Some remarks on topological full groups
of Cantor minimal systems

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Abstract

Giordano, Putnam and Skau showed that topological full groups of Cantor minimal systems are complete invariants for flip conjugacy. We will completely determine the structure of normal subgroups of the topological full group. Moreover, a necessary and sufficient condition for the topological full group to be finitely generated will be given.

1 Introduction

The notion of full group was introduced by Henry Dye in the measurable dynamical setting. He showed that if G_1 and G_2 are two countable groups of measure preserving transformations acting ergodically on a Lebesgue space, any group isomorphism between the full groups $[G_1]$ and $[G_2]$ is implemented by an orbit equivalence of G_1 and G_2. Giordano, Putnam and Skau studied its topological analogue in [GPS2]. They introduced several kinds of full groups for Cantor minimal systems and showed that they are complete invariants for orbit equivalence, strong orbit equivalence and flip conjugacy. Here, by a Cantor minimal system, we mean a pair of a Cantor set X and a minimal homeomorphism $\varphi \in \mathrm{Homeo}(X)$. The Cantor set X is characterized as a topological space which is compact, metrizable, perfect and totally disconnected. A homeomorphism φ is said to be minimal, if there is no non-trivial closed φ-invariant closed subset.

The purpose of this paper is to clarify the structure of topological full groups associated with Cantor minimal systems. The topological full group $[[\varphi]]$ consists of all homeomorphisms γ that are locally equal to some power of the minimal homeomorphism φ (see Definition 2.6 below). As the number of clopen sets in X is only countable, the set of $\mathbb{Z}$-valued continuous functions on X denoted by $C(X, \mathbb{Z})$ is also countable. Thus $[[\varphi]]$ is a countable group. This is a great difference from $[\varphi]$: the full group $[\varphi]$ is always uncountable, and a complete invariant for orbit equivalence, although $[[\varphi]]$ is a complete invariant for flip conjugacy. Since the topological full group $[[\varphi]]$ has full information of (X, φ) and is also countable, it is natural and important to investigate its various properties as a group. The present paper is the first attempt at such a study.

Now we give an overview of each section below. In Section 2, we will recall several results from [GPS1], [GPS2] and [GPS3]. We write the kernel of the index map $I : [[\varphi]] \to \mathbb{Z}$ by $[[\varphi]]_0$ (see Definition 2.7 below). In Section 3, the notion of AF full groups is introduced and the structure of its normal subgroups is discussed. Based on the results obtained in Section 3, we will define the signature on $[[\varphi]]_0$ in Section 4. The signature takes its value in $K^0(X, \varphi)/2K^0(X, \varphi)$, where $K^0(X, \varphi)$ is defined as

$$K^0(X, \varphi) = C(X, \mathbb{Z})/\{f - f\varphi^{-1} : f \in C(X, \mathbb{Z})\}.$$

Let $[f]$ denote the equivalence class of a function f in $K^0(X, \varphi)$. This K^0-group played a central role in [GPS1]. They showed that K^0-groups of two Cantor minimal systems are unital order isomorphic if and only if the two systems are strong orbit equivalent. In the present paper we show that the quotient of $[[\varphi]]_0$ by its commutator subgroup $D([[\varphi]]_0)$ is isomorphic to the $\mathbb{Z}_2$-valued dynamical cohomology $K^0(X, \varphi) \otimes \mathbb{Z}_2 \cong K^0(X, \varphi)/2K^0(X, \varphi)$. In addition, the commutator subgroup $D([[\varphi]]_0)$ is shown to be simple, which was mentioned in [Sk2, Theorem 4.11] without proof. In Section 5, it is proved that $D([[\varphi]]_0)$ is finitely generated if and only if (X, φ) is a minimal subshift. We need the result of Section 4 to determine when $[[\varphi]]$ is finitely generated. It is also shown that $D([[\varphi]]_0)$ never be finitely presented. In Section 6, two examples of minimal subshifts are given.

2 Preliminaries

We would like to collect basic facts which are needed in later sections. We will consider a countable equivalence relation $\mathcal{R}$ on a compact metrizable zero-dimensional space X, that is, X is homeomorphic to a compact subset of $\{0, 1\}^{\mathbb{N}}$ with the product topology and $\mathcal{R} \subset X \times X$ is an equivalence relation so that each equivalence class $[x]_{\mathcal{R}} = \{y \in X : (x, y) \in \mathcal{R}\}$ is countable for $x \in X$. If $(x, y), (y, z) \in \mathcal{R}$, then these pairs are said to be composable and their product is defined as $(x, y)(y, z) = (x, z)$. For $(x, y) \in \mathcal{R}$ its inverse is defined as $(x, y)^{-1} = (y, x)$. With this structure $\mathcal{R}$ is a (principal) groupoid. We say that $\mathcal{R}$ is minimal, if $[x]_{\mathcal{R}}$ is dense in X for all $x \in X$.

Definition 2.1 ([GPS3, Definition 2.1, 3.1, 3.7]). Let X be a compact metrizable zero-dimensional space and let $\mathcal{R} \subset X \times X$ be a countable equivalence relation.

(i) Suppose that $\mathcal{R}$ is equipped with a Hausdorff locally compact, second countable topology $\mathcal{T}$ so that the product of composable pairs is continuous and and the inverse map is a homeomorphism. We say that $(\mathcal{R}, \mathcal{T})$ is an étale equivalence relation on X, if the range map r defined by $\mathcal{R} \ni (x, y) \mapsto x \in X$ is a local homeomorphism.

(ii) An étale equivalence relation $(\mathcal{R}, \mathcal{T})$ on X is called a compact étale equivalence relation (CEER for short), if $\mathcal{R}$ is compact.

(iii) Let $\{(\mathcal{R}_n, \mathcal{T}_n)\}_{n=1}^{\infty}$ be a sequence of CEER's on X, so that $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ and $\mathcal{R}_n \in \mathcal{T}_{n+1}$. Let $(\mathcal{R}, \mathcal{T})$ be the inductive limit of $\{(\mathcal{R}_n, \mathcal{T}_n)\}_{n=1}^{\infty}$ with the inductive limit topology $\mathcal{T}$. We say that $(\mathcal{R}, \mathcal{T})$ is an AF equivalence relation and use the notation $(\mathcal{R}, \mathcal{T}) = \lim(\mathcal{R}_n, \mathcal{T}_n)$.

We remark that if $(\mathcal{R}, \mathcal{T})$ is a CEER, $\mathcal{T}$ is the relative topology from $X \times X$ and the cardinality of each equivalence class is uniformly finite.

Definition 2.2 ([GPS3, Example 2.7(ii)]). We say that $B = (V, E)$ is a Bratteli diagram when $V = \bigcup_{n=0}^{\infty} V_n$ and $E = \bigcup_{n=1}^{\infty} E_n$ are disjoint unions of finite sets of vertices and edges with source maps $s : E_n \to V_{n-1}$ and range maps $r : E_n \to V_n$ both of which are surjective. We always assume that V_0 consists of one point v_0. For a Bratteli diagram $B = (V, E)$,

$$X_B = \{(e_n)_n : e_n \in E_n, r(e_n) = s(e_{n+1}) \text{ for all } n \in \mathbb{N}\}$$

is called the infinite path space of B. The space X_B is endowed with the natural product topology.

We say that $B = (V, E)$ is simple when for every $v \in V_n$ there exists $m > n$ such that all vertices of V_m are connected to v.

When $B = (V, E)$ is a Bratteli diagram, the infinite path space X_B is a compact metrizable zero-dimensional space. Observe that if B is simple, X_B is a finite set or a Cantor set.

Definition 2.3 ([GPS3, Example 2.7(ii)]). Let $B = (V, E)$ be a Bratteli diagram. Two infinite paths are said to be tail equivalent, if they agree from some level on. We denote the tail equivalence relation by $\mathcal{R}_B$. Clearly $\mathcal{R}_B$ is an increasing union of

$$\mathcal{R}_N = \{((e_n)_n, (f_n)_n) \in X_B \times X_B : e_m = f_m \text{ for all } m \geq N\}$$

for $N \in \mathbb{N}$. Given the relative topology of $X \times X$, $\mathcal{R}_N$ is a CEER. Give $\mathcal{R}_B$ the inductive limit topology $\mathcal{T}_B$. We call $(\mathcal{R}_B, \mathcal{T}_B)$ the AF equivalence relation associated with B.

The following theorem asserts that every AF equivalence relation can be expressed by a Bratteli diagram.

