N)^{-1} = KK(\delta_N)$.

Proof. The proof is by induction. It is easy to see that

$$z_1^{-1} \cdot KK(\delta_1) \cdot (z_1 \otimes z_1) = \sigma(\emptyset, \emptyset) + \sigma(\{1\}, \emptyset) + \sigma(\{1\}, \{1\}) = \sigma_1.$$

The conclusion follows from

$$\begin{aligned}
&\sigma_{n-1} \otimes \sigma_1 \\
&= \sum_{J \subset I \subset [n-1]} \sigma(I, J) \otimes \sigma(\emptyset, \emptyset) + \sigma(I, J) \otimes \sigma(\{n\}, \emptyset) + \sigma(I, J) \otimes \sigma(\{n\}, \{n\}) \\
&= \sum_{J \subset I \subset [n-1]} \sigma(I, J) + \sum_{J \subset I \subset [n], n \in I, n \notin J} \sigma(I, J) + \sum_{J \subset I \subset [n], n \in J} \sigma(I, J) \\
&= \sigma_n.
\end{aligned}$$

□

Let $e \in \mathcal{T}^{\otimes N}$ be the unit of $S_\emptyset \cong \mathbb{C}$. Let $H'_{\mathbb{T}^N}(B, B)$ be the subset of $KK(A \otimes \mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N})$ consisting of x satisfying

$$K_0(x)([1 \otimes e]) = [1 \otimes e]$$

and

$$x \cdot (1_A \otimes \sigma_N) = (1_A \otimes \sigma_N) \cdot (x \otimes 1_{\mathcal{T}^{\otimes N}}),$$

and let $H'_{\mathbb{T}^N}(B, B)^{-1}$ be the subset of $H'_{\mathbb{T}^N}(B, B)$ consisting of invertible elements.

From the above two lemmas, we have the following.

Lemma 6.11. (1) *The KK-equivalence $w \cdot (1_A \otimes z_N) \in KK(B, A \otimes \mathcal{T}^{\otimes N})$ induces a bijective correspondence between $H_{\mathbb{T}^N}(B,B)$ and $H'_{\mathbb{T}^N}(B,B)$.*

(2) *The KK-equivalence $w \cdot (1_A \otimes z_N) \in KK(B, A \otimes \mathcal{T}^{\otimes N})$ induces a bijective correspondence between $H_{\mathbb{T}^N}(B,B)^{-1}$ and $H'_{\mathbb{T}^N}(B,B)^{-1}$.*

Since

$$A \otimes \mathcal{T}^{\otimes N} = \bigoplus_{I \subset [N]} A \otimes S_I,$$

any element $x \in KK(A \otimes \mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N})$ can be written as

$$x = \sum_{I,J \subset [N]} x(I,J),$$

where $x(I,J)$ is in $KK(A \otimes S_I, A \otimes S_J)$. Now we calculate $H'_{\mathbb{T}^N}(B,B)$ and $H'_{\mathbb{T}^N}(B,B)^{-1}$ as follows.

Lemma 6.12. *For*

$$x = \sum_{I,J \subset [N]} x(I,J) \in KK(A \otimes \mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N}),$$

the following are equivalent.

(1) *x belongs to $H'_{\mathbb{T}^N}(B,B)$.*

(2) *For each $K \subset [N]$, there exists $y_K \in KK(A \otimes S_K, A)$ such that $K_0(y_\emptyset)([1]) = [1]$ and*

$$x(I,J) = \begin{cases} y_{I \setminus J} \otimes 1_{S_J} & \text{if } J \subset I \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, x is invertible if and only if $y_\emptyset$ is invertible.

Proof. The implication from (2) to (1) is straightforward. Let us show the other implication. Notice that $KK(A \otimes \mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N} \otimes \mathcal{T}^{\otimes N})$ is isomorphic to

$$\bigoplus_{I,J,J' \subset [N]} KK(A \otimes S_I, A \otimes S_{J'} \otimes S_J).$$

For any z in $KK(A \otimes \mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N} \otimes \mathcal{T}^{\otimes N})$, we denote its $KK(A \otimes S_I, A \otimes S_{J'} \otimes S_J)$ component by $z(I,J',J)$. One can see

$$\begin{aligned} &(x \cdot (1_A \otimes \sigma_N))(I,J',J) \\ &= \begin{cases} x(I, J' \cup J) \cdot (1_A \otimes \sigma(J' \cup J, J)) & \text{if } J' \cap J = \emptyset \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} &((1_A \otimes \sigma_N) \cdot (x \otimes 1_{\mathcal{T}^{\otimes N}}))(I,J',J) \\ &= \begin{cases} (1_A \otimes \sigma(I,J)) \cdot (x(I \setminus J, J') \otimes 1_{S_J}) & \text{if } J \subset I \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By letting $J' = \emptyset$, we have

$$\begin{aligned} &x(I,J)\cdot(1_A\otimes\sigma(J,J)) \\ &= \begin{cases} (1_A\otimes\sigma(I,J))\cdot(x(I\setminus J,\emptyset)\otimes 1_{\mathcal{T}^{\otimes N}}) & \text{if } J\subset I \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, if J is not contained in I, then $x(I,J)=0$. If J is a subset of I, then

$$\begin{aligned} &x(I,J) \\ &= (1_A\otimes\sigma(I,J))\cdot(x(I\setminus J,\emptyset)\otimes 1_{\mathcal{T}^{\otimes N}})\cdot(1_A\otimes\sigma(J,J))^{-1} \\ &= x(I\setminus J,\emptyset)\otimes 1_{S_J}, \end{aligned}$$

which implies (2). In addition, it is easy to see that

$$x=\sum_{J\subset I\subset[N]} x(I,J)$$

is of the form of an upper triangular matrix. Therefore, x is invertible if and only if the diagonal part is invertible. Since the diagonal part is given by

$$\sum_{I\subset[N]} y_\emptyset\otimes 1_{S_I},$$

the proof is completed. □

Notice that, for $x_1, x_2 \in H'_{\mathbb{T}^N}(B,B)$, the product $x_1\cdot x_2$ is computed as follows:

$$(x_1\cdot x_2)(I,\emptyset)=\sum_{J\subset I} x_1(I,J)\cdot x_2(J,\emptyset)=\sum_{J\subset I}(x_1(I\setminus J,\emptyset)\otimes 1_{S_J})\cdot x_2(J,\emptyset).$$

From Lemma 6.8, 6.11, 6.12 and this observation, we get the following proposition. For

$$y\in KK(A\otimes\mathcal{T}^{\otimes N},A)=\bigoplus_{I\subset[N]} KK(A\otimes S_I,A),$$

its $KK(A\otimes S_I,A)$ component is denoted by y_I.

Proposition 6.13. *There exists a surjective map* Θ *from* $\mathrm{Hom}_{\mathbb{T}^N}(B,B)$ *to*

$$H''=\{y\in KK(A\otimes\mathcal{T}^{\otimes N},A)\mid K_0(y_\emptyset)([1])=[1]\}$$

satisfying the following.

(1) $\rho_1,\rho_2\in\mathrm{Hom}_{\mathbb{T}^N}(B,B)$ *are* $\mathbb{T}^N$*-asymptotically unitarily equivalent if and only if* $\Theta(\rho_1)=\Theta(\rho_2)$.

(2) *If* y *is in* H'' *and* $y_\emptyset\in KK(A,A)$ *is invertible, then there exists* $\rho\in\mathrm{Aut}_{\mathbb{T}^N}(B)$ *such that* $\Theta(\rho)=y$.

(3) *For* $\rho_1, \rho_2 \in \mathrm{Hom}_{\mathbb{T}^N}(B, B)$, $\Theta(\rho_2 \circ \rho_1)$ *is given by*

$$\Theta(\rho_2 \circ \rho_1)_I = \sum_{J \subset I} (\Theta(\rho_1)_{I \setminus J} \otimes 1_{S_J}) \cdot \Theta(\rho_2)_J, \qquad I \subset [N].$$

Remark 6.14. When $A = \mathcal{O}_\infty$,

$$\{y \in H'' \mid y_\emptyset \text{ is invertible}\}$$

is isomorphic to $\mathbb{Z}^{2^{N-1}-1}$. For any $\rho \in \mathrm{Aut}_{\mathbb{T}^N}(B)$, $\alpha : \mathbb{Z}^N \curvearrowright A$ and $\rho|A$ give rise to a cocycle action of $\mathbb{Z}^{N+1}$ on A. Hence, when $N \geq 2$, there exist infinitely many outer cocycle $\mathbb{Z}^{N+1}$-actions on $\mathcal{O}_\infty$ which are not mutually equivalent. In other words, $\mathcal{O}_\infty \otimes \mathbb{K}$ admits infinitely many outer $\mathbb{Z}^{N+1}$-actions which are not mutually cocycle conjugate. Moreover, there exist infinitely many cocycle $\mathbb{Z}^{N+1}$-actions on $\mathcal{O}_\infty$ which are not equivalent to a genuine action, because outer $\mathbb{Z}^{N+1}$-actions on $\mathcal{O}_\infty$ are unique up to cocycle conjugacy by Theorem 5.4.

Let $H_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$ be the subset of $KK(C^*(\mathbb{Z}^N), B)$ consisting of x satisfying

$$K_0(x)([1]) = [1]$$

and

$$x \cdot KK(\hat{\alpha}) = KK(\delta_N) \cdot (x \otimes 1_{C^*(\mathbb{Z}^N)}).$$

There exists a natural map from $\mathrm{Hom}_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$ to $H_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$.

Let $H'_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$ be the subset of $KK(\mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N})$ consisting of a satisfying

$$K_0(a)([e]) = [1 \otimes e]$$

and

$$a \cdot (1_A \otimes \sigma_N) = \sigma_N \cdot (a \otimes 1_{\mathcal{T}^{\otimes N}}).$$

From Lemma 6.9 and 6.10, one has the following.

