Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems

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Abstract

We show that every minimal action of any finitely generated abelian group on the Cantor set is (topologically) orbit equivalent to an AF relation. As a consequence, this extends the classification up to orbit equivalence of minimal dynamical systems on the Cantor set to include AF relations and $\mathbb{Z}^d$-actions.

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1 Introduction

In this paper we continue the study, undertaken in [GPS1], [GPS3], [M1], [M2] and [GMPS1], of the orbit equivalence of minimal dynamical systems on a Cantor set X, i.e. a compact, totally disconnected, metrizable space with no isolated points. By a dynamical system, we mean to include actions of countable groups as well as étale equivalence relations (we may call an equivalence relation just a relation). We recall the definition of étale equivalence relations (see [R, GPS2] for more information).

Definition 1.1. An equivalence relation R on X endowed with a topology $\mathcal{O}$ is said to be étale if $(R, \mathcal{O})$ is a locally compact Hausdorff r-discrete groupoid. In other words, $(R, \mathcal{O})$ is a locally compact Hausdorff groupoid, and the two canonical projections from R to X are open and are local homeomorphisms.

For an action φ of a countable discrete group G on X by homeomorphisms, the orbit relation R_φ is defined by

$$R_\varphi = \{(x, \varphi^g(x)) \in X \times X \mid x \in X,\ g \in G\},$$

and if φ is free (i.e. $\{g \in G \mid \varphi^g(x){=}x\} = \{e\}$ for all $x \in X$) then R_φ is étale with the topology obtained by transferring the product topology on $X \times G$. Another rich and tractable class of étale equivalence relations is the so-called AF equivalence relations, which have played an important rôle in earlier work and will do so in this paper as well. Briefly, an étale equivalence relation R is AF (or approximately finite) if it can be written as an increasing union of compact open subrelations. Such relations have a nice presentation by means of a combinatorial object called a Bratteli diagram (see [GPS1] for more information). Recall also that an equivalence relation R on X (even without R having any topology itself) is minimal if $R[x] = \{y \in X \mid (x, y) \in R\}$ is dense in X for any $x \in X$.

As already mentioned our primary interest is the notion of orbit equivalence. We recall the following definition which generalizes the usual definition for group actions.

Definition 1.2. Let X and X' be two topological spaces and let R and R' be equivalence relations on X and X', respectively. We say that (X, R) and (X', R') are orbit equivalent if there is a homeomorphism $h : X \to X'$ such that $(h \times h)(R) = R'$.

In measurable dynamics, the study of orbit equivalence, initiated by Dye [D], was developed by Krieger [K], Ornstein-Weiss [OW] and Connes-Feldman-Weiss [CFW] among many others in the amenable case. The strategy of their proofs consisted of showing that any amenable measurable equivalence relation is orbit equivalent to a hyperfinite measurable equivalence relation and classifying these ones. In the non-singular case, the complete invariant of orbit equivalence is an ergodic flow, the so-called associated flow ([K]). There has also been considerable work done with the same strategy in the category of Borel equivalence relations. Hyperfiniteness of Borel $\mathbb{Z}^n$-actions was proved by Weiss. More generally, Jackson-Kechris-Louveau [JKL] showed hyperfiniteness of Borel actions of any finitely generated groups with polynomial growth. Recently, it was also proved that Borel actions of any countable abelian groups are hyperfinite ([GJ]). Complete classification of hyperfinite Borel equivalence relations up to orbit equivalence was given in [DJK] (see also [KM]).

Our strategy in the topological case is similar:

1) Provide an invariant of (topological) orbit equivalence, which is an ordered abelian group whose definition is given in Section 2.

2) Show that this invariant is complete for minimal AF relations on the Cantor set. This classification was obtained in [GPS1] (see [P] for a new proof).

3) Prove that more general minimal equivalence relations are *affable* i.e. orbit equivalent to an AF equivalence relation. A fortiori, this extends the classification to a larger family of equivalence relations. A key technical ingredient for this step is the absorption theorem for minimal AF relations, which states that a 'small' extension of a minimal AF relation is orbit equivalent to the AF relation.

The main contribution of this paper is to establish such a result for minimal actions of finitely generated abelian groups on a Cantor set.

Let φ be a free minimal action of $\mathbb{Z}^d$ as homeomorphisms on the Cantor set X and let R_φ be the associated étale equivalence relation. Notice that if $\{\mathcal{T}(x) \mid x \in X\}$ is a family of tessellations of $\mathbb{R}^d$ with compact cells and such that $\mathcal{T}(\varphi^n(x)) = \mathcal{T}(x)+n$ for $n \in \mathbb{Z}^d$, then we can associate to it a finite subrelation $R_\mathcal{T}$ of R_φ by stating that

$$(\varphi^p(x), \varphi^q(x)) \in R_\mathcal{T} \text{ if and only if } p, q \text{ belongs to the same cell of } \mathcal{T}(x).$$

Therefore the first step of the proof of the affability of R_φ is to construct a family of nested sequences $\{\mathcal{T}_l(x) \mid x \in X\}_{l\geq 1}$ of tessellations of $\mathbb{R}^d$ such that the associated sequence $(R_{\mathcal{T}_l})_{l\geq 1}$ of finite equivalence relations defines a minimal AF subrelation of R_φ satisfying the assumptions of the absorption theorem ([M3, Theorem 3.2]).