Theorem 2.4 ([GPS3, Theorem 3.9]). *Let $(\mathcal{R}, \mathcal{T})$ be an AF equivalence relation on X. There exists a Bratteli diagram $B = (V, E)$ such that $(\mathcal{R}, \mathcal{T})$ is isomorphic to the AF equivalence relation $(\mathcal{R}_B, \mathcal{T}_B)$ associated with B. Thus, there exists a homeomorphism $\pi : X \to X_B$ such that $\pi \times \pi$ gives a homeomorphism from $\mathcal{R}$ to $\mathcal{R}_B$. Furthermore, $\mathcal{R}$ is minimal if and only if B is simple.*

Let us recall that a dimension group $K_0(B)$ is associated to a Bratteli diagram $B = (V, E)$. The notation is derived from the K-theory of C^*-algebras: the Bratteli diagram $B = (V, E)$ gives rise to an approximately finite-dimensional (AF) C^*-algebra and its K_0-group is isomorphic to $K_0(B)$. The reader may refer to Section 3 of [GPS1] for the connection to K-theory of C^*-algebras. The dimension group $K_0(B)$ is defined as the inductive limit of a free abelian group $\mathbb{Z}^{V_n}$ of finite rank. The connecting map from $\mathbb{Z}^{V_n}$ to $\mathbb{Z}^{V_{n+1}}$ is given by

$$\mathbb{Z}^{V_n} \ni v \mapsto \sum_{w \in V_{n+1}} m_{v,w} w \in \mathbb{Z}^{V_{n+1}},$$

where the canonical basis is denoted by the vertices and $m_{v,w}$ is the number of edges which go from $v \in V_n$ to $w \in V_{n+1}$. Endowed with a positive cone $K_0(B)^+$ and an order unit, $K_0(B)$ is a unital ordered group. We do not go to the detail, because it is not needed in this paper.

We will review relation between Cantor minimal systems and AF equivalence relations. Let φ be a minimal homeomorphism on the Cantor set X. As explained in [GPS3, Example 2.7(i)], an étale equivalence relation $(\mathcal{R}, \mathcal{T})$ is associated with (X, φ). More precisely,

$$\mathcal{R} = \{(x, \varphi^n(x)) : x \in X, \ n \in \mathbb{Z}\}$$

and $\mathcal{T}$ is given by transferring the product topology on $X \times \mathbb{Z}$ to $\mathcal{R}$ via the bijection $(x, n) \mapsto (x, \varphi^n(x))$.

A subset $Y \subset X$ is said to be wandering if $Y \cap \varphi^n(Y)$ is empty for all $n \in \mathbb{N}$. We consider the subequivalence relation $\mathcal{R}_Y$ of $\mathcal{R}$ defined by

$$\mathcal{R}_Y = \{(x, \varphi^n(x)), (\varphi^n(x), x) : x \in X, \ n \geq 0, \ \varphi^k(x) \notin Y \ \text{ if } 0 \leq k < n - 1\}$$

for a closed wandering subset Y. Thus $\mathcal{R}_Y$ is obtained from $\mathcal{R}$ by splitting the φ-orbits meeting Y in the forward and backward orbits at Y. Let $\mathcal{T}_Y$ denote the relative topology from $\mathcal{T}$. Although $\mathcal{R}_Y$ is defined in Section 4 of [GPS3] in the case that Y is not necessarily wandering, we do not need that case here. If Y is a singleton $\{x\}$, we write $(\mathcal{R}_x, \mathcal{T}_x)$ instead of $(\mathcal{R}_{\{x\}}, \mathcal{T}_{\{x\}})$.

Theorem 2.5 ([GPS1, Lemma 5.1][GPS3, Theorem 4.6]). *Let (X, φ) be a Cantor minimal system and let $(\mathcal{R}, \mathcal{T})$ be the associated étale equivalence relation. When $Y \subset X$ is a non-empty closed wandering set, the equivalence relation $(\mathcal{R}_Y, \mathcal{T}_Y)$ is AF.*

The key of the above theorem is that $(\mathcal{R}_x, \mathcal{T}_x)$ is an AF equivalence relation. This fact goes back to [HPS] and it enables us to represent a minimal homeomorphism on the Cantor set as a Bratteli-Vershik map on the infinite path space of a Bratteli diagram. Besides, under this representation, we can see that the dimension group associated with the Bratteli diagram is isomorphic to $K^0(X, \varphi)$. We refer the reader to [HPS].

In the last part of this section, we will review three kinds of full groups which were introduced in [GPS2].

Definition 2.6 ([GPS2, Definition 1.1, 2.1]). Let (X, φ) be a Cantor minimal system.

(i) The full group $[\varphi]$ of (X, φ) is the subgroup of all homeomorphisms $\gamma \in \text{Homeo}(X)$ such that $\gamma(x) \in \{\varphi^n(x) : n \in \mathbb{Z}\}$ for all $x \in X$. To any $\gamma \in [\varphi]$ is associated a map $n : X \to \mathbb{Z}$, defined by $\gamma(x) = \varphi^{n(x)}(x)$ for $x \in X$.

(ii) The topological full group $[[\varphi]]$ of (X, φ) is the subgroup of all homeomorphisms $\gamma \in \text{Homeo}(X)$, whose associated map $n : X \to \mathbb{Z}$ is continuous.

(iii) For every $x \in X$, let $[[\varphi]]_x$ denote the subgroup of all elements $\gamma \in [[\varphi]]$ that preserve the forward orbit $\{\varphi^n(x) : n \in \mathbb{N}\}$ and the backward orbit $\{\varphi^{1-n}(x) : n \in \mathbb{N}\}$ respectively.

Let $(\mathcal{R}, \mathcal{T})$ be the equivalence relation associated with a Cantor minimal system (X, φ). Then $[\varphi]$ consists of homeomorphisms γ which preserves every equivalence class $[x]_{\mathcal{R}}$, and $[[\varphi]]$ consists of the homeomorphism γ such that $X \ni x \mapsto (x, \gamma(x)) \in \mathcal{R}$ is well-defined and continuous. For every $x \in X$, $[[\varphi]]_x$ is the set of all homeomorphisms $\gamma \in [[\varphi]]$ that preserves every $\mathcal{R}_x$-equivalence class.

Definition 2.7 ([GPS2, Section 5]). Let (X, φ) be a Cantor minimal system and μ be a φ-invariant probability measure on X. For $\gamma \in [[\varphi]]$, we define

$$I(\gamma) \;=\; \int n_\gamma \, d\mu,$$

where n_γ is the continuous map from X to $\mathbb{Z}$ associated with γ. Then $I(\gamma)$ does not depend on the choice of μ and I is a surjective homomorphism from $[[\varphi]]$ to $\mathbb{Z}$. We call the homomorphism I the index map and denote its kernel by $[[\varphi]]_0$.

The reason why the value of the index map falls into the integers is that the K_1-group of the crossed product C^*-algebras arising from (X, φ) is isomorphic to the integers. We refer the reader to [GPS2, Section 5] for details. We also remark that $[[\varphi]]_x$ is contained in $[[\varphi]]_0$ because every element of $[[\varphi]]_x$ is of finite order.

The following is the main theorem of [GPS2].

Theorem 2.8 ([GPS2, Corollary 4.4, 4.6, 4.11, 5.18]). *Let (X, φ) and (Y, ψ) be Cantor minimal systems.*

(i) *The two Cantor minimal systems are orbit equivalent if and only if $[\varphi]$ and $[\psi]$ are isomorphic as abstract groups.*

(ii) *The two Cantor minimal systems are flip conjugate if and only if $[[\varphi]]$ and $[[\psi]]$ are isomorphic as abstract groups.*

(iii) *The two Cantor minimal systems are flip conjugate if and only if $[[\varphi]]_0$ and $[[\psi]]_0$ are isomorphic as abstract groups.*

(iv) *Let $x \in X$ and $y \in Y$. The two Cantor minimal systems are strong orbit equivalent if and only if $[[\varphi]]_x$ and $[[\psi]]_y$ are isomorphic as abstract groups.*

3   AF full groups

In this section we will introduce AF full groups and study their normal subgroups. The signature map defined on the AF full group will play a crucial role in the next section.

Definition 3.1. Let X be a compact metrizable zero-dimensional space and let $(\mathcal{R}, \mathcal{T})$ be an étale equivalence relation on X. We put

$$[\mathcal{R}, \mathcal{T}] \ = \ \{\ \gamma \in \mathrm{Homeo}(X) : X \ni x \mapsto (x, \gamma(x)) \in \mathcal{R} \text{ is well-defined and continuous }\}$$

and call it the topological full group associated with $(\mathcal{R}, \mathcal{T})$.

Since the diagonal $\{(x, x) : x \in X\}$ is a clopen subset in $\mathcal{R}$, the fixed point set of each element $\gamma \in [\mathcal{R}, \mathcal{T}]$ is clopen. When $(\mathcal{R}, \mathcal{T})$ is the étale equivalence relation associated with a Cantor minimal system (X, φ), the topological full group $[\mathcal{R}, \mathcal{T}]$ defined above equals $[[\varphi]]$ defined in Section 2. It is also obvious that $[[\varphi]]_x$ equals $[\mathcal{R}_x, \mathcal{T}_x]$ for all $x \in X$.

The following proposition asserts that AF equivalence relations can be characterized by their associated topological full groups. Although the idea of the proof has already appeared in [D] or [K] and this fact is folklore for experts, we would like to give a proof without using Theorem 2.4 for the reader's convenience.