Lemma 6.15. *The map*

$$x \mapsto z_N^{-1} \cdot x \cdot w \cdot (1_A \otimes z_N)$$

gives a bijective correspondence between $H_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$ *and* $H'_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$.

For $a \in H'_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$ and $I, J \subset [N]$, we denote the $KK(S_I, A \otimes S_J)$ component of a by $a(I, J)$. Let $h : \mathbb{C} \to A$ be the unital homomorphism. In a similar fashion to Lemma 6.12, we have the following.

Lemma 6.16. *For*

$$a = \sum_{I,J \subset [N]} a(I, J) \in KK(\mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N}),$$

the following are equivalent.

(1) a *belongs to* $H'_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$.

(2) *For each* $K \subset [N]$*, there exists* $b_K \in KK(S_K, A)$ *such that* $b_\emptyset = KK(h)$ *and*

$$a(I,J) = \begin{cases} b_{I\setminus J} \otimes 1_{S_J} & \text{if } J \subset I \\ 0 & \text{otherwise.} \end{cases}$$

We let $\iota : C^*(\mathbb{Z}^N) \to B$ denote the canonical embedding. When $\{u_g\}_{g\in\mathbb{Z}^N}$ is an α-cocycle in A, one can define a homomorphism $\iota_u : C^*(\mathbb{Z}^N) \to B$ by $\iota_u(\lambda_g) = u_g\lambda_g^\alpha$ for $g \in \mathbb{Z}^N$. Clearly ι_u belongs to $\mathrm{Hom}_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$.

Lemma 6.17. *If* A *satisfies the universal coefficient theorem and*

$$K_i(A) = \{f([1]) \mid f \in \mathrm{Hom}(K_0(A), K_i(A))\}$$

for each $i = 0, 1$*, then for any* α*-cocycle* $\{u_g\}_{g\in\mathbb{Z}^N}$ *in* A*, there exists an automorphism* $\mu \in \mathrm{Aut}_{\mathbb{T}^N}(B)$ *such that* $KK(\mu|A) = 1_A$ *and* $KK(\mu\circ\iota) = KK(\iota_u)$*. When* $N = 1$*, we only need the assumption on* $K_1(A)$*.*

Proof. For $I \subset [N]$, we let $|I|$ denote the cardinality of I. There exists a natural isomorphism between $KK(S_I, A)$ and $K_i(A)$, where $i = 0$ if $|I|$ is even and $i = 1$ if $|I|$ is odd. From the assumption, we have the following: for any $q \in KK(S_I, A)$, there exists $p \in KK(A \otimes S_I, A)$ such that $KK(h \otimes \mathrm{id}_{S_I}) \cdot p = q$.

By Lemma 6.15,

$$a = z_N^{-1} \cdot KK(\iota) \cdot w \cdot (1_A \otimes z_N)$$

and

$$c = z_N^{-1} \cdot KK(\iota_u) \cdot w \cdot (1_A \otimes z_N)$$

are in $H'_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$. We claim that there exists $x \in H'_{\mathbb{T}^N}(B, B)$ such that $a \cdot x = c$.

We construct $y_I \in KK(A \otimes S_I, A)$ for each $I \subset [N]$ by using the induction on $|I|$. First, we let $y_\emptyset = 1_A \in KK(A, A)$. Suppose that y_I has been chosen for every $I \subset [N]$ with $|I| < n$. Take $I \subset [N]$ such that $|I| = n$. Put

$$q = c(I,\emptyset) - \sum_{J\subset I, J\neq I} a(I,J) \cdot y_J \in KK(S_I, A).$$

Then there exists $p \in KK(A \otimes S_I, A)$ such that $KK(h \otimes \mathrm{id}_{S_I}) \cdot p = q$. We let $y_I = p$. Define $x \in KK(A \otimes \mathcal{T}^{\otimes N}, A \otimes \mathcal{T}^{\otimes N})$ by

$$x(I,J) = \begin{cases} y_{I\setminus J} \otimes 1_{S_J} & \text{if } J \subset I \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 6.12, x belongs to $H'_{\mathbb{T}^N}(B, B)$. For every $I \subset [N]$,

$$\begin{aligned} (a \cdot x)(I,\emptyset) &= \sum_{J\subset I} a(I,J) \cdot x(J,\emptyset) \\ &= \sum_{J\subset I, J\neq I} a(I,J) \cdot y_J + a(I,I) \cdot y_I \\ &= c(I,\emptyset), \end{aligned}$$

because $a(I,I) = KK(h) \otimes 1_{S_I}$ by Lemma 6.16. Since $a \cdot x$ is in $H'_{\mathbb{T}^N}(C^*(\mathbb{Z}^N), B)$, we get $a \cdot x = c$ by Lemma 6.16. Notice that $x(\emptyset, \emptyset) = 1_A$.

It follows from Lemma 6.8 and 6.11 that there exists an automorphism $\mu \in \mathrm{Aut}_{\mathbb{T}^N}(B)$ such that $KK(\mu \circ j_\alpha) = KK(j_\alpha)$ and

$$KK(\mu) = w \cdot (1_A \otimes z_N) \cdot x \cdot (1_A \otimes z_N)^{-1} \cdot w^{-1}.$$

Then one has

$$\begin{aligned} KK(\mu \circ \iota) &= (z_N \cdot a \cdot (1_A \otimes z_N)^{-1} \cdot w^{-1}) \cdot (w \cdot (1_A \otimes z_N) \cdot x \cdot (1_A \otimes z_N)^{-1} \cdot w^{-1}) \\ &= z_N \cdot a \cdot x \cdot (1_A \otimes z_N)^{-1} \cdot w^{-1} \\ &= z_N \cdot c \cdot (1_A \otimes z_N)^{-1} \cdot w^{-1} \\ &= KK(\iota_u). \end{aligned}$$

By Theorem 2.6 (2), $\mu \circ j_\alpha$ and j_α are asymptotically unitarily equivalent in $\mathrm{Hom}(A, B)$. Since $\alpha : \mathbb{Z}^N \curvearrowright A$ is asymptotically representable, by using an argument similar to the proof of Theorem 4.8, one can show that $\mu \circ j_\alpha | A$ and $j_\alpha | A$ are asymptotically unitarily equivalent in $\mathrm{Hom}(A, A)$, which implies $KK(\mu|A) = 1_A$. □

Theorem 6.18. *Let A be a unital Kirchberg algebra satisfying the universal coefficient theorem and suppose that*

$$K_i(A) = \{f([1]) \mid f \in \mathrm{Hom}(K_0(A), K_i(A))\}$$

for each $i = 0, 1$. Then any asymptotically representable outer actions of $\mathbb{Z}^N$ on A are strongly KK-trivially cocycle conjugate to each other. When $N = 1$, we only need the assumption on $K_1(A)$.

Proof. Let α and β be asymptotically representable outer actions of $\mathbb{Z}^N$ on A. By Theorem 6.6, we may assume that there exists an α-cocycle $\{u_g\}_{g \in \mathbb{Z}^N}$ in A such that $\beta = \alpha^u$. Let $B = A \rtimes_\alpha \mathbb{Z}^N$. It follows from Lemma 6.17 that there exists an automorphism $\mu \in \mathrm{Aut}_{\mathbb{T}^N}(B)$ such that $KK(\mu|A) = 1_A$ and $KK(\mu \circ \iota) = \iota_u$. By [4, Theorem 1.7], $\mu \circ \iota$ and ι_u are approximately unitarily equivalent. Hence the conclusion follows from Corollary 4.10 (2) and its proof. □

Lemma 6.19. *Let A be a unital Kirchberg algebra such that the following two conditions hold.*

(i) *If $y \in KK(A, A)$ satisfies $K_0(y)([1]) = 0$, then $y = 0$.*

(ii) *$KK(A \otimes S, A) = 0$.*

If $\mu \in \mathrm{Hom}_{\mathbb{T}^N}(B, B)$ satisfies $KK(\mu \circ \iota) = KK(\iota)$, then $KK(\mu) = 1_B$.

Proof. For $I \subset [N]$, we let $|I|$ denote the cardinality of I. There exists a natural isomorphism between $KK(A \otimes S_I, A)$ and $KK(A, A)$. When $|I|$ is even, from condition (i), we have the following: if $y \in KK(A \otimes S_I, A)$ satisfies $KK(h \otimes \mathrm{id}_{S_I}) \cdot y = 0$, then $y = 0$.

By Lemma 6.15 and 6.16,

$$a = z_N^{-1} \cdot KK(\iota) \cdot w \cdot (1_A \otimes z_N)$$

is in $H'_{\mathbb{T}^N}(C^*(G), B)$ and $a(I, I) = KK(h \otimes \mathrm{id}_{S_I})$ for every $I \subset [N]$. By Lemma 6.8 and 6.11,

$$x = (1_A \otimes z_N)^{-1} \cdot w^{-1} \cdot KK(\mu) \cdot w \cdot (1_A \otimes z_N)$$

is in $H'_{\mathbb{T}^N}(B, B)$. From the hypothesis, we have $a \cdot x = a$. By Lemma 6.12, there exists $y_I \in KK(A \otimes S_I, A)$ for each $I \subset [N]$ such that $K_0(y_\emptyset)([1]) = [1]$ and

$$x(I, J) = \begin{cases} y_{I \setminus J} \otimes 1_{S_J} & \text{if } J \subset I \\ 0 & \text{otherwise.} \end{cases}$$

Notice that condition (i) implies $y_\emptyset = 1_A$ and that condition (ii) implies $y_I = 0$ if $|I|$ is odd.

We prove $y_I = 0$ for every non-empty $I \subset [N]$ by using the induction on $|I|$. Suppose that we have shown $y_I = 0$ for every I such that $0 < |I| < n$ and n is even. Take $I \subset [N]$ such that $|I| = n$. From $a \cdot x = a$, one has

$$\begin{aligned} a(I, \emptyset) &= (a \cdot x)(I, \emptyset) \\ &= \sum_{J \subset I} a(I, J) \cdot x(J, \emptyset) \\ &= \sum_{J \subset I, J \neq \emptyset, J \neq I} a(I, J) \cdot x(J, \emptyset) + a(I, \emptyset) \cdot x(\emptyset, \emptyset) + a(I, I) \cdot x(I, \emptyset) \\ &= a(I, \emptyset) \cdot 1_A + KK(h \otimes \mathrm{id}_{S_I}) \cdot y_I, \end{aligned}$$

and so $KK(h \otimes \mathrm{id}_{S_I}) \cdot y_I = 0$. Hence we get $y_I = 0$.

Consequently

$$x = \sum_{I \subset [N]} y_\emptyset \otimes 1_{S_I} = 1_{\mathcal{T}^{\otimes N}},$$

which implies $KK(\mu) = 1_B$. □

Theorem 6.20. *Suppose $A = M_n \otimes C \otimes \mathcal{O}_\infty$, where C is a UHF algebra satisfying $C \cong C \otimes C$ and $n \in \mathbb{N}$. Then, any outer $\mathbb{Z}^N$-actions on A are cocycle conjugate to each other. In particular, they are asymptotically representable.*

Proof. Note that A satisfies the hypothesis of Lemma 6.19. We use induction on N. Assume that the claim has been shown for $N - 1$.

Let α be an outer $\mathbb{Z}^N$-action on A. Let α' be the $\mathbb{Z}^{N-1}$-action generated by the first $N - 1$ generators of α. By the induction hypothesis, $\alpha' : \mathbb{Z}^{N-1} \curvearrowright A$ is asymptotically representable. Let $B = A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$ and let $\iota : C^*(\mathbb{Z}^{N-1}) \to B$ be the canonical embedding. We denote the N-th generator of α by α_N. The automorphism α_N of A naturally extends to an automorphism $\tilde{\alpha}_N$ of B. Clearly we have $\tilde{\alpha}_N \circ \iota = \iota$. It follows from Lemma 6.19 that $KK(\tilde{\alpha}_N)$ is equal to 1_B. The rest of the proof is the same as that of Theorem 5.4 or Theorem 6.6. □

7 Cocycle actions of $\mathbb{Z}^2$

In this section, we consider when a given cocycle action of $\mathbb{Z}^2$ on a unital Kirchberg algebra is equivalent to a genuine action. Our basic philosophy goes back to Adrian Ocneanu's

idea in the case of von Neumann algebras, while we have to handle K-theory with a special care in the case of C^*-algebras.

We put $S = C_0((0,1)) \cong C^*(\mathbb{R})$. For a unital C^*-algebra A, the connected component of the identity in $U(A)$ is denoted by $U(A)_0$.

Lemma 7.1. *Let A be a unital Kirchberg algebra and let $u : [0,1] \to U(A)$ be a continuous path of unitaries such that $u(0) = u(1) = 1$. If the K_1-class of u is zero in $K_1(S \otimes A)$, then there exists a continuous map $v : [0,1] \times [0,1] \to U(A)$ such that*

$$v(0,t) = 1,\ v(1,t) = u(t),\ \mathrm{Lip}(v(\cdot,t)) < 7 \text{ and } v(s,0) = v(s,1) = 1$$

for all $s, t \in [0,1]$.

Proof. We regard $S \otimes A$ as a subalgebra of $C(\mathbb{T}) \otimes A$. Since A is a Kirchberg algebra, there exists a nonzero projection $p \in A$ such that $p \neq 1$ and $\|[u, 1 \otimes p]\| < 1$, where u is regarded as an element of the unitization of $S \otimes A$. Note that $1 \otimes p$ and $1 - 1 \otimes p$ are properly infinite and full in $C(\mathbb{T}) \otimes A$. By [2, Lemma 2.4 (ii)] (and its proof), we can find $v : [0,1] \times [0,1] \to U(A)$ such that

$$v(0,t) = 1,\ v(1,t) = u(t) \text{ and } v(s,0) = v(s,1) = 1$$

for all $s, t \in [0,1]$. Then the conclusion follows from [23, Theorem 3.1]. □

Lemma 7.2. *Let A be a unital Kirchberg algebra and let α be an automorphism of A with the Rohlin property.*

(1) *For any unitary $u \in U(A)_0$ and $\varepsilon > 0$, there exists a unitary $v \in U(A)_0$ such that $\|u - v\alpha(v)^*\| < \varepsilon$.*

(2) *Let $u : [0,1] \to U(A)$ be a continuous path of unitaries such that $u(0) = u(1) = 1$. Suppose that the K_1-class of u is zero in $K_1(S \otimes A)$. Then, for any $\varepsilon > 0$, there exists a continuous path of unitaries $v : [0,1] \to U(A)$ such that $v(0) = v(1) = 1$ and $\|u(t) - v(t)\alpha(v(t))^*\| < \varepsilon$ for all $t \in [0,1]$.*

Proof. (1) This is a special case of [20, Lemma 8].

(2) This also follows from [20, Lemma 8] and its proof by using the lemma above. □

Lemma 7.3. *Let ε be a sufficiently small positive real number. Let A be a unital Kirchberg algebra and let α be an automorphism of A with the Rohlin property. If $x : [0,1] \to U(A)$ is a path of unitaries satisfying*

$$\|x(t) - \alpha(x(t))\| < \frac{\varepsilon}{27}$$

for all $t \in [0,1]$, then there exists a path of unitaries $y : [0,1] \to U(A)$ such that

$$y(0) = x(0), \quad y(1) = x(1), \quad \|y(t) - \alpha(y(t))\| < \varepsilon$$

for all $t \in [0,1]$ and $\mathrm{Lip}(y)$ is not greater than 6π.

Proof. This follows from [20, Theorem 7] and its proof. Note that the α-fixed point subalgebra of A_ω contains a unital copy of $\mathcal{O}_\infty$ by [20, Corollary 11]. □

Let α be an automorphism of a unital C^*-algebra A. For a unitary $u \in U(A)_0$ satisfying

$$\|u - \alpha(u)\| < 2,$$

we associate an element $\kappa(u, \alpha)$ in $\mathrm{Coker}(\mathrm{id} - K_0(\alpha))$ as follows. Take a path of unitaries $x : [0, 1] \to U(A)$ such that $x(0) = 1$ and $x(1) = u$. Then, we can find another path of unitaries $v : [0, 1] \to U(A)$ such that $v(0) = v(1) = 1$ and

$$\|x(t)\alpha(x(t))^* - v(t)\| < 2$$

for all $t \in [0, 1]$. Under the identification of $K_1(S \otimes A)$ with $K_0(A)$, the unitary v determines an element of $K_0(A)$. We let $\kappa(u, \alpha)$ be the equivalence class of this element in $\mathrm{Coker}(\mathrm{id} - K_0(\alpha))$.

Lemma 7.4. *In the setting above, $\kappa(u, \alpha) \in \mathrm{Coker}(\mathrm{id} - K_0(\alpha))$ does not depend on the choice of x and v.*

Proof. We let log be the standard branch defined on the complement of the negative real axis. Suppose that $x : [0, 1] \to U(A)$ is a path satisfying $x(0) = 1$ and $x(1) = u$, and that $v : [0, 1] \to U(A)$ is a path satisfying $v(0) = v(1) = 1$ and $\|x(t)\alpha(x(t))^* - v(t)\| < 2$ for all $t \in [0, 1]$.

First, we show that $\kappa(u, \alpha)$ does not depend on the choice of v. We define $w : [0, 1] \to U(A)$ by

$$w(t) = x(t)\alpha(x(t))^* e^{-t \log(u\alpha(u)^*)}, \qquad t \in [0, 1].$$

Then $\|x(t)\alpha(x(t))^* - w(t)\|$ is less than 2. It is easy to see that v and w are homotopic in the unitary group of the unitization of $S \otimes A$ via the homotopy

$$(s, t) \mapsto v(t) e^{s \log(v(t)^* x(t) \alpha(x(t))^*)} e^{-st \log(u\alpha(u)^*)}, \qquad s, t \in [0, 1].$$

Hence $\kappa(u, \alpha)$ does not depend on the choice of v.

Suppose that $y : [0, 1] \to U(A)$ is another path satisfying $y(0) = 1$ and $y(1) = u$. Then $z(t) = y(t)x(t)^*$ is a unitary in the unitization of $S \otimes A$, and

$$\|y(t)\alpha(y(t))^* - z(t)v(t)\alpha(z(t))^*\| = \|z(t)(x(t)\alpha(x(t))^* - v(t))\alpha(z(t))^*\| < 2.$$

In addition, $[zv(\mathrm{id}_S \otimes \alpha)(z)^*]$ belongs to $[v] + \mathrm{Im}(\mathrm{id} - K_1(\mathrm{id}_S \otimes \alpha))$ in $K_1(S \otimes A)$, which means that $\kappa(u, \alpha)$ does not depend on the choice of x. □

Lemma 7.5. *If $u_1, u_2 \in U(A)_0$ satisfy $\|u_1 - \alpha(u_1)\| + \|u_2 - \alpha(u_2)\| < 2$, then we have $\kappa(u_1u_2, \alpha) = \kappa(u_1, \alpha) + \kappa(u_2, \alpha)$.*

Proof. For each $i = 1, 2$, we choose $x_i : [0, 1] \to U(A)$ and $v_i : [0, 1] \to U(A)$ such that $x_i(0) = 1$, $x_i(1) = u_i$, $v_i(0) = v_i(1) = 1$ and

$$\|x_i(t)\alpha(x_i(t))^* - v_i(t)\| \leq \|u_i - \alpha(u_i)\|$$

for all $t \in [0, 1]$. Then $t \mapsto x_1(t)x_2(t)$ is a path of unitaries from 1 to u_1u_2. Put $v(t) = x_1(t)v_2(t)x_1(t)^*v_1(t)$ so that

$$\|x_1(t)x_2(t)\alpha(x_1(t)x_2(t))^* - v(t)\| \leq \|u_1 - \alpha(u_1)\| + \|u_2 - \alpha(u_2)\| < 2.$$

Clearly $[v]$ equals $[v_1] + [v_2]$ in $K_1(S \otimes A)$, and so we have $\kappa(u_1u_2, \alpha) = \kappa(u_1, \alpha) + \kappa(u_2, \alpha)$. □

Lemma 7.6. *Let $0 < \varepsilon < 2$. Let A be a unital Kirchberg algebra and let α be an automorphism of A with the Rohlin property. If a unitary $u \in U(A)_0$ satisfies*

$$\|u - \alpha(u)\| < \varepsilon, \quad \kappa(u, \alpha) = 0,$$

then there exists a path of unitaries $x : [0,1] \to U(A)$ such that $x(0) = 1$, $x(1) = u$ and

$$\|x(t) - \alpha(x(t))\| < \varepsilon$$

for all $t \in [0,1]$.