Proposition 3.2. *For an étale equivalence relation $(\mathcal{R}, \mathcal{T})$ on the Cantor set X, the following are equivalent.*

(i) *$(\mathcal{R}, \mathcal{T})$ is an AF equivalence relation.*

(ii) *$[\mathcal{R}, \mathcal{T}]$ is locally finite.*

Proof. (i)⇒(ii) Let $(\mathcal{R}, \mathcal{T})$ be an AF equivalence relation. Suppose that $(\mathcal{R}, \mathcal{T})$ is the inductive limit of CEER's $\{(\mathcal{R}_n, \mathcal{T}_n)\}_n$. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be elements of $[\mathcal{R}, \mathcal{T}]$ and let H be the subgroup they generate. We must show that H is finite. Put $F = \{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}, \ldots, \gamma_m, \gamma_m^{-1}\}$. Since X is compact, $\{(x, \gamma_i(x)) : x \in X\}$ and their union are also compact in $\mathcal{R}$. Hence there exists n such that $\mathcal{R}_n$ contains $(x, \gamma_i(x))$ for all $x \in X$ and $i = 1, 2, \ldots, m$. Let d be a metric on X. There is $\varepsilon > 0$ such that $d(x, x') > 2\varepsilon$ if $(x, x') \in \mathcal{R}_n$ and $x \neq x'$, because $(\mathcal{R}_n, \mathcal{T}_n)$ is a CEER. Choose a real number $\delta > 0$ smaller than ε so that $d(x, y) < \delta$ implies $d(\gamma(x), \gamma(y)) < \varepsilon$ for all $\gamma \in F$. For every $x \in X$, let U_x be the open set defined by

$$U_x \ = \ \{y \in X : d(y', [x]_{\mathcal{R}_n}) < \delta \text{ for all } y' \in [y]_{\mathcal{R}_n}\}.$$

If $y \in U_x$, there exists a unique $x' \in [x]_{\mathcal{R}_n}$ with $d(y, x') < \delta$. Furthermore, by the choice of ε, we have $d(\gamma(y), \gamma(x')) < \delta$ for all $\gamma \in F$, which implies $d(\gamma(y), \gamma(x')) < \delta$ for all $\gamma \in H$. Every element γ belonging to H induces a permutation on the finite set $[x]_{\mathcal{R}_n}$. The above argument shows that if γ induces the identity on $[x]_{\mathcal{R}_n}$, then γ is the identity on U_x. Since X is compact, it can be covered by finitely many U_x's. Then we obtain an injective homomorphism from H to a finite direct sum of permutation groups, which completes the proof.

(ii)⇒(i) Suppose that $[\mathcal{R}, \mathcal{T}]$ is an increasing union of finite subgroups $G_1, G_2, \ldots$ and let $\mathcal{R}_n = \{(x, \gamma(x)) : x \in X, \gamma \in G_n\}$. Let $\mathcal{T}_n$ be the relative topology from $\mathcal{T}$. Then $(\mathcal{R}_n, \mathcal{T}_n)$ is a

CEER, because $\{(x, \gamma(x)) : x \in X\}$ is compact for all $\gamma \in G_n$. To complete the proof, it suffices to show $\mathcal{R} = \bigcup_n \mathcal{R}_n$. Let $(x, y) \in \mathcal{R}$. We may assume $x \neq y$. Choose a clopen neighborhood U of (x, y) so that the range map and the source map on U are local homeomorphisms, and $r(U)$ and $s(U)$ are disjoint. Define $\gamma \in \mathrm{Homeo}(X)$ by $\gamma(z) = s_0 r_0^{-1}(z)$ if $z \in r(U)$, $\gamma(z) = r_0 s_0^{-1}(z)$ if $z \in s(U)$ and $\gamma(z) = z$ otherwise, where s_0 and r_0 denote the restriction to U of s and r, respectively. Then $X \ni z \mapsto (z, \gamma(z)) \in \mathcal{R}$ is clearly a well-defined continuous map. It follows that we have $\gamma \in G_n$ for some n and $(x, y) = (x, \gamma(x)) \in \mathcal{R}_n$. $\qquad\square$

We call $[\mathcal{R}, \mathcal{T}]$ an AF full group if $(\mathcal{R}, \mathcal{T})$ is an AF equivalence relation.

Let $(\mathcal{R}, \mathcal{T})$ be an AF equivalence relation on X and let $B = (V, E)$ be a Bratteli diagram such that $(\mathcal{R}, \mathcal{T})$ is isomorphic to the AF equivalence relation associated with B. We identify the infinite path space X_B with X and $(\mathcal{R}_B, \mathcal{T}_B)$ with $(\mathcal{R}, \mathcal{T})$. Put $G = [\mathcal{R}, \mathcal{T}]$. We would like to examine the group structure of G.

For each $n \in \mathbb{N}$,

$$\mathcal{R}_n = \{ (x, y) \in X \times X : x \text{ and } y \text{ agree except for the initial } n \text{ edges}\}$$

is a subrelation of $\mathcal{R}$. Give $\mathcal{R}_n$ the relative topology $\mathcal{T}_n$ of $X \times X$. Then $(\mathcal{R}_n, \mathcal{T}_n)$ is a CEER and we have $(\mathcal{R}, \mathcal{T}) = \lim(\mathcal{R}_n, \mathcal{T}_n)$. Take $\gamma \in G$. As in the proof of the above proposition, we can find $n \in \mathbb{N}$ such that $(x, \gamma(x))$ is in $\mathcal{R}_n$ for all $x \in X$. The infinite paths x and $\gamma(x)$ agree from the $(n+1)$-st level on. Since γ is continuous, for sufficiently large $m > n$, we may assume that the initial n edges of $\gamma(x)$ depend only on the initial m edges of x. Then, if the initial m edges of x and y coincide, then the initial m edges of $\gamma(x)$ and $\gamma(y)$ also coincide, because γ preserves the k-th edges if $k > n$. Hence γ induces a permutation on the set of paths from the top vertex v_0 to each $v \in V_m$. These permutations form a subgroup G_m in G, which is isomorphic to $\bigoplus_{v \in V_m} S_{h(v)}$, where $h(v)$ denotes the number of paths from the top vertex v_0 to v. Clearly G_m is contained in G_{m+1} and G equals the union of all the G_m's. The inclusion $G_m \subset G_{m+1}$ is described by the edge set E_{m+1}: if there are k edges between $v \in V_m$ and $v' \in V_{m+1}$, then $S_{h(v')} \subset G_{m+1}$ contains k copies of $S_{h(v)}$.

We can summarize the conclusion just obtained as follows.

Proposition 3.3. *When $B = (V, E)$ is a Bratteli diagram and $(\mathcal{R}, \mathcal{T})$ is the associated AF equivalence relation, the AF full group $G = [\mathcal{R}, \mathcal{T}]$ can be written as a increasing union of subgroups G_m, which is isomorphic to $\bigoplus_{v \in V_m} S_{h(v)}$.*

Let p be a path from v_0 to $v \in V_m$ and let $U(p)$ be a clopen set of infinite paths whose initial paths agree with p. The zero-dimensional topology in X is generated by clopen sets of this form. If γ is in G_m, then $1_{U(p)} \gamma^{-1} = 1_{\gamma(U(p))}$ is equal to $U(q)$ for another path q from v_0 to v. Define the K^0-group associated with the AF equivalence relation $(\mathcal{R}, \mathcal{T})$ by

$$K^0(\mathcal{R}, \mathcal{T}) = C(X, \mathbb{Z})/\{f - f\gamma^{-1} : \gamma \in [\mathcal{R}, \mathcal{T}]\}.$$

We denote the equivalence class of $f \in C(X, \mathbb{Z})$ by $[f]$. Obviously $K^0(\mathcal{R}, \mathcal{T})$ has a natural unital ordered group structure and, by the above observation, it is unital order isomorphic to $K_0(B)$.

Let us consider normal subgroups of $G = [\mathcal{R}, \mathcal{T}]$. The symmetric group $S_{h(v)}$ has the normal subgroup $A_{h(v)}$, and so G_m has the normal subgroup H_m isomorphic to $\bigoplus_{v \in V_m} A_{h(v)}$.

Lemma 3.4. *In the above setting, the commutator subgroup $D(G)$ equals to the union $\bigcup H_m$. Furthermore, $D(G)$ is simple if the Bratteli diagram B is simple, or equivalently $\mathcal{R}$ is minimal.*

Proof. The first half of the statement is obvious.

To prove the latter half, let N be a normal subgroup of $D(G)$ and let $\gamma \in N$ be a non-trivial element. Suppose that γ is contained in H_m and its v-summand is non-trivial for some $v \in V_m$.

If $l > m$ is sufficiently large, then, for all $w \in V_l$, there exists a path from v to w. Therefore the w-summand of γ is non-trivial for all $w \in V_l$. By taking commutators, we can see that every $A_{h(w)}$ is contained in N, confirming that H_l is contained in N, if l is sufficiently large. It follows that we have $N = D(G)$.