Proof. Choose a path of unitaries $u : [0,1] \to U(A)$ such that $u(0) = 1$ and $u(1) = u$. Since $\|u - \alpha(u)\| < \varepsilon < 2$, there exists a path of unitaries $v : [0,1] \to U(A)$ such that $v(0) = v(1) = 1$ and

$$\|u(t)\alpha(u(t))^* - v(t)\| \leq \|u - \alpha(u)\|$$

for all $t \in [0,1]$. We can regard v as a unitary in the unitization of $S \otimes A$. From $\kappa(u, \alpha) = 0$, we may assume that the K_1-class of v in $K_1(S \otimes A)$ is zero. Hence, by Lemma 7.2 (2), we can find $w : [0,1] \to U(A)$ such that $w(0) = w(1) = 1$ and

$$\|v(t) - w(t)\alpha(w(t))^*\| < \varepsilon - \|u - \alpha(u)\|$$

for all $t \in [0,1]$. Therefore, one gets

$$\|u(t)\alpha(u(t))^* - w(t)\alpha(w(t))^*\| < \varepsilon$$

for all $t \in [0,1]$. Thus, $x(t) = w(t)^* u(t)$ meets the requirements. □

Lemma 7.7. *Let A be a unital Kirchberg algebra and let α be an automorphism of A with the Rohlin property. For any $x \in K_0(A)$ and $\varepsilon > 0$, there exists $v \in U(A)_0$ such that $\|v - \alpha(v)\| < \varepsilon$ and $\kappa(v, \alpha) = x + \mathrm{Im}(\mathrm{id} - K_0(\alpha))$.*

Proof. Choose $n \in \mathbb{N}$ so that $2\pi/n < \varepsilon$. By [20, Corollary 11], we may replace (A, α) with $(A \otimes \mathcal{O}_\infty, \alpha \otimes \gamma)$, where γ is an aperiodic automorphism of $\mathcal{O}_\infty$. We may further assume that there exists a projection $e \in \mathcal{O}_\infty$ such that $[e] = [1]$, $e\gamma^k(e) = 0$ for $k = 1, 2, \ldots, n-1$ and $e = \gamma^n(e)$. Put $e_0 = 1 - (e + \gamma(e) + \cdots + \gamma^{n-1}(e))$. There exists a path of unitaries $u : [0,1] \to U(A)$ such that $u(0) = u(1) = 1$, $\mathrm{Lip}(u) \leq 2\pi$ and $[u] \in K_1(S \otimes A) \cong K_0(A)$ equals x. We define $v \in U(A)_0$ by

$$v = \sum_{k=0}^{n-1} \alpha^k(u(k/n)) \otimes \gamma^k(e) + 1 \otimes e_0.$$

One can verify $\|v - (\alpha \otimes \gamma)(v)\| < \varepsilon$ and $\kappa(v, \alpha \otimes \gamma) = x + \mathrm{Im}(\mathrm{id} - K_0(\alpha \otimes \gamma))$ easily. □

Lemma 7.8. *Let A be a unital Kirchberg algebra and let α be an automorphism of A with the Rohlin property. For any unitary $w \in A$ with $\|w - 1\| < 2$ and $\varepsilon > 0$, there exists $u \in U(A)_0$ such that*

$$\|uw\alpha(u)^* - 1\| < \varepsilon, \quad \|u - \alpha(u)\| < 2 \quad \textit{and} \quad \kappa(u, \alpha) = 0.$$

Proof. From Lemma 7.2 (1), it is easy to find a unitary $u \in U(A)_0$ such that

$$\|uw\alpha(u)^* - 1\| < \min\{2 - \|w-1\|,\ \varepsilon\}.$$

Then one has $\|u - \alpha(u)\| < 2$. By the lemma above, we can find a unitary v in A which is almost fixed by α and $\kappa(v, \alpha)$ equals $-\kappa(u, \alpha)$. Thanks to Lemma 7.5, by replacing u with vu, we get the desired unitary. □

Lemma 7.9. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let α and β be automorphisms of a unital Kirchberg algebra A such that $\alpha^m \circ \beta^n$ is outer for all $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Let u and w be unitaries in A such that*

$$\beta \circ \alpha = \mathrm{Ad}\, w \circ \alpha \circ \beta, \quad \|w - 1\| < \delta$$

$$\|u - \alpha(u)\| < \delta, \quad u \in U(A)_0 \ \text{ and } \ \kappa(u, \alpha) = 0.$$

Then, there exists a unitary $v \in A$ such that

$$\|v - \alpha(v)\| < \varepsilon \ \text{ and } \ \|u - v\beta(v)^*\| < \varepsilon.$$

Proof. Choose $N \in \mathbb{N}$ and $\delta > 0$ so that $6\pi/N < \varepsilon$ and $(86N + 25)\delta < \varepsilon$. Suppose that α, β, u, w are given.

The automorphisms α, β give an outer cocycle action of $\mathbb{Z}^2$. By Corollary 6.4 (2) (see also Theorem 3.6), there exists a family of mutually orthogonal projections

$$\{E_k^{(0)} \mid k = 0, 1, \dots, N-1\} \cup \{E_k^{(1)} \mid k = 0, 1, \dots, N\}$$

in $(A_\omega)^\alpha$ such that

$$\sum_{k=0}^{N-1} E_k^{(0)} + \sum_{k=0}^{N} E_k^{(1)} = 1 \ \text{ and } \beta(E_k^{(i)}) = E_{k+1}^{(i)},$$

where $E_N^{(0)}$ and $E_{N+1}^{(1)}$ are understood as $E_0^{(0)}$ and $E_0^{(1)}$, respectively.

Define $u_k \in U(A)$ for $k = 0, 1, \dots$ by $u_0 = 1$ and $u_{k+1} = u\beta(u_k)$. By an elementary estimate, we obtain

$$\|u_k - \alpha(u_k)\| < (3k - 2)\delta$$

for any $k \in \mathbb{N}$. Moreover, from Lemma 7.5, we can see $\kappa(u_k, \alpha) = 0$. By applying Lemma 7.6 and Lemma 7.3 to u_N and u_{N+1}, we obtain a path of unitaries y and z such that the following hold.

- $y(1) = z(1) = 1$, $y(0) = u_N$ and $z(0) = u_{N+1}$.
- $\|y(t) - \alpha(y(t))\| < 27(3N - 2)\delta$ and $\|z(t) - \alpha(z(t))\| < 27(3N + 1)\delta$ for all $t \in [0, 1]$.
- Both $\mathrm{Lip}(y)$ and $\mathrm{Lip}(z)$ are less than 6π.

Define a unitary $V \in A^\omega$ by

$$V = \sum_{k=0}^{N-1} u_k \beta^k(y(k/N)) E_k^{(0)} + \sum_{k=0}^{N} u_k \beta^k(z(k/(N+1))) E_k^{(1)}.$$

We can easily see $\|u - V\beta(V)^*\| < 6\pi/N < \varepsilon$. Furthermore,

$$\|V - \alpha(V)\| < (3N-2)\delta + 2N\delta + 27(3N+1)\delta = (86N+25)\delta < \varepsilon,$$

which completes the proof. □

Proposition 7.10. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let A be a unital Kirchberg algebra. Suppose that $\alpha, \beta \in \mathrm{Aut}(A)$ and $w \in U(A)$ satisfy*

$$\beta \circ \alpha = \mathrm{Ad}\, w \circ \alpha \circ \beta \quad \textit{and} \quad \|w - 1\| < \delta.$$

Suppose further that $\alpha^m \circ \beta^n$ is outer for all $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Then, there exist $a, b \in U(A)$ such that

$$\|a - 1\| < \varepsilon, \quad \|b - 1\| < \varepsilon$$

and

$$b\beta(a)w\alpha(b)^* a^* = 1.$$

In particular,

$$(\mathrm{Ad}\, a \circ \alpha) \circ (\mathrm{Ad}\, b \circ \beta) = (\mathrm{Ad}\, b \circ \beta) \circ (\mathrm{Ad}\, a \circ \alpha).$$

Proof. Choose a decreasing sequence of positive real numbers $\varepsilon_1, \varepsilon_2, \dots$ so that $\sum \varepsilon_k < \varepsilon$. By applying Lemma 7.9 to $\varepsilon_k > 0$, we obtain $\delta_k > 0$. We show that $\delta = \delta_1$ meets the requirements.

Suppose that $\alpha, \beta \in \mathrm{Aut}(A)$ and $w \in U(A)$ are given. From Lemma 7.8, there exists $u_1 \in U_0(A)$ satisfying

$$\|u_1 w \alpha(u_1)^* - 1\| < \min\{\delta_1 - \|w-1\|,\ \delta_2\}, \quad \|u_1 - \alpha(u_1)\| < \delta_1$$

and $\kappa(u_1, \alpha) = 0$. From Lemma 7.9, we can find a unitary v_1 in A such that

$$\|v_1 - \alpha(v_1)\| < \varepsilon_1 \quad \text{and} \quad \|u_1 - v_1\beta(v_1)^*\| < \varepsilon_1.$$

Define unitaries a_1 and b_1 by $a_1 = v_1^* \alpha(v_1)$ and $b_1 = v_1^* u_1 \beta(v_1)$. Clearly $\|a_1 - 1\|$ and $\|b_1 - 1\|$ are less than ε_1. Put $\alpha_2 = \mathrm{Ad}\, a_1 \circ \alpha$ and $\beta_2 = \mathrm{Ad}\, b_1 \circ \beta$. Then, it is straightforward to see

$$\beta_2 \circ \alpha_2 = \mathrm{Ad}\, w_2 \circ \alpha_2 \circ \beta_2,$$

where $w_2 = v_1^* u_1 w \alpha(u_1)^* v_1$ and $\|w_2 - 1\| < \delta_2$.

By repeating this argument, one obtains unitaries $\{a_k\}_k$, $\{b_k\}_k$, $\{w_k\}_k$ and automorphisms $\{\alpha_k\}_k$, $\{\beta_k\}_k$ which satisfy the following.

$$\|a_k - 1\| < \varepsilon_k, \quad \|b_k - 1\| < \varepsilon_k, \quad \|w_k - 1\| < \delta_k,$$

$$\alpha_{k+1} = \mathrm{Ad}\, a_k \circ \alpha_k, \quad \beta_{k+1} = \mathrm{Ad}\, b_k \circ \beta_k, \quad w_{k+1} = v_k^* u_k w_k \alpha_k(u_k)^* v_k$$

and

$$\beta_k \circ \alpha_k = \operatorname{Ad} w_k \circ \alpha_k \circ \beta_k.$$

Define $a, b \in U(A)$ by

$$a = \lim_{k\to\infty} a_k a_{k-1} \dots a_2 a_1 \quad \text{and} \quad b = \lim_{k\to\infty} b_k b_{k-1} \dots b_2 b_1.$$

Since $\sum_k \varepsilon_k < \varepsilon$, we get $\|a - 1\| < \varepsilon$ and $\|b - 1\| < \varepsilon$. Besides, one can check

$$w_{k+1} = (b_k \dots b_2 b_1)\beta(a_k \dots a_2 a_1) w \alpha(b_k \dots b_2 b_1)^* (a_k \dots a_2 a_1)^*$$

inductively. Therefore, we have

$$b\beta(a) w \alpha(b)^* a^* = 1$$

and

$$(\operatorname{Ad} a \circ \alpha) \circ (\operatorname{Ad} b \circ \beta) = (\operatorname{Ad} b \circ \beta) \circ (\operatorname{Ad} a \circ \alpha),$$

because w_k converges to the identity. □

Theorem 7.11. *Let (α, u) be an outer cocycle action of $\mathbb{Z}^2$ on a unital Kirchberg algebra A. If the cohomology class of $[u(\cdot,\cdot)]$ is trivial in $H^2(\mathbb{Z}^2, K_1(A))$, then (α, u) is equivalent to a genuine action.*

Proof. We may assume that $[u(g,h)] = 0$ in $K_1(A)$ for all $g, h \in \mathbb{Z}^2$. Hence there exists $w \in U(A)$ such that

$$\alpha_{(1,0)} \circ \alpha_{(0,1)} = \operatorname{Ad} w \circ \alpha_{(0,1)} \circ \alpha_{(1,0)}$$

and $[w] = 0$ in $K_1(A)$. By Theorem 2.8, the automorphism $\alpha_{(1,0)}$ has the Rohlin property. It follows from Lemma 7.2 (1) that, for any $\delta > 0$, we can find $v \in U(A)_0$ such that $\|w - \alpha_{(1,0)}(v^*)v\| < \delta$. Thus, by replacing $\alpha_{(0,1)}$ with $\operatorname{Ad} v \circ \alpha_{(0,1)}$, we may further assume $\|w - 1\| < \delta$. Then, the conclusion follows from Proposition 7.10. □

Corollary 7.12. *Any outer cocycle action of $\mathbb{Z}^2$ on a unital Kirchberg algebra A in the Cuntz standard form is equivalent to a genuine action.*

Proof. Let (α, u) be an outer cocycle action of $\mathbb{Z}^2$ on a unital Kirchberg algebra A in the Cuntz standard form. There exists a unitary $w \in A$ such that

$$\alpha_{(1,0)} \circ \alpha_{(0,1)} = \operatorname{Ad} w \circ \alpha_{(0,1)} \circ \alpha_{(1,0)}.$$

Let B be the crossed product of A by $\alpha_{(1,0)}$. We denote the implementing unitary in B by λ. Consider the Pimsner-Voiculescu exact sequence for $\alpha_{(1,0)}$. Since $[\lambda] \in K_1(B)$ is sent to $[1] \in K_0(A)$ and $[1] = 0$ in $K_0(A)$, there exists a unitary $x \in A$ such that $[x] = [\lambda]$ in $K_1(B)$. The automorphism $\alpha_{(0,1)}$ of A extends to $\tilde{\alpha}_{(0,1)} \in \operatorname{Aut}(B)$ by $\tilde{\alpha}_{(0,1)}(\lambda) = w^*\lambda$. Then we have

$$[\alpha_{(0,1)}(x)] = [\tilde{\alpha}_{(0,1)}(\lambda)] = [w^*\lambda] = [w^* x],$$

that is, $[w] = [x] - [\alpha_{(0,1)}(x)]$ in $K_1(B)$. It follows that there exists a unitary $y \in A$ such that

$$[w] = [x] - [\alpha_{(0,1)}(x)] + [y] - [\alpha_{(1,0)}(y)]$$

in $K_1(A)$. This means that there exists a unitary $v \in A$ such that $[v] = 0$ in $K_1(A)$ and $w = \alpha_{(1,0)}(y)^* xvy\alpha_{(0,1)}(x)^*$, which implies

$$\operatorname{Ad} x^* \circ \alpha_{(1,0)} \circ \operatorname{Ad} y \circ \alpha_{(0,1)} = \operatorname{Ad} v \circ \operatorname{Ad} y \circ \alpha_{(0,1)} \circ \operatorname{Ad} x^* \circ \alpha_{(1,0)}.$$

By the theorem above, (α, u) is equivalent to a genuine action. □

8 Classification of locally KK-trivial $\mathbb{Z}^2$-actions

In this section, we classify locally KK-trivial outer $\mathbb{Z}^2$-actions on a Kirchberg algebra up to KK-trivially cocycle conjugacy. We denote by $\mathbb{K}$ the C^*-algebra of all compact operators on $\ell^2(\mathbb{Z})$. We put $S = C_0((0,1))$ and regard it as a subalgebra of $C(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

From Remark 4.6, Corollary 6.5, Proposition 6.13 (2) and Theorem 7.11, we can see that outer locally KK-trivial $\mathbb{Z}^2$-actions on a unital Kirchberg algebra A can be classified by

$$\{x \in KK^1(A, A) \mid K_0(x)([1]) = 0\}$$

up to KK-trivially cocycle conjugacy. Meanwhile, the results of [11] and [27] show that uniformly outer $\mathbb{Z}^2$-actions on a UHF algebra A are classified by the fundamental group $\pi_1(\operatorname{Aut}(A))$ up to cocycle conjugacy. On the other hand, when A is a unital Kirchberg algebra, the group $\pi_1(\operatorname{Aut}(A \otimes \mathbb{K}))$ is isomorphic to $KK^1(A, A)$ by [5]. This is not a coincidence, and we first show that our classification invariant also lives in the group $\pi_1(\operatorname{Aut}(A \otimes \mathbb{K}))$. This also allows us to see the topological origin of the invariant (see Remark 8.1 below).

For a locally KK-trivial $\mathbb{Z}^2$-action α on a unital Kirchberg algebra A, we introduce an invariant $\Phi(\alpha) \in KK(A, S \otimes A)$ as follows. By Theorem 2.6 (2), $\alpha_{(1,0)}$ is homotopic to an inner automorphism. Since the unitary group of the multiplier algebra of a stable C^*-algebra is path-connected, $\alpha_{(1,0)} \otimes \mathrm{id}_{\mathbb{K}}$ is homotopic to the identity in $\operatorname{Aut}(A \otimes \mathbb{K})$. Let $\{\gamma_t\}_{t \in [0,1]}$ be a path from $\mathrm{id}_{A \otimes \mathbb{K}}$ to $\alpha_{(1,0)} \otimes \mathrm{id}_{\mathbb{K}}$. Define a homomorphism $\pi : A \otimes \mathbb{K} \to C(\mathbb{T}) \otimes (A \otimes \mathbb{K})$ by

$$\pi(a)(t) = (\gamma_t^{-1} \circ (\alpha_{(0,1)} \otimes \mathrm{id}_{\mathbb{K}}) \circ \gamma_t)(a)$$

for $a \in A \otimes \mathbb{K}$ and $t \in [0,1]$. We can see that the homotopy class of π does not depend on the choice of the path $\{\gamma_t\}_t$, because $\alpha_{(0,1)} \otimes \mathrm{id}_{\mathbb{K}}$ is homotopic to the identity. Let $j_A : A \otimes \mathbb{K} \to C(\mathbb{T}) \otimes (A \otimes \mathbb{K})$ be the homomorphism defined by $j_A(a) = 1_{C(\mathbb{T})} \otimes a$. We put

$$\Phi(\alpha) = KK(\pi) - KK(j_A)$$

and regard it as an element of $KK(A, S \otimes A)$.