Conversely, suppose that B is not simple. We can find a vertex $v \in V_m$ such that $V_l' = \{w \in V_l : w \text{ is connected to } v\}$ is a proper subset of V_l for all $l > m$. Let N_l be the direct sum of all w-summands for $w \in V_l'$ in H_l. Then $N = \bigcup_{l>m} N_l$ is a non-trivial normal subgroup of H. This completes the proof. $\square$

The quotient group G_m/H_m is isomorphic to $\mathbb{Z}_2^{V_m}$ unless $h(v)$ is one for some $v \in V_m$, where $h(v)$ denotes the number of paths from the top vertex v_0 to v. Suppose that $\#[x]_{\mathcal{R}}$ is more than one for all $x \in X$. Then $h(v)$ is more than one for all $v \in V_m$, if m is sufficiently large. Therefore $G/D(G)$ is isomorphic to the inductive limit of $G_m/H_m \cong \mathbb{Z}_2^{V_m}$, where the connecting homomorphism from $\mathbb{Z}_2^{V_m}$ to $\mathbb{Z}_2^{V_{m+1}}$ is given by the edge set E_m. Hence we obtain the following under the identification of $K^0(\mathcal{R}, \mathcal{T}) \otimes \mathbb{Z}_2$ with $K^0(\mathcal{R}, \mathcal{T})/2K^0(\mathcal{R}, \mathcal{T})$.

Lemma 3.5. *If $\#[x]_{\mathcal{R}}$ is more than one for all $x \in X$, the quotient group $G/D(G)$ is isomorphic to $K^0(\mathcal{R}, \mathcal{T})/2K^0(\mathcal{R}, \mathcal{T})$.*

Definition 3.6. We denote the composition of the quotient map from G to $G/D(G)$ and the isomorphism in the lemma above by $\mathrm{sgn} : G \to K^0(\mathcal{R}, \mathcal{T})/2K^0(\mathcal{R}, \mathcal{T})$ and call it the signature map. For $\gamma \in G = [\mathcal{R}, \mathcal{T}]$, we call $\mathrm{sgn}(\gamma) \in K^0(\mathcal{R}, \mathcal{T})/2K^0(\mathcal{R}, \mathcal{T})$ the signature of γ.

We would like to give an explicit procedure for computing the signature. Let γ be an element of $[\mathcal{R}, \mathcal{T}]$. For $x \in X$, let $f(x)$ be the least natural number such that $\gamma^{f(x)}(x) = x$. Then f is a well-defined continuous function. If $f^{-1}(n)$ is not empty, we can find a clopen subset $V_n \subset f^{-1}(n)$ so that $V_n, \gamma(V_n), \ldots, \gamma^{n-1}(V_n)$ are mutually disjoint and $f^{-1}(n) = V_n \cup \gamma(V_n) \cup \cdots \cup \gamma^{n-1}(V_n)$. Put

$$g(x) \;=\; \sum_{n \in 2\mathbb{N}} 1_{V_n}(x)$$

for $x \in X$. The following lemma claims that $\mathrm{sgn}(\gamma)$ depends only on the dynamical behavior of γ.

Lemma 3.7. *In the above setting, $\mathrm{sgn}(\gamma)$ is equal to $[g] + 2K^0(\mathcal{R}, \mathcal{T})$.*

Proof. Put $U = \bigcup_{n \in 2\mathbb{N}} V_n$. For $z \in X$, we define

$$\gamma'(z) = \begin{cases} \gamma(z) & z \in U \\ \gamma^{-1}(z) & z \in \gamma(U) \\ z & \text{otherwise.} \end{cases}$$

Then it is not hard to see that γ' belongs to G. In addition, $\gamma^{-1}\gamma'$ is in $D(G)$, and so it suffices to show $\mathrm{sgn}(\gamma') = [1_U] + 2K^0(\mathcal{R}, \mathcal{T})$. Let us assume that γ' belongs to $G_m \cong \bigoplus_{v \in V_m} S_{h(v)}$. For $v \in V_m$, let P_v be the set of paths p from v_0 to v such that $U(p) \subset U$. The $S_{h(v)}$ component of $\gamma' \in G_m$ is a product of $\#P_v$ transpositions. Hence

$$\mathrm{sgn}(\gamma') = \sum_{\#P_v \notin 2\mathbb{N}} [1_{U(p_v)}] + 2K^0(\mathcal{R}, \mathcal{T}),$$

where p_v is a path from v_0 to v. It follows that

$$
\begin{aligned}
\operatorname{sgn}(\gamma') &= \sum_{v \in V_m} \#P_v [1_{U(p_v)}] + 2K^0(\mathcal{R}, \mathcal{T}) \\
&= \sum_{v \in V_m} \sum_{p \in P_v} [1_{U(p)}] + 2K^0(\mathcal{R}, \mathcal{T}) \\
&= [1_U] + 2K^0(\mathcal{R}, \mathcal{T}),
\end{aligned}
$$

which completes the proof. $\qquad\square$

Note that the construction of the function g is possible, if γ is of finite order.

4 Signatures

Let (X, φ) be a Cantor minimal system. In this section we would like to investigate $[[\varphi]]_0$ by using the signature map.

Lemma 4.1. *When x and y have distinct orbits in X, we have $[[\varphi]]_0 = [[\varphi]]_x [[\varphi]]_y$, that is, for every $\gamma \in [[\varphi]]_0$ there exist $\gamma_1 \in [[\varphi]]_x$ and $\gamma_2 \in [[\varphi]]_y$ such that $\gamma = \gamma_1 \gamma_2$.*

Proof. Let us denote the forward and backward orbit of x by $O^+(x) = \{\varphi^n(x); n \in \mathbb{N}\}$ and $O^-(x) = \{\varphi^{1-n}(x); n \in \mathbb{N}\}$, respectively. Put

$$
A \;=\; \{n \in \mathbb{N} : \gamma(\varphi^n(x)) \in O^-(x)\}
$$

and

$$
B \;=\; \{n \in \mathbb{N} : \gamma(\varphi^{1-n}(x)) \in O^+(x)\},
$$

then A and B have the same cardinalities as $\gamma \in [[\varphi]]_0$, according to [GPS2, Remark 5.6]. Let l be the maximum of $n \in A \cup B$. Choose a small clopen neighborhood U of x so that $\varphi^{-l}(U), \varphi^{-l+1}(U), \dots, \varphi^l(U)$ are mutually disjoint and

$$
V \;=\; \bigcup_{n \in B} \varphi^{1-n}(U) \cup \bigcup_{n \in A} \varphi^n(U)
$$

does not contain $y, \varphi(y), \dots, \varphi^{2l-1}(y)$. Let π be a bijective map from A to B. Define $\gamma_2 \in \mathrm{Homeo}(X)$ by

$$
\gamma_2(z) \;=\; \begin{cases} \varphi^{1-\pi(n)-n}(z) & z \in \varphi^n(U) \text{ for some } n \in A \\ \varphi^{n-1+\pi^{-1}(n)}(z) & z \in \varphi^{1-n}(U) \text{ for some } n \in B \;. \\ z & z \in X \setminus V \end{cases}
$$

Then, it is not hard to see $\gamma_2 \in [[\varphi]]_y$ and $\gamma\gamma_2 \in [[\varphi]]_x$. $\qquad\square$

The signature map has been already defined on $[[\varphi]]_x$ for every $x \in X$, taking its value in $K^0(\mathcal{R}_x, \mathcal{T}_x) \otimes \mathbb{Z}_2$. Since $K^0(\mathcal{R}_x, \mathcal{T}_x)$ can be identified with $K^0(X, \varphi)$ (see [GPS1, Theorem 3.7]), we may regard $\operatorname{sgn}(\gamma) \in K^0(X, \varphi) \otimes \mathbb{Z}_2$.

We need to know the behavior of the signature map on the intersection $[[\varphi]]_x \cap [[\varphi]]_y$.

Lemma 4.2. *For $\gamma \in [[\varphi]]_x \cap [[\varphi]]_y$, the signature $\operatorname{sgn}(\gamma) \in K^0(X, \varphi)/2K^0(X, \varphi)$ is well-defined.*

Proof. It is clear from Lemma 3.7. $\qquad\square$

Lemma 4.3. *The signature map* sgn *from* $[[\varphi]]_x \cap [[\varphi]]_y$ *to* $K^0(X, \varphi)/2K^0(X, \varphi)$ *is surjective.*

Proof. The subequivalence relation $\mathcal{R}_{\{x,y\}}$ is also an AF equivalence relation by Theorem 2.5 and $[[\varphi]]_x \cap [[\varphi]]_y = [\mathcal{R}_{\{x,y\}}, \mathcal{T}_{\{x,y\}}]$. The image of the signature map defined on $[\mathcal{R}_{\{x,y\}}, \mathcal{T}_{\{x,y\}}]$ is the whole of $K^0(\mathcal{R}_{\{x,y\}}, \mathcal{T}_{\{x,y\}}) \otimes \mathbb{Z}_2$ by Lemma 3.5. There exists a canonical surjective map from $K^0(\mathcal{R}_{\{x,y\}}, \mathcal{T}_{\{x,y\}})$ to $K^0(X, \varphi)$ (see [GPS1, Theorem 1.17]), thereby completing the proof. $\square$

Lemma 4.4. *Let x and y be lying in distinct orbits. For $\gamma \in [[\varphi]]_0$, let $\gamma = \gamma_1\gamma_2$ be the decomposition described in Lemma 4.1, where $\gamma_1 \in [[\varphi]]_x$ and $\gamma_2 \in [[\varphi]]_y$. Put $s = \mathrm{sgn}(\gamma_1) + \mathrm{sgn}(\gamma_2) \in K^0(X, \varphi)/2K^0(X, \varphi)$.*

   (i) *s does not depend on the choice of γ_1 and γ_2.*

   (ii) *s does not depend on the choice of x and y lying in distinct orbits.*

Proof. (1) follows from Lemma 4.2 and the fact that sgn is a homomorphism on AF full groups. Let us prove (2). Suppose that x, y and z have distinct orbits. In the proof of Lemma 4.1 it is easy to construct γ_2 so that γ_2 does not move $z, \varphi(z), \ldots, \varphi^{l-1}(z)$. Therefore we can make γ_2 in $[[\varphi]]_y \cap [[\varphi]]_z$, which means that the signature takes the same value if one uses x and z instead of x and y for the definition. In the case of using z and y to define the signature, we can show the assertion similarly. $\square$

Definition 4.5. Let $\gamma \in [[\varphi]]_0$. We denote the value $s \in K^0(X, \varphi)/2K^0(X, \varphi)$ described in the lemma above by $\mathrm{sgn}(\gamma)$. By the lemma above, sgn is a well-defined map from $[[\varphi]]_0$ to $K^0(X, \varphi)/2K^0(X, \varphi)$. We call sgn the signature map.

Proposition 4.6. *The signature map from $[[\varphi]]_0$ to $K^0(X, \varphi)/2K^0(X, \varphi)$ is a group homomorphism.*

Proof. It suffices to show $\mathrm{sgn}(\gamma\tau) = \mathrm{sgn}(\gamma) + \mathrm{sgn}(\tau)$ for $\gamma, \tau \in [[\varphi]]_0$. Suppose that $x, y \in X$ have distinct orbits. Let $\tau = \tau_1\tau_2$ be the decomposition as in Lemma 4.1. Put

$$m = \max\{n \in \mathbb{N} : \tau_1^{-1}(\varphi^n(y)) \in O^-(y) \text{ or } \tau_1^{-1}(\varphi^{1-n}(y)) \in O^+(y)\}.$$

When we apply Lemma 4.1 to γ, it is possible to choose the clopen set U so that V does not contain $\varphi^k(y)$ for all $1 - m \leq k \leq m$. Then, the obtained $\gamma_2 \in [[\varphi]]_y$ satisfies $\tau_1^{-1}\gamma_2\tau_1 \in [[\varphi]]_y$. Because the signature map defined on an AF full group is a homomorphism (Definition 3.6), we have $\mathrm{sgn}(\gamma_1\tau_1) = \mathrm{sgn}(\gamma_1) + \mathrm{sgn}(\tau_1)$. Similarly we get $\mathrm{sgn}((\tau_1^{-1}\gamma_2\tau_1)\tau_2) = \mathrm{sgn}(\tau_1^{-1}\gamma_2\tau_1) + \mathrm{sgn}(\tau_2) = \mathrm{sgn}(\gamma_2) + \mathrm{sgn}(\tau_2)$, where the last equation follows from Lemma 3.7. Hence we have

$$\mathrm{sgn}(\gamma\tau) = \mathrm{sgn}(\gamma_1\tau_1) + \mathrm{sgn}(\tau_1^{-1}\gamma_2\tau_1\tau_2)$$
$$= \mathrm{sgn}(\gamma_1) + \mathrm{sgn}(\tau_1) + \mathrm{sgn}(\gamma_2) + \mathrm{sgn}(\tau_2)$$
$$= \mathrm{sgn}(\gamma) + \mathrm{sgn}(\tau),$$

proving that sgn is a homomorphism. $\square$

Lemma 4.7. *Let U be a clopen neighborhood of $x \in X$. Then there exists a clopen neighborhood V of x such that $V \subset U$ and $[1_V]$ is 2-divisible in $K^0(X, \varphi)$.*

Proof. Choose a small neighborhood V_0 contained in U so that $2\mu(V_0)$ is less than $\mu(U)$ for all invariant probability measure μ. By applying [GW, Lemma 2.5] with $A = U \setminus V_0$ and $B = V_0$, we obtain $V_1 \subset U \setminus V_0$ satisfying $[1_{V_0}] = [1_{V_1}]$ in $K^0(X, \varphi)$. Then $V_0 \cup V_1$ does the work. $\square$

   Now we are ready to prove the main theorems of this section.

Theorem 4.8. *Let (X, φ) be a Cantor minimal system. Then the kernel of the signature map equals the commutator $D([[\varphi]]_0)$. Thus* sgn *establishes an isomorphism between $[[\varphi]]_0/D([[\varphi]]_0)$ and $K^0(X, \varphi)/2K^0(X, \varphi)$.*

Proof. It is clear that the commutator subgroup $D([[\varphi]]_0)$ is contained in the kernel of sgn. Let us prove the other inclusion. Take $\gamma \in [[\varphi]]_0$ with $\text{sgn}(\gamma) = 0$. In the proof of Lemma 4.1 it is easy to choose the clopen set U so that $[1_U]$ is 2-divisible in $K^0(X, \varphi)$ by using Lemma 4.7. Then, by Lemma 3.7, we can see that $\text{sgn}(\gamma_2) = 0$. As $\text{sgn}(\gamma) = 0$, the signature of γ_1 is equal to zero, too. Lemma 3.5 tells us that γ_1 and γ_2 are contained in the commutator subgroups of AF full groups. Hence γ is in the commutator subgroup $D([[\varphi]]_0)$.

Since the signature map is surjective, it is easily verified that $[[\varphi]]_0/D([[\varphi]]_0)$ is isomorphic to $K^0(X, \varphi)/2K^0(X, \varphi)$. $\qquad\square$

Note that $\text{sgn}(\varphi\gamma\varphi^{-1})$ is equal to $\text{sgn}(\gamma)$ for all $\gamma \in D([[\varphi]]_0)$. It follows that the commutator subgroup $D([[\varphi]]_0)$ coincides with $D([[\varphi]])$.

Theorem 4.9. *Let (X, φ) be a Cantor minimal system. The commutator subgroup $D([[\varphi]]_0)$ is simple.*

Proof. Suppose that $N \subset D([[\varphi]]_0)$ is a nontrivial normal subgroup. It suffices to show that N contains $D([[\varphi]]_x)$ for every $x \in X$. Take a nontrivial element $\gamma \in N$. Let l be the maximum of $n(z)$, where n is the continuous function from X to $\mathbb{Z}$ satisfying $\gamma(z) = \varphi^{n(z)}(z)$. In view of $\gamma \neq \text{id}, \varphi$ we can select $y \in X$ so that x and y have distinct orbits and $\gamma(y) \neq y, \varphi(y)$. Choose a clopen neighborhood U of y which does not intersect with $\varphi(U)$ and does not contain $\varphi(y), \gamma(y), \varphi^{-1}\gamma(y)$ and $\varphi^k(x)$ with $|k| \leq l+1$. From Lemma 4.7 we may further assume that $[1_U]$ is 2-divisible in $K^0(X, \varphi)$. Define $\sigma \in D([[\varphi]]_0)$ by

$$\sigma(z) = \begin{cases} \varphi(z) & z \in U \\ \varphi^{-1}(z) & z \in \varphi(U) \\ z & \text{otherwise.} \end{cases}$$

From the construction, we infer that the commutator $\sigma\gamma^{-1}\sigma\gamma$ belongs to $\ker \text{sgn} \cap [[\varphi]]_x = D([[\varphi]]_x)$. Besides, on account of $\sigma\gamma^{-1}\sigma\gamma(y) = \sigma\gamma^{-1}\gamma(y) = \sigma(y) = \varphi(y)$, the commutator $\sigma\gamma^{-1}\sigma\gamma$ is not the identity. Then using Lemma 3.4 shows that N contains $D([[\varphi]]_x)$. $\qquad\square$

Combining Theorem 4.8 and 4.9, we get the following.

Corollary 4.10. *For a Cantor minimal system (X, φ), $[[\varphi]]_0$ is simple if and only if $K^0(X, \varphi)$ is 2-divisible.*

We would like to conclude this section by giving an example of a computation of the signature. Let U be a nonempty clopen subset of X and let φ_U be the first return map on U. We can extend φ_U to a homeomorphism on X by putting $\varphi_U(x) = x$ for all $x \in X \setminus U$. We remark that the index of φ_U is one.