Remark 8.1. Sergey Neshveyev kindly informed the authors of the following fact: to the $\mathbb{Z}^2$-action α, one can associate a principal $\operatorname{Aut}(A \otimes \mathbb{K})$-bundle over $\mathbb{T}^2$ in a standard way, and our invariant $\Phi(\alpha)$ is essentially the usual obstruction class in $H^2(\mathbb{T}^2, \pi_1(\operatorname{Aut}(A \otimes \mathbb{K})))$ for the existence of a continuous section. We will come back to this point elsewhere.

Lemma 8.2. *The above KK^1-class $\Phi(\alpha)$ satisfies $K_0(\Phi(\alpha))([1]) = 0$.*

Proof. Since $\alpha_{(1,0)}$ is homotopic to an inner automorphism, say $\operatorname{Ad} w_0$, we may assume that the homotopy γ is of the following form:

$$\gamma_t = \begin{cases} \operatorname{Ad} w(t), & 0 \le t \le 1/2, \\ \gamma'_t \otimes \mathrm{id}_{\mathbb{K}}, & 1/2 \le t \le 1, \end{cases}$$

where $\{w(t)\}_{0\le t\le 1/2}$ is a continuous path from $1\otimes 1$ to $w_0\otimes 1$ in the unitary group of $M(A\otimes\mathbb{K})$, and $\{\gamma'_t\}_{1/2\le t\le 1}$ is a continuous path from $\operatorname{Ad} w_0$ to $\alpha_{(1,0)}$ in $\operatorname{Aut}(A)$. Let e be a minimal projection of $\mathbb{K}$. Then all we have to show is that the projection loop $\pi(1\otimes e)(t)$ is homotopic to the constant loop $1\otimes e$. For $1/2\le t\le 1$, there is nothing to show. For $0\le t\le 1/2$, we have $\pi(1\otimes e)(t) = \operatorname{Ad} w'(t)(1\otimes e)$, where $w'(t) = w(t)^*(\alpha_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})(w(t))$. Note that $w'(1/2) = w_0^*\alpha_{(0,1)}(w_0)\otimes 1$, and $w_0^*\alpha_{(0,1)}(w_0)$ is trivial in $K_1(A)$. Since the unitary group of $M(A\otimes\mathbb{K})$ is contractible [28, Theorem 16.8], its fundamental group is trivial, and so the continuous path $\{w'(t)\}_{0\le t\le 1/2}$ is homotopic to that of the form $\{w''(t)\otimes 1\}_{0\le t\le 1/2}$ within the set of continuous paths from $1\otimes 1$ to $w_0^*\alpha_{(0,1)}(w_0)\otimes 1$. This finishes the proof. □

Lemma 8.3. *Let $\alpha,\beta : \mathbb{Z}^2 \curvearrowright A$ be two locally KK-trivial $\mathbb{Z}^2$-actions on a unital Kirchberg algebra. If α and β are KK-trivially cocycle conjugate, then $\Phi(\alpha) = \Phi(\beta)$.*

Proof. We may assume that there exists an α-cocycle $\{u_n\}_{n\in\mathbb{Z}^2}$ in A such that $\beta_n = \operatorname{Ad} u_n\circ\alpha_n$ for all $n\in\mathbb{Z}^2$. Let w be the unitary operator on $\ell^2(\mathbb{Z})$ defined by $w = \sum_p e_{p+1,p}$, where $\{e_{p,q}\}_{p,q}$ is a family of matrix units. Take a path of automorphisms $\{\gamma_t\}_{t\in[0,1]}$ of $A\otimes\mathbb{K}\otimes\mathbb{K}$ from the identity to $\alpha_{(1,0)}\otimes\operatorname{Ad} w\otimes\mathrm{id}_{\mathbb{K}}$. Define a homomorphism π from $A\otimes\mathbb{K}\otimes\mathbb{K}$ to $C(\mathbb{T})\otimes(A\otimes\mathbb{K}\otimes\mathbb{K})$ by

$$\pi(a)(t) = (\gamma_t^{-1}\circ(\alpha_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}}\otimes\operatorname{Ad} w)\circ\gamma_t)(a).$$

Since there exists a path of unitaries from w to 1, it is easily seen that $\Phi(\alpha)$ is equal to $KK(\pi) - KK(j_A)$. Put

$$v = \sum_{p,q\in\mathbb{Z}} u_{(p,q)}\otimes e_{p,p}\otimes e_{q,q}.$$

Then one has

$$v(\alpha_{(1,0)}\otimes\operatorname{Ad} w\otimes\mathrm{id}_{\mathbb{K}})(v^*) = u_{(1,0)}\otimes 1\otimes 1$$

and

$$v(\alpha_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}}\otimes\operatorname{Ad} w)(v^*) = u_{(0,1)}\otimes 1\otimes 1,$$

which imply

$$\pi(a)(t) = (\gamma_t^{-1}\circ\operatorname{Ad} v^*\circ(\beta_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}}\otimes\operatorname{Ad} w)\circ\operatorname{Ad} v\circ\gamma_t)(a),$$

and $\{\operatorname{Ad} v\circ\gamma_t\circ\operatorname{Ad} v^*\}_t$ is a path from the identity to $\beta_{(1,0)}\otimes\operatorname{Ad} w\otimes\mathrm{id}_{\mathbb{K}}$. Consequently $\Phi(\beta) = KK(\pi) - KK(j_A)$, which completes the proof. □

Proposition 8.4. *Let A be a unital Kirchberg algebra. For any $x\in KK(A, S\otimes A)$ with $K_0(x)([1]) = 0$, there exists a locally KK-trivial outer action $\alpha : \mathbb{Z}^2 \curvearrowright A$ such that $\Phi(\alpha) = x$.*

Proof. Choose an aperiodic automorphism $\gamma \in \mathrm{Aut}(\mathcal{O}_\infty)$ which is homotopic to the identity and let $\{\gamma_t\}_{t\in[0,1]}$ be the homotopy from the identity to γ. By [5, Proposition 5.7], we can find a path of automorphisms $\{\sigma_t\}_{t\in[0,1]}$ of A such that $\sigma_0 = \sigma_1$ and the KK-class of the homomorphism

$$A \ni a \mapsto (\sigma_t(a))_{t\in[0,1]} \in C(\mathbb{T}) \otimes A$$

is equal to $x + KK(j_A)$. We regard $(\sigma_t)_t$ as an automorphism of $C([0,1]) \otimes A$. By replacing A and $(\sigma_t)_t$ with $A \otimes \mathcal{O}_\infty$ and $(\sigma_t \otimes \gamma)_t$, we may assume that $(\sigma_t)_t$ has the Rohlin property. Let ρ be an aperiodic automorphism of A which is homotopic to the identity. Then the automorphism $\mathrm{id}_{C([0,1])} \otimes \rho$ of $C([0,1]) \otimes A$ also has the Rohlin property. Moreover, the two homomorphisms

$$A \ni a \mapsto (\sigma_t(a))_t \in C([0,1]) \otimes A$$

and

$$A \ni a \mapsto 1 \otimes \rho(a) \in C([0,1]) \otimes A$$

clearly have the same KK-class. By Theorem 2.6 (2), they are asymptotically unitarily equivalent. It follows that $(\sigma_t)_t$ and $\mathrm{id}_{C([0,1])} \otimes \rho$ are asymptotically unitarily equivalent as automorphisms of $C([0,1]) \otimes A$. Then we can apply the argument of [20, Theorem 5] and conclude that $(\sigma_t)_t$ and $\mathrm{id}_{C([0,1])} \otimes \rho$ are KK-trivially cocycle conjugate (note that $(C([0,1]) \otimes A)_\omega$ contains a unital copy of $\mathcal{O}_\infty$ and [20, Theorem 7] works for $C([0,1]) \otimes A$). Hence there exist a path of automorphisms $\{\mu_t\}_{t\in[0,1]}$ of A and a path of unitaries $\{u_t\}_{t\in[0,1]}$ such that $KK(\mu_t) = 1$ and

$$\mathrm{Ad}\, u_t \circ \rho = \mu_t \circ \sigma_t \circ \mu_t^{-1}$$

for all $t \in [0,1]$. By replacing μ_t and σ_t with $\mu_t \circ \mu_0^{-1}$ and $\mu_0 \circ \sigma_t \circ \mu_0^{-1}$, we may assume that $\mu_0 = \mathrm{id}_A$. Furthermore, by replacing u_t and ρ with $u_t u_0^*$ and $\mathrm{Ad}\, u_0 \circ \rho$, we may also assume that $u_0 = 1$. Thus, $\rho = \sigma_0 = \sigma_1$ and

$$\mathrm{Ad}\, u_1 \circ \rho = \mu_1 \circ \sigma_1 \circ \mu_1^{-1} = \mu_1 \circ \rho \circ \mu_1^{-1},$$

which means that ρ and μ_1 give a cocycle action of $\mathbb{Z}^2$. By replacing A, ρ and μ_t with $A \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty$, $\rho \otimes \gamma \otimes \mathrm{id}_{\mathcal{O}_\infty}$ and $\mu_t \otimes \mathrm{id}_{\mathcal{O}_\infty} \otimes \gamma_t$, we may further assume that ρ and μ_1 give an outer cocycle action. Since $[u_1] = 0$ in $K_1(A)$, Proposition 7.10 implies that there exist unitaries $a, b \in A$ such that

$$b\mu_1(a)u_1\rho(b)^* a^* = 1.$$

In addition, from the construction, we can see that $[b] = 0$ in $K_1(A)$. Let $\{b_t\}_{t\in[0,1]}$ be a path of unitaries from 1 to b. By replacing ρ, σ_t, μ_t and u_t with $\mathrm{Ad}\, a \circ \rho$, $\mathrm{Ad}\, a \circ \sigma_t$, $\mathrm{Ad}\, b_t \circ \mu_t$ and $b_t \mu_t(a) u_t \rho(b_t)^* a^*$, we get $\rho \circ \mu_1 = \mu_1 \circ \rho$ and $u_1 = 1$.