Lemma 4.11. *When U and V are nonempty clopen subsets, the signature of $\varphi_U\varphi_V^{-1}$ is equal to $[1_U] + [1_V] + 2K^0(X, \varphi)$.*

Proof. We may assume that V is contained in U, because $\varphi_U\varphi_V^{-1} = (\varphi_{U\cup V}\varphi_U^{-1})^{-1}(\varphi_{U\cup V}\varphi_V^{-1})$. Then, it is easily checked that $\gamma = \varphi_U\varphi_V^{-1}$ is in $[[\varphi]]_x$ for every $x \in V$. For $x \in X$, let $f(x)$ be the least natural number such that $\gamma^{f(x)}(x) = x$. Put $V_n = V \cap f^{-1}(n)$. Since

$$U = \bigcup_{n \in \mathbb{N}} \bigcup_{k=1}^{n} \gamma^k(V_n),$$

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by virtue of Lemma 3.7, we obtain

$$
\begin{aligned}
\operatorname{sgn}(\gamma) &= \sum_{n \in 2\mathbb{N}} [1_{V_n}] + 2K^0(X, \varphi) \\
&= \sum_{n \in 2\mathbb{N}} [1_{V_n}] + \sum_{n \in 2\mathbb{N}} \sum_{k=1}^{n} [1_{\gamma^k(V_n)}] + \sum_{n \in 2\mathbb{N}-1} \left([1_{V_n}] + \sum_{k=1}^{n} [1_{\gamma^k(V_n)}] \right) + 2K^0(X, \varphi) \\
&= \sum_{n \in \mathbb{N}} [1_{V_n}] + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n} [1_{\gamma^k(V_n)}] + 2K^0(X, \varphi) \\
&= [1_V] + [1_U] + 2K^0(X, \varphi).
\end{aligned}
$$

$\square$

5 Finite generatedness

Let (X, φ) be a Cantor minimal system. In this section we will give a necessary and sufficient condition for $D([[\varphi]]_0)$ to be finitely generated. As pointed out in the proof of Theorem 4.8, every $\gamma \in D([[\varphi]]_0)$ is a product of some $\gamma_1 \in D([[\varphi]]_x)$ and $\gamma_2 \in D([[\varphi]]_y)$. Thus, in order to prove the finite generatedness, it suffices to show that there exists a finite number of elements of $D([[\varphi]]_0)$ generating $D([[\varphi]]_x)$ for all $x \in X$.

Lemma 5.1. *Let n be a natural number more than two. The alternating group A_n is generated by $(1, 2, 3), (2, 3, 4), \ldots, (n-2, n-1, n)$.*

Proof. The proof is by induction. For $n = 3, 4$ the assertion follows from the straightforward computation. Let us assume that A_n is generated by $(1, 2, 3), (2, 3, 4), \ldots, (n-2, n-1, n)$.
Suppose $k \in \{1, 2, \ldots, n-2\}$. Then

$$
((k, n-1)(n-1, n))(n-1, n, n+1)((k, n-1)(n-1, n))^{-1} = (n, k, n+1).
$$

Similarly $(j, k, n+1)$ can be generated for all distinct $j, k \in \{1, 2, \ldots, n\}$. $\square$

Let U be a nonempty clopen subset of X such that $\varphi^{-1}(U), U$ and $\varphi(U)$ are mutually disjoint. Then we define a homeomorphism γ_U of order three by

$$
\gamma_U(x) = \begin{cases} \varphi(x) & x \in \varphi^{-1}(U) \cup U \\ \varphi^{-2}(x) & x \in \varphi(U) \\ x & \text{otherwise.} \end{cases}
$$

For convenience, we put $\gamma_\emptyset = \mathrm{id}$.
When $\varphi^{-2}(U), \varphi^{-1}(U), U, \varphi(U)$ and $\varphi^2(U)$ are mutually disjoint, we define a homeomorphism τ_U of order five by $\tau_U = \gamma_{\varphi^{-1}(U)} \gamma_{\varphi(U)}$. Observe that all γ_U and τ_U belong to the commutator subgroup $D([[\varphi]]_0)$, because they have odd order (see Lemma 3.7).

Lemma 5.2. *The commutator subgroup $D([[\varphi]]_0)$ is generated by*

$$
\Gamma = \{ \gamma_U \in D([[\varphi]]_0) : \varphi^{-1}(U), U \text{ and } \varphi(U) \text{ are mutually disjoint} \}.
$$

Proof. It suffices to show that for all $x \in X$, $D([[\varphi]]_x)$ is generated by elements of Γ. Lemma 3.4 tells us that $D([[\varphi]]_x)$ is the inductive limit of finite groups which are isomorphic to direct sums of alternating groups. In these alternating groups, every 3-cyclic permutation of the form $(i-1, i, i+1)$ corresponds to some γ_U. Thanks to Lemma 5.1, we get the conclusion. $\square$

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Lemma 5.3. *Let U and V be clopen subsets of X.*

(i) *If $\varphi^{-2}(V), \varphi^{-1}(V), V, \varphi(V)$ and $\varphi^2(V)$ are mutually disjoint and U is contained in V, then we have $\tau_V \gamma_U \tau_V^{-1} = \gamma_{\varphi(U)}$ and $\tau_V^{-1} \gamma_U \tau_V = \gamma_{\varphi^{-1}(U)}$.*

(ii) *If $\varphi^{-1}(U), U, \varphi(U) \cup \varphi^{-1}(V), V$ and $\varphi(V)$ are mutually disjoint, then we have $\gamma_V \gamma_U^{-1} \gamma_V^{-1} \gamma_U = \gamma_{\varphi(U) \cap \varphi^{-1}(V)}$.*

Proof. Straightforward computations. □

The following is the main theorem of this section. The reader, who is not familiar with the rudiments of the theory of subshifts, may refer to [LM].

Theorem 5.4. *Let (X, φ) be a Cantor minimal system. The commutator subgroup $D([[\varphi]]_0)$ is finitely generated if and only if (X, φ) is conjugate to a minimal subshift.*

Proof. The 'only if' part is easy to show. Let us assume that $D([[\varphi]]_0)$ is generated by finitely many elements $\gamma_1, \gamma_2, \ldots, \gamma_m$. Let $n_i : X \to \mathbb{Z}$ be the continuous function such that $\gamma_i(x) = \varphi^{n_i(x)}(x)$ for all $x \in X$. Then, $\mathcal{P}_i = \{n_i^{-1}(k)\}_{k \in \mathbb{Z}}$ is a finite clopen partition for each $i = 1, 2, \ldots, m$. Take a finite clopen partition $\mathcal{P}$ which is finer than all the $\mathcal{P}_i$'s. We equip $\mathcal{P}^{\mathbb{Z}}$ with the product topology and denote the shift map by σ. Define a continuous map π from X to $\mathcal{P}^{\mathbb{Z}}$ so that $\varphi^k(x) \in \pi(x)_k$. Then π is a factor map from (X, φ) to the minimal subshift $(\pi(X), \sigma)$. For each $i = 1, 2, \ldots, m$, we can define a homeomorphism $\tau_i \in \text{Homeo}(\pi(X))$ by $\tau_i(\xi) = \sigma^k(\xi)$ when $\xi_0 \subset n_i^{-1}(k)$. It is an easy observation that τ_i is an element of $[[\sigma]]$ and $\pi \gamma_i = \tau_i \pi$. We would like to prove that π is injective. Suppose that x and y are distinct points in X and $\pi(x) = \pi(y)$. There exists $\gamma \in D([[\varphi]]_0)$ such that $\gamma(x) \neq x$ and $\gamma(y) = y$. Since $D([[\varphi]]_0)$ is generated by $\gamma_1, \gamma_2, \ldots, \gamma_m$, we can find $r_1, r_2, \ldots, r_l \in \mathbb{Z}$ and $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, m\}$ such that $\gamma = \gamma_{i_1}^{r_1} \gamma_{i_2}^{r_2} \ldots \gamma_{i_l}^{r_l}$. This together with $\pi \gamma_i = \tau_i \pi$ shows that

$$\pi\gamma(x) = \pi \gamma_{i_1}^{r_1} \gamma_{i_2}^{r_2} \ldots \gamma_{i_l}^{r_l}(x) = \tau_{i_1}^{r_1} \tau_{i_2}^{r_2} \ldots \tau_{i_l}^{r_l} \pi(x)$$
$$= \tau_{i_1}^{r_1} \tau_{i_2}^{r_2} \ldots \tau_{i_l}^{r_l} \pi(y) = \pi \gamma_{i_1}^{r_1} \gamma_{i_2}^{r_2} \ldots \gamma_{i_l}^{r_l}(y)$$
$$= \pi\gamma(y) = \pi(y) = \pi(x),$$

which means $\sigma^k \pi(x) = \pi \varphi^k(x) = \pi(x)$ for some nonzero $k \in \mathbb{Z}$. This is a contradiction because σ is minimal.

Let us prove the 'if' part. Suppose that (X, φ) is conjugate to a minimal subshift. We may assume that X is a shift invariant closed subset of $A^{\mathbb{Z}}$, where A is a finite set. We may further assume that $\xi_i \neq \xi_j$ for all $\xi \in X$ and $i, j \in \mathbb{Z}$ with $|i - j| \leq 4$. We define the cylinder sets by

$$[a_{-m} \ldots a_{-1} a_0 . a_1 \ldots a_n] = \{\xi \in X : \xi_i = a_i \text{ for all } i = -m, \ldots, n-1, n\}$$

for $n, m \in \mathbb{N} \cup \{0\}$ and $a_{-m}, \ldots, a_n \in A$. Note that $\varphi^{-2}(U), \varphi^{-1}(U), U, \varphi(U)$ and $\varphi^2(U)$ are mutually disjoint, if U is a cylinder set. Since every clopen set can be written as a disjoint union of these cylinder sets, it suffices to show that γ_U can be generated for every cylinder set U by virtue of Lemma 5.2. Put

$$F = \{\gamma_U : U = [ab.c], \ a, b, c \in A\}.$$

We claim that F generates $D([[\varphi]]_0)$. It is not hard to see that $\tau_{[a.]}$ can be generated for all $a \in A$, because $\gamma_{\varphi^{-1}([a.])}$ and $\gamma_{\varphi([a.])}$ are generated. When U is a cylinder set $[a_{-m} \ldots a_0 . \ldots a_n]$, by Lemma 5.3 (i), it is easily verified that

$$\tau_{[a_0.]} \gamma_U \tau_{[a_0.]}^{-1} = \gamma_{\varphi(U)}$$

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and

$$\tau_{[a_0.]}^{-1} \gamma_U \tau_{[a_0.]} = \gamma_{\varphi^{-1}(U)}.$$

Therefore, it suffices to show that γ_U can be generated for every cylinder set $[a_{-m} \dots a_0.a_1]$. This is inductively done by using Lemma 5.3 (ii) with $U = [a_{-m} \dots a_0.a_1]$ and $V = [a_1a_2.]$. □

This result allows us to conclude the following.

Corollary 5.5. *When (X, φ) is a Cantor minimal system, the following are equivalent.*

   (i) *(X, φ) is a minimal subshift and $K^0(X, \varphi) \otimes \mathbb{Z}_2$ is a finite group.*

   (ii) *$[[\varphi]]_0$ is finitely generated.*

   (iii) *$[[\varphi]]$ is finitely generated.*

Proof. (i)⇒(ii). It follows from Theorem 4.8 and Theorem 5.4.

   (ii)⇒(iii). It follows from $[[\varphi]]/[[\varphi]]_0 \cong \mathbb{Z}$.

   (iii)⇒(i). It is clear that $[[\varphi]]/D([[\varphi]]_0) \cong \mathbb{Z} \oplus (K^0(X, \varphi) \otimes \mathbb{Z}_2)$ should be finitely generated. By the same proof as the 'only if' part of Theorem 5.4, we can conclude that (X, φ) is a minimal subshift. □

There exist uncountably many minimal subshifts (X, φ) whose K^0-group $K^0(X, \varphi)$ is 2-divisible by [Su, Theorem 1.1]. Combining Theorem 2.8 (iii), Corollary 4.10 and the result obtained above, we get uncountably many finitely generated simple groups.

We would like to show that the topological full group is never finitely presented. In the following lemma, topological entropy is denoted by $h(\cdot)$. A dynamical system (X, φ) is said to be embeddable into another dynamical system (Y, ψ), if there exists a continuous injection $\pi : X \to Y$ with $\pi\varphi = \psi\pi$.

Lemma 5.6. *Let (X, φ) be a minimal subshift. Let (Y, ψ) be an irreducible subshift of finite type and let $p \in \mathbb{N}$ be its period. Then (X, φ) can be embedded into (Y, ψ) if and only if $h(\varphi) < h(\psi)$ and $[1_X]$ is divisible by p in $K^0(X, \varphi)$.*

Proof. When p is one, this is a direct consequence of Corollary 10.1.9 of [LM]. Let $p \geq 2$. There exists a clopen ψ^p-invariant subset U of Y and Y is the disjoint union of $U, \psi(U), \dots, \psi^{p-1}(U)$.

Suppose that (X, φ) is embedded into (Y, ψ). Evidently we have $h(\varphi) < h(\psi)$ (see [LM, Corollary 4.4.9]). Define $V = U \cap X$. Then $[1_X]$ is equal to $p[1_V]$.

Let us prove the converse. We can find a clopen subset $V \subset X$ such that X is the disjoint union of $V, \varphi(V), \dots, \varphi^{p-1}(V)$, because $[1_X]$ is divisible by p in $K^0(X, \varphi)$. Then (V, φ^p) is a minimal subshift. Since (U, ψ^p) is a mixing shift of finite type and $h(\varphi^p) = ph(\varphi) < ph(\psi) = h(\psi^p)$, by Corollary 10.1.9 of [LM], (V, φ^p) is embedded into (U, ψ^p). Therefore (X, φ) is embedded into (Y, ψ). □

In the proof of the following theorem, we use the fact that the commutator subgroup of the topological full group is a complete invariant for flip conjugacy. This fact was mentioned in [Sk2, Theorem 4.11] without proof.

Theorem 5.7. *Let (X, φ) be a minimal subshift. Then $D([[\varphi]]_0)$ is never finitely presented.*

Proof. Suppose that X is a subshift in $A^{\mathbb{Z}}$, where A is a finite set. We will freely use the notation of Theorem 5.4. We may assume that X is contained in

$$Y \;=\; \{\xi \in A^{\mathbb{Z}} : \xi_i \neq \xi_j \text{ for all } i, j \in \mathbb{Z} \text{ with } |i - j| \leq 4\}.$$

Notice that Y is a subshift of finite type. To avoid the ambiguity, denote the shift map on Y by φ and its restriction to a shift invariant subspace Z by $\varphi|_Z$. We define the cylinder sets as clopen subsets of Y. If U is a clopen set of Y and U, $\varphi(U)$, $\varphi^2(U)$ are disjoint, then we define $\gamma_U \in \mathrm{Homeo}(Y)$ in the same way as described before. Put

$$F \;=\; \{\gamma_U \in \mathrm{Homeo}(Y) : U = [ab.c], \; a, b, c \in A\}.$$

The proof is by contradiction. Suppose that $D([[\varphi|_X]]_0)$ is finitely presented. According to the proof of Theorem 5.4, $D([[\varphi|_X]]_0)$ is generated by $\{\gamma|_X : \gamma \in F\}$, and moreover, if Z is another minimal subshift in Y, then $D([[\varphi|_Z]]_0)$ is also generated by $\{\gamma|_Z : \gamma \in F\}$. By [H, V.2], $D([[\varphi|_X]]_0)$ may be assumed to have a finite presentation $\langle (\gamma|_X)_{\gamma \in F}, (r_i)_{i \in I} \rangle$, where I is a finite set and r_i is an element of the free group over F. Let $\tilde{r}_i$ denote the canonical image of r_i in $\mathrm{Homeo}(Y)$. It is not hard to see that the fixed point set $\{y \in Y : \tilde{r}_i(y) = y\}$ is clopen in Y for all $i \in I$. Since $\tilde{r}_i|_X = \mathrm{id}$, the clopen subset

$$Y_0 \;=\; \{y \in Y : \tilde{r}_i(y) = y \text{ for all } i \in I\}$$

contains X. Hence X is a shift invariant closed subset of $Y_1 = \bigcap_{n \in \mathbb{Z}} \varphi^n(Y_0)$, which is also a shift of finite type. By taking the irreducible component containing X, we may further assume that Y_1 is irreducible. Let Z be any minimal subshift in Y_1. Then $D([[\varphi|_Z]]_0)$ is a quotient of $D([[\varphi|_X]]_0) = \langle (\gamma|_X)_{\gamma \in F}, (r_i)_{i \in I} \rangle$, because $\tilde{r}_i|_Z = \mathrm{id}$. Therefore we have $D([[\varphi|_Z]]_0) \cong D([[\varphi|_X]]_0)$ by Theorem 4.9, and so $\varphi|_Z$ is flip conjugate to $\varphi|_X$. This contradicts the lemma above. $\square$

Corollary 5.8. *Let (X, φ) be a minimal subshift. Then $[[\varphi]]$ and $[[\varphi]]_0$ are never finitely presented.*

Proof. If $K^0(X, \varphi)/2K^0(X, \varphi)$ is infinite, we have nothing to do because of Corollary 5.5. Suppose that $K^0(X, \varphi)/2K^0(X, \varphi)$ is finite. Combining the theorem above and V.5.(i) of [H], we see that $[[\varphi]]_0$ is not finitely presented. As for $[[\varphi]]$, we get the conclusion from [H, V.15]. $\square$

6 Examples

Example 6.1. We will consider an example of Cantor minimal systems represented by a stationary ordered Bratteli diagram, in other words, minimal subshifts arising from substitution rules. We refer the reader to [DHS] for the details of these topics.

Let $B = (V, E)$ be the stationary Bratteli diagram given by the following figure, that is, every vertex set V_n except for V_0 consists of two vertices, every edge set E_n except for E_1 is a copy of the finite set $\{a, b, c, \ldots, h\}$, and the range and source map are defined in the same way at every level.
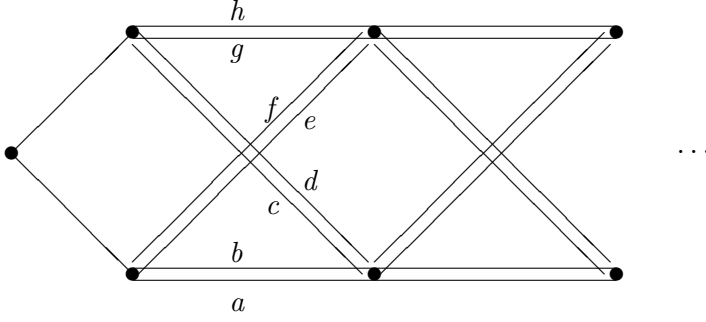
We give an ordering on E_n for $n \geq 2$ by $a < b < c < d$ and $e < g < f < h$. Then $B = (V, E)$ is a stationary ordered Bratteli diagram with the unique minimal path $(a, a, \dots)$ and the unique maximal path $(h, h, \dots)$. Let (X, φ) be the associated Cantor minimal system. This Cantor minimal system (X, φ) was presented in the final section of [Sk1] by Skau in order to show that there are two topologically orbit equivalent dynamical systems that are not conjugate. Actually, (X, φ) is strong orbit equivalent to the odometer system of type 2^∞, because $K^0(X, \varphi)$ is isomorphic to $\mathbb{Z}[1/2]$ (see [GPS1, Theorem 2.1]). But (X, φ) is not isomorphic to the odometer system.