Consequently, one has

$$\mu_t^{-1} \circ \rho \circ \mu_t = \mathrm{Ad}\, \mu_t^{-1}(u_t^*) \circ \sigma_t$$

for all $t \in [0,1]$ and $(\mu_t^{-1}(u_t^*))_t$ is a unitary in $C(\mathbb{T}) \otimes A$. Therefore, the KK-class of

$$A \ni a \mapsto ((\mu_t^{-1} \circ \rho \circ \mu_t)(a))_t \in C(\mathbb{T}) \otimes A$$

is equal to $x + KK(j_A)$. The $\mathbb{Z}^2$-action generated by ρ and μ_1 is the desired one. □

Using Proposition 8.5 below and working on $\mathrm{Hom}_{\mathbb{T}}(A \rtimes \mathbb{Z}, A \rtimes \mathbb{Z})$, we can prove Proposition 8.4 without using [5, Proposition 5.7].

Next we will prove Proposition 8.5. We must recall a few basic facts about crossed products. Let ρ be an automorphism of a unital C^*-algebra A. We let $A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{R}$ denote the crossed product of $A \rtimes_\rho \mathbb{Z}$ by $\hat{\rho}$ which is regarded as an $\mathbb{R}$-action, and let $A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{T}$ be the usual crossed product by the dual action $\hat{\rho} : \mathbb{T} \curvearrowright A \rtimes_\rho \mathbb{Z}$. As described in [1, Proposition 10.3.2], there exists an isomorphism between $A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{R}$ and the mapping torus

$$M_{\hat{\hat{\rho}}} = \{f : [0,1] \to A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{T} \mid f(1) = \hat{\hat{\rho}}(f(0))\}.$$

Let w be the unitary operator on $\ell^2(\mathbb{Z})$ defined by $w = \sum_p e_{p+1,p}$, where $\{e_{p,q}\}_{p,q}$ is a family of matrix units. Then, by Takesaki-Takai duality, $(A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{T}, \hat{\hat{\rho}})$ is conjugate to $(A \otimes \mathbb{K}, \rho \otimes \mathrm{Ad}\, w)$, and so there exists an isomorphism φ from $A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{R}$ to the mapping torus

$$M_{\rho \otimes \mathrm{Ad}\, w} = \{f : [0,1] \to A \otimes \mathbb{K} \mid f(1) = (\rho \otimes \mathrm{Ad}\, w)(f(0))\}.$$

Suppose σ is in $\mathrm{Hom}_{\mathbb{T}}(A \rtimes_\rho \mathbb{Z}, A \rtimes_\rho \mathbb{Z})$. We denote its canonical extension to $A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{R}$ by the same symbol σ. Define a unitary $v \in M(A \otimes \mathbb{K})$ by

$$v = \sum_{p \in \mathbb{Z}} \sigma(\lambda_p^\rho) \lambda_{-p}^\rho \otimes e_{p,p}.$$

It is not so hard to see

$$(\varphi \circ \sigma \circ \varphi^{-1})(f)(t) = (\mathrm{Ad}\, v^* \circ (\sigma \otimes \mathrm{id}_{\mathbb{K}}))(f(t)).$$

Proposition 8.5. *Let $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ be two locally KK-trivial outer actions of $\mathbb{Z}^2$ on a unital Kirchberg algebra. If $\Phi(\alpha) = \Phi(\beta)$, then α and β are KK-trivially cocycle conjugate.*

Proof. Put $\rho = \alpha_{(1,0)}$. By Lemma 2.10, the $\mathbb{Z}$-action induced by ρ is asymptotically representable. Let φ be the isomorphism between $A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{R}$ and the mapping torus

$$M_{\rho \otimes \mathrm{Ad}\, w} = \{f : [0,1] \to A \otimes \mathbb{K} \mid f(1) = (\rho \otimes \mathrm{Ad}\, w)(f(0))\}.$$

as described above. Take a path of automorphisms $\{\gamma_t\}_{t \in [0,1]}$ such that

$$\gamma_0 = \mathrm{id}_{A \otimes \mathbb{K}} \text{ and } \gamma_1 = \rho \otimes \mathrm{Ad}\, w = \alpha_{(1,0)} \otimes \mathrm{Ad}\, w.$$

Then, $\psi(f)(t) = \gamma_t^{-1}(f(t))$ gives an isomorphism from $M_{\rho \otimes \mathrm{Ad}\, w}$ to $C(\mathbb{T}) \otimes A \otimes \mathbb{K}$. Letting

$$x \in KK(S \otimes (A \rtimes_\rho \mathbb{Z}), A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{R})$$

be the Thom element, we have a KK-equivalence

$$z = x \cdot KK(\psi \circ \varphi) \in KK(S \otimes (A \rtimes_\rho \mathbb{Z}), C(\mathbb{T}) \otimes A).$$

We let $\tilde{\alpha}_{(0,1)}$ denote the canonical extension of $\alpha_{(0,1)}$ to $A \rtimes_\rho \mathbb{Z} = A \rtimes_{\alpha_{(1,0)}} \mathbb{Z}$. The automorphism $\tilde{\alpha}_{(0,1)}$ further extends to $A \rtimes_\rho \mathbb{Z} \rtimes_{\hat{\rho}} \mathbb{R}$ and we use the same symbol for it. It is easy to see that

$$x^{-1} \cdot KK(\mathrm{id}_S \otimes \tilde{\alpha}_{(0,1)}) \cdot x = KK(\tilde{\alpha}_{(0,1)})$$

and

$$(\psi\circ\varphi\circ\tilde{\alpha}_{(0,1)}\circ\varphi^{-1}\circ\psi^{-1})(f)(t)=(\gamma_t^{-1}\circ(\alpha_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\gamma_t)(f(t))$$

for all $f\in C(\mathbb{T})\otimes A\otimes\mathbb{K}$ and $t\in[0,1]$. It follows that $z^{-1}\cdot KK(\mathrm{id}_S\otimes\tilde{\alpha}_{(0,1)})\cdot z$ equals the KK-class of the automorphism $(\gamma_t^{-1}\circ(\alpha_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\gamma_t)_t$ of $C(\mathbb{T})\otimes A\otimes\mathbb{K}$.

We now turn to the action β. By Theorem 2.9, we may assume that there exists an $\alpha_{(1,0)}$-cocycle $\{u_n\}_{n\in\mathbb{Z}}$ such that $\beta_{(n,0)}=\mathrm{Ad}\,u_n\circ\alpha_{(n,0)}$ for all $n\in\mathbb{Z}$. One can extend $\beta_{(0,1)}$ to $\tilde{\beta}_{(0,1)}\in\mathrm{Hom}_{\mathbb{T}}(A\rtimes_\rho\mathbb{Z},A\rtimes_\rho\mathbb{Z})$ by setting

$$\tilde{\beta}_{(0,1)}(\lambda_n^\rho)=\beta_{(0,1)}(u_n^*)u_n\lambda_n^\rho.$$

Define unitaries $u,v\in M(A\otimes\mathbb{K})$ by

$$u=\sum_{p\in\mathbb{Z}}u_p\otimes e_{p,p}\quad\text{and}\quad v=(\beta_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})(u^*)u.$$

The automorphism $\tilde{\beta}_{(0,1)}$ further extends to $A\rtimes_\rho\mathbb{Z}\rtimes_{\hat{\rho}}\mathbb{R}$ and we use the same symbol for it. It is easy to see that

$$x^{-1}\cdot KK(\mathrm{id}_S\otimes\tilde{\beta}_{(0,1)})\cdot x=KK(\tilde{\beta}_{(0,1)}).$$

By the observation before this proposition, we also have

$$\begin{aligned}&(\psi\circ\varphi\circ\tilde{\beta}_{(0,1)}\circ\varphi^{-1}\circ\psi^{-1})(f)(t)\\&=(\gamma_t^{-1}\circ\mathrm{Ad}\,v^*\circ(\beta_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\gamma_t)(f(t))\\&=(\gamma_t^{-1}\circ\mathrm{Ad}\,u^*\circ(\beta_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\mathrm{Ad}\,u\circ\gamma_t)(f(t))\end{aligned}$$

for all $f\in C(\mathbb{T})\otimes A\otimes\mathbb{K}$ and $t\in[0,1]$. Put $\sigma_t=\mathrm{Ad}\,u\circ\gamma_t\circ\mathrm{Ad}\,u^*$. Then $\{\sigma_t\}_{t\in[0,1]}$ is a path from the identity to $\beta_{(1,0)}\otimes\mathrm{Ad}\,w$, and $z^{-1}\cdot KK(\mathrm{id}_S\otimes\tilde{\beta}_{(0,1)})\cdot z$ equals the KK-class of the automorphism $(\sigma_t^{-1}\circ(\beta_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\sigma_t)_t$ of $C(\mathbb{T})\otimes A\otimes\mathbb{K}$.

From $\Phi(\alpha)=\Phi(\beta)$, the two homomorphisms

$$A\otimes\mathbb{K}\ni a\mapsto((\gamma_t^{-1}\circ(\alpha_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\gamma_t)(a))_t\in C(\mathbb{T})\otimes A\otimes\mathbb{K}$$

and

$$A\otimes\mathbb{K}\ni a\mapsto((\sigma_t^{-1}\circ(\beta_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\sigma_t)(a))_t\in C(\mathbb{T})\otimes A\otimes\mathbb{K}$$

have the same KK-class in $KK(A,C(\mathbb{T})\otimes A)$, and so they are asymptotically unitarily equivalent by Theorem 2.6 (2). Therefore the two automorphisms $(\gamma_t^{-1}\circ(\alpha_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\gamma_t)_t$ and $(\sigma_t^{-1}\circ(\beta_{(0,1)}\otimes\mathrm{id}_{\mathbb{K}})\circ\sigma_t)_t$ of $C(\mathbb{T})\otimes A\otimes\mathbb{K}$ have the same KK-class. Thus,

$$z^{-1}\cdot KK(\mathrm{id}_S\otimes\tilde{\alpha}_{(0,1)})\cdot z=z^{-1}\cdot KK(\mathrm{id}_S\otimes\tilde{\beta}_{(0,1)})\cdot z,$$

and so $KK(\tilde{\alpha}_{(0,1)})=KK(\tilde{\beta}_{(0,1)})$ in $KK(A\rtimes_\rho\mathbb{Z},A\rtimes_\rho\mathbb{Z})$. Applying Theorem 4.11, we can conclude that α and β are outer conjugate by an automorphism μ such that $KK(\mu)=1_A$. Corollary 6.5 (and its proof) tells us that α and β are KK-trivially cocycle conjugate. □

By Proposition 8.4 and 8.5, we get the following.