Since $K^0(X, \varphi) \cong \mathbb{Z}[1/2]$ is 2-divisible, $[[\varphi]]_0$ is a simple group by Corollary 4.10. Moreover, Proposition 16 of [DHS] tells us that (X, φ) is isomorphic to the minimal subshift arising from the substitution rule $0 \mapsto 0011$ and $1 \mapsto 0101$, and so $[[\varphi]]_0$ is a finitely generated group by Corollary 5.5.

To specify generators of $[[\varphi]]_0$, we define a clopen set $U(a)$ in the infinite path space X by

$$ U(a) \;=\; \{(e_n)_{n \in \mathbb{N}} \in X : e_2 = a\} $$

and put

$$ \sigma_a(x) \;=\; \begin{cases} \varphi(x) & x \in U(a) \\ \varphi^{-1}(x) & x \in \varphi(U(a)) \\ x & \text{otherwise.} \end{cases} $$

Clopen sets $U(b), U(c), \dots, U(h)$ and homeomorphisms $\sigma_b, \sigma_c, \dots, \sigma_h$ are defined in a similar fashion. Let $\sigma_1 = \sigma_a \sigma_e$, $\sigma_2 = \sigma_b \sigma_g$, $\sigma_3 = \sigma_c \sigma_f$ and $\sigma_4 = \sigma_d \sigma_h$.

We infer that $[[\varphi]]_0$ is generated by four elements $\sigma_1, \sigma_1 \sigma_2 \sigma_3, \sigma_4$ and σ_d. In order to check it, let G be the subgroup generated by this four elements. By the equations

$$ (\sigma_1 \sigma_2 \sigma_3) \sigma_1 (\sigma_1 \sigma_2 \sigma_3)^{-1} = \sigma_2, \qquad (\sigma_1 \sigma_2 \sigma_3) \sigma_2 (\sigma_1 \sigma_2 \sigma_3)^{-1} = \sigma_3, $$
$$ (\sigma_2 \sigma_3 \sigma_4)^{-1} \sigma_d (\sigma_2 \sigma_3 \sigma_4) = \sigma_c, \qquad (\sigma_2 \sigma_3 \sigma_4)^{-1} \sigma_c (\sigma_2 \sigma_3 \sigma_4) = \sigma_b, $$
$$ (\sigma_1 \sigma_2 \sigma_3)^{-1} \sigma_b (\sigma_2 \sigma_3 \sigma_4) = \sigma_a, \qquad \sigma_e = \sigma_a \sigma_1, \sigma_g = \sigma_b \sigma_2, \sigma_f = \sigma_c \sigma_3, \sigma_h = \sigma_d \sigma_4, $$

we get $\sigma_a, \sigma_b, \dots, \sigma_h \in G$. Besides, by $\sigma_a \sigma_4 \sigma_a \sigma_4 = \gamma_{U(a)}$, we have $\gamma_{U(a)} \in G$. In the same way $\gamma_{U(b)}, \gamma_{U(c)}, \dots, \gamma_{U(h)}$ are also obtained. As shown in Lemma 5.2, to prove $G = [[\varphi]]_0$, it suffices to show that every γ_U is contained in G. In order to get γ_U for a smaller clopen set U, we use the similar method to the proof of Theorem 5.4. For example, $\gamma_{U(a) \cap \varphi(U(d))}$ is obtained as the commutator $\gamma_{U(b)} \sigma_d \gamma_{U(b)}^{-1} \sigma_d$. We leave the details to the reader.

Evidently, the topological full group $[[\varphi]]$ is generated by $[[\varphi]]_0$ and φ. Note that $[[\varphi]]$ has three generators φ, σ_1 and σ_d, because we have $\varphi^i \sigma_1 \varphi^{-i} = \sigma_{i+1}$ for $i = 1, 2, 3$.

Example 6.2. Let $\alpha \in [0, 1)$ be an irrational number and let R_α be the translation on $[0, 1)$ defined by $R_\alpha(t) = t + \alpha \pmod{1}$. We define the map $\pi : [0, 1) \to \{0, 1\}$ by $\pi(t) = 0$ if $t \in [0, \alpha)$

and $\pi(t) = 1$ otherwise. Let X be the closure of $\{(\pi(R_\alpha^n(t)))_{n\in\mathbb{Z}} : t \in [0,1)\}$ in $\{0,1\}^{\mathbb{Z}}$ and let φ be the two-sided shift map on X. The dynamical system (X, φ) is called a Sturmian shift arising from the α-rotation R_α on $[0,1)$, and known to be a minimal subshift. This class of shifts was introduced by Morse and Hedlund in 1940 and has been studied by several authors. On this subject the reader can consult Section 13.7 of [LM].

We will borrow the notation from the proof of Theorem 5.4. Let $U = [0.]$ and let μ be a unique φ-invariant probability measure on X. Then $\mu(U) = \alpha$. Since (X, φ) is a kind of Denjoy system, we know that its K^0-group $K^0(X, \varphi)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ by [PSS, Theorem 5.3]. The integration by μ gives the isomorphism from $K^0(X, \varphi)$ to $\mathbb{Z} + \mathbb{Z}\alpha \subset \mathbb{R}$, and $[1_X]$ and $[1_U]$ are generators of $K^0(X, \varphi)$. By Theorem 4.8 and 4.9, the commutator subgroup $D([[\varphi]]_0)$ is simple and $[[\varphi]]_0/D([[\varphi]]_0)$ is isomorphic to $K^0(X, \varphi)/2K^0(X, \varphi) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Furthermore, $[[\varphi]]$ is finitely generated by Corollary 5.5.

We will verify that $[[\varphi]]$ is generated by three elements. (The number of generators is at least three, because $[[\varphi]]/D([[\varphi]]) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is of rank three.) A Sturmian shift arising from the $(-\alpha)$-rotation is flip conjugate to (X, φ), and so we may assume that α is less than one half without loss of generality. Then U and $\varphi(U)$ are disjoint. Let n be the maximum natural number satisfying $(n+1)\alpha < 1$. If 0 appears in a two-sided sequence $x \in X$, then at least n 1's follow it. We denote consecutive n 1's by 1^n simply. When we put

$$V = [0 \overbrace{11\ldots 1}^{n} 1.] = [01^n 1.],$$

it is easily seen that V and $\varphi(V)$ are disjoint, and $\mu(V) = 1 - (n+1)\alpha$. Define $\sigma_U, \sigma_V \in [[\varphi]]_0$ by

$$\sigma_U(x) = \begin{cases} \varphi(x) & x \in U \\ \varphi^{-1}(x) & x \in \varphi(U) \\ x & \text{otherwise} \end{cases}$$

and

$$\sigma_V(x) = \begin{cases} \varphi(x) & x \in V \\ \varphi^{-1}(x) & x \in \varphi(V) \\ x & \text{otherwise.} \end{cases}$$

Then the two elements $\mathrm{sgn}(\sigma_U) = [1_U] + 2K^0(X, \varphi)$ and $\mathrm{sgn}(\sigma_V) = [1_V] + 2K^0(X, \varphi)$ generate $K^0(X, \varphi)/2K^0(X, \varphi) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We claim that $[[\varphi]]$ is generated by σ_U, σ_V and φ. Let us consider the case $n \geq 2$ at first. Taking the commutator of σ_U and $\varphi \sigma_U \varphi^{-1}$, we have $\gamma_{\varphi(U)}$. The equations

$$\varphi^{n-1} \gamma_{\varphi(U)} \varphi^{1-n} = \gamma_{\varphi^n(U)} = \gamma_{[01^n.]}$$
$$\sigma_U \gamma_{[01^n.]}^{-1} \sigma_U \gamma_{[01^n.]} = \gamma_{[01^n 0.]}$$

and

$$\sigma_V \gamma_{[01^n.]}^{-1} \sigma_V \gamma_{[01^n.]} = \gamma_{[01^n 1.0]}$$

are easily checked. Conjugating $\gamma_{[01^n 0.]}$ and $\gamma_{[01^n 1.0]}$ by some power of φ, we get $\gamma_{[01^n 01^n.]}$ and $\gamma_{[01^n 101^n.]}$. By repeating this procedure, for every cylinder set W, γ_W will be obtained. Thereby $[[\varphi]]$ is generated. In the case $n = 1$ we can do the same argument, starting from $\sigma_U \sigma_V \sigma_U \sigma_V = \gamma_{[0110.]}$.

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