Theorem 8.6. *Let A be a unital Kirchberg algebra. There exists a bijective correspondence between the following two sets.*

(1) *KK-trivially cocycle conjugacy classes of locally KK-trivial outer $\mathbb{Z}^2$-actions on A.*

(2) $\{x \in KK(A, S \otimes A) \mid K_0(x)([1]) = 0\}$.

Example 8.7. When A is the Cuntz algebra $\mathcal{O}_n$,

$$\{x \in KK(A, S \otimes A) \mid K_0(x)([1]) = 0\}$$

is isomorphic to $\mathrm{Ext}(\mathbb{Z}/(n-1)\mathbb{Z}, \mathbb{Z}/(n-1)\mathbb{Z}) \cong \mathbb{Z}/(n-1)\mathbb{Z}$. It follows that the cocycle conjugacy classes of outer $\mathbb{Z}^2$-actions on $\mathcal{O}_n$ correspond to $\mathbb{Z}/(n-1)\mathbb{Z}$.

Next we consider the non-unital case. For a locally KK-trivial $\mathbb{Z}^2$-action $\alpha : \mathbb{Z}^2 \curvearrowright A$ on a non-unital Kirchberg algebra A, one can define $\Phi(\alpha) \in KK(A, S \otimes A)$ in a similar fashion to the unital case. In the same way as Lemma 8.3, one can show that $\Phi(\alpha)$ is an invariant of KK-trivially cocycle conjugacy.

Theorem 8.8. *Let A be a non-unital Kirchberg algebra. There exists a bijective correspondence between the following two sets.*

(1) *KK-trivially cocycle conjugacy classes of locally KK-trivial outer $\mathbb{Z}^2$-actions on A.*

(2) $KK(A, S \otimes A)$.

Proof. We may assume that $A = A_0 \otimes \mathbb{K}$ and A_0 is a unital Kirchberg algebra in the Cuntz standard form. We let $\{e_{p,q}\}_{p,q \in \mathbb{Z}}$ denote a family of matrix units of $\mathbb{K}$.

For any $x \in KK(A, S \otimes A) = KK(A_0, S \otimes A_0)$, by Proposition 8.4, there exists a locally KK-trivial outer $\mathbb{Z}^2$-action α on A_0 such that $\Phi(\alpha) = x$. Then clearly $\Phi(\alpha \otimes \mathrm{id}_{\mathbb{K}}) = x$.

Let $\alpha : \mathbb{Z}^2 \curvearrowright A$ be a locally KK-trivial outer action. Choose partial isometries u_0, v_0 in A so that

$$u_0 u_0^* = 1 \otimes e_{0,0}, \qquad u_0^* u_0 = \alpha_{(1,0)}(1 \otimes e_{0,0})$$

and

$$v_0 v_0^* = 1 \otimes e_{0,0}, \qquad v_0^* v_0 = \alpha_{(0,1)}(1 \otimes e_{0,0}).$$

Define unitaries $u, v \in M(A)$ by

$$u = \sum_{p \in \mathbb{Z}} (1 \otimes e_{p,0}) u_0 \alpha_{(1,0)}(1 \otimes e_{0,p}) \quad \text{and} \quad v = \sum_{p \in \mathbb{Z}} (1 \otimes e_{p,0}) v_0 \alpha_{(0,1)}(1 \otimes e_{0,p}).$$

Then $(\mathrm{Ad}\, u \circ \alpha_{(1,0)})(1 \otimes e_{p,q}) = 1 \otimes e_{p,q}$ and $(\mathrm{Ad}\, v \circ \alpha_{(0,1)})(1 \otimes e_{p,q}) = 1 \otimes e_{p,q}$ for all $p, q \in \mathbb{Z}$. In addition, $u\alpha_{(1,0)}(v)\alpha_{(0,1)}(u^*)v^*$ commutes with $1 \otimes e_{p,q}$. It follows that there exist $\rho, \sigma \in \mathrm{Aut}(A_0)$ and $w \in U(A)$ such that

$$\mathrm{Ad}\, u \circ \alpha_{(1,0)} = \rho \otimes \mathrm{id}_{\mathbb{K}}, \qquad \mathrm{Ad}\, v \circ \alpha_{(0,1)} = \sigma \otimes \mathrm{id}_{\mathbb{K}}$$

and

$$u\alpha_{(1,0)}(v)\alpha_{(0,1)}(u^*)v^* = w \otimes 1, \qquad \rho \circ \sigma = \mathrm{Ad}\, w \circ \sigma \circ \rho.$$

By Corollary 7.12, there exists $a, b \in U(A_0)$ such that

$$(\operatorname{Ad} a \circ \rho) \circ (\operatorname{Ad} b \circ \sigma) = (\operatorname{Ad} b \circ \sigma) \circ (\operatorname{Ad} a \circ \rho)$$

and

$$a\rho(b)w\sigma(a^*)b^* = 1.$$

Let β be a $\mathbb{Z}^2$-action on A_0 induced by $\operatorname{Ad} a \circ \rho$ and $\operatorname{Ad} b \circ \sigma$. The two unitaries $(a \otimes 1)u$ and $(b \otimes 1)v$ of $M(A)$ give rise to an α-cocycle and $\beta \otimes \mathrm{id}_{\mathbb{K}}$ is the cocycle perturbation of α. Since $\Phi(\alpha) = \Phi(\beta \otimes \mathrm{id}_{\mathbb{K}}) = \Phi(\beta)$, we can complete the proof by Proposition 8.5. □

References

[1] B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.

[2] E. Blanchard, R. Rohde and M. Rørdam, *Properly infinite $C(X)$-algebras and K_1-injectivity*, J. Noncommut. Geom. **2** (2008), 263–282. arXiv:0704.1554

[3] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1981), 181–197.

[4] M. Dadarlat, *Approximately unitarily equivalent morphisms and inductive limit C^*-algebras*, K-Theory **9** (1995), 117–137.

[5] M. Dadarlat, *The homotopy groups of the automorphism group of Kirchberg algebras*, J. Noncommut. Geom. **1** (2007), 113–139.

[6] U. Haagerup and M. Rørdam, *Perturbations of the rotation C^*-algebras and of the Heisenberg commutation relation*, Duke Math. J. **77** (1995), 627–656.

[7] M. Izumi, *The Rohlin property for automorphisms of C^*-algebras*, Mathematical Physics in Mathematics and Physics (Siena, 2000), 191–206, Fields Inst. Commun., **30**, Amer. Math. Soc., Providence, RI, 2001.

[8] M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. I*, Duke Math. J. **122** (2004), 233–280.

[9] M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. II*, Adv. Math. **184** (2004), 119–160.

[10] Y. Katayama, *A duality for an action of a countable amenable group on a hyperfinite II_1-factor*, J. Operator Theory **21** (1989), 297–314.

[11] T. Katsura and H. Matui, *Classification of uniformly outer actions of $\mathbb{Z}^2$ on UHF algebras*, Adv. Math. **218** (2008), 940–968. arXiv:0708.4073

[12] E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theorem*, preprint, 1994.

[13] E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine Angew. Math. **525** (2000), 17–53.

[14] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Comm. Math. Phys. **81** (1981), 429–435.

[15] A. Kishimoto, *Automorphisms of AT algebras with the Rohlin property*, J. Operator Theory **40** (1998), 277–294.

[16] A. Kishimoto, *Unbounded derivations in AT algebras*, J. Funct. Anal. **160** (1998), 270–311.

[17] A. Kishimoto and A. Kumjian, *Crossed products of Cuntz algebras by quasi-free automorphisms*, Operator algebras and their applications (Waterloo, ON, 1994/1995), 173–192. Fields Inst. Commun., **13**, Amer. Math. Soc., Providence, RI, 1997.

[18] H. Matui, *Classification of outer actions of $\mathbb{Z}^N$ on $\mathcal{O}_2$*, Adv. Math. **217** (2008), 2872–2896. arXiv:0708.4074

[19] H. Nakamura, *The Rohlin property for $\mathbb{Z}^2$-actions on UHF algebras*, J. Math. Soc. Japan **51** (1999), 583–612.

[20] H. Nakamura, *Aperiodic automorphisms of nuclear purely infinite simple C^*-algebras*, Ergodic Theory Dynam. Systems **20** (2000), 1749–1765.

[21] D. Olesen, G. K. Pedersen and M. Takesaki, *Ergodic actions of compact abelian groups*, J. Operator Theory **3** (1980), 237–269.

[22] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114.

[23] N. C. Phillips, *Real rank and exponential length of tensor products with $\mathcal{O}_\infty$*, J. Operator Theory **47** (2002), 117–130.

[24] M. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Free probability theory (Waterloo, ON, 1995), 189–212. Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.

[25] M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Classification of nuclear C^*-algebras. Entropy in operator algebras, 1–145. Encyclopaedia Math. Sci., **126**, Springer, Berlin, 2002.

[26] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431–474.

[27] K. Thomsen, *The homotopy type of the group of automorphisms of a UHF-algebra*, J. Funct. Anal. **72** (1987), 182–207.

[28] N. E. Wegge-Olsen, *K-theory and C^*-algebras: A friendly approach*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.