For the case $d = 2$, this construction (done in [GMPS2]) involved a precise control on the geometry of the cells of the tessellations. More precisely, given a Delaunay set P (i.e. a both uniformly discrete and relatively dense subset) of $\mathbb{R}^2$, we had to modify the Voronoi tessellation associated to P to ensure that the disjoint cells of the tessellation are well separated.

For $d \geq 3$, three new issues arise:

a) the first one is the geometry of the tessellations. The argument used to modify the Voronoi tessellations in dimension two does not work for $d \geq 3$. For example, given a Delaunay subset P of $\mathbb{R}^3$, its Delaunay triangulation may contain slivers. These are tetrahedron whose four vertices lie close to a plane and whose projection to the plane is a convex quadrilateral with no short edge. In [CDEFT], Cheng, Dey, Edelsbrunner, Facello and Teng proved the existence of a triangulation whose vertices are P and with no slivers. More precisely, they showed that there exists an assignment of weights so that the weighted Delaunay triangulation (defined in Section 3) contains no slivers. We generalize their argument to triangulations of $\mathbb{R}^d$ for any $d > 2$ in Section 3 and 4.

b) the second one is the combinatorics related to the nesting of the tessellations which is substantially more complicated being a lot less geometrically intuitive. This construction of families of nested sequences of tessellations is done in Section 6.

c) the third one is the application of the absorption theorem. The first version of this theorem was stated and shown in [GPS2]. In [GMPS2], it was generalized to the case that the 'small' extension is given by a compact relation transverse to the AF equivalence relation. This version was sufficient for the study of $\mathbb{Z}^2$-actions, but too restricted for successive applications of the theorem. Building on the idea of [GMPS2], the absorption theorem was further generalized in [M3] and it is this last version which is used inductively in Section 7 of this paper.

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2 Main results

Let Ω be a compact metrizable space and let G be a locally compact group. A group homomorphism $\varphi : G \to \mathrm{Homeo}(\Omega)$ is called an action, if $G \times \Omega \ni (g, x) \mapsto \varphi^g(x) \in G$ is continuous. The action φ is said to be free, if $\varphi^g(x) \neq x$ for any $x \in X$ and $g \in G \setminus \{e\}$, where e is the neutral element of G. The action φ is said to be minimal, if $\{\varphi^g(x) \in X \mid g \in G\}$ is dense in X for every $x \in X$.

We denote by $\mathbb{R}^d$ the usual d-dimensional Euclidean space. For $p \in \mathbb{R}^d$ and $r > 0$, $B(p, r)$ denotes the open ball of radius r centred at p.

Definition 2.1. Let $d \in \mathbb{N}$. Let φ be a free action of $\mathbb{R}^d$ on a compact metrizable space Ω. We call a closed subset $X \subset \Omega$ a flat Cantor transversal, if the following are satisfied.

(1) X is homeomorphic to a Cantor set.

(2) For any $x \in \Omega$, $\varphi^p(x)$ is in X for some $p \in \mathbb{R}^d$.

(3) There exists a positive real number $M > 0$ such that

$$C = \{\varphi^p(x) \mid x \in X, p \in B(0, M)\}$$

is open in Ω and

$$X \times B(0, M) \ni (x, p) \mapsto \varphi^p(x) \in C$$

is a homeomorphism.

(4) For any $x \in X$ and $r > 0$, there exists an open neighbourhood $U \subset X$ of x in X such that $\{p \in B(0,r) \mid \varphi^p(x) \in X\} = \{p \in B(0,r) \mid \varphi^p(y) \in X\}$ for all $y \in U$.

We note that the third property means that Ω is locally homeomorphic to the product of X and $\mathbb{R}^d$. If φ is minimal, then (2) follows from (3) automatically.

When $X \subset \Omega$ is a flat Cantor transversal, we define an equivalence relation R_φ on X by

$$R_\varphi = \{(x, \varphi^p(x)) \mid x \in X, \varphi^p(x) \in X, p \in \mathbb{R}^d\}.$$

If φ is a minimal action on Ω, then R_φ is a minimal equivalence relation on X. Indeed, for any $x \in X$ and a non-empty open subset $U \subset X$, (3) implies that $\{\varphi^q(y) \mid y \in U, q \in B(0,M)\}$ is open in Ω and the minimality of φ implies that $\varphi^p(x)$ belongs to this open set for some $p \in \mathbb{R}^d$, and so $\varphi^{p-q}(x) \in U$ for some $q \in B(0,M)$. We provide R_φ with a topology whose basis is given by the sets of the form $\{(x, \varphi^p(x)) \mid x \in U\} \cap R_\varphi$, where U is an open subset of X and p is in $\mathbb{R}^d$. It is not hard to see that R_φ is an étale equivalence relation with this topology.

We have two important classes of examples of flat Cantor transversals.

Example 2.2. Let X be a Cantor set and let φ be a free minimal action of $\mathbb{Z}^d$ on X. Let Ω denote the quotient space of $X \times \mathbb{R}^d$ by the equivalence relation $\{((x,p),(\varphi^n(x), p+n)) \mid x \in X, p \in \mathbb{R}^d, n \in \mathbb{Z}^d\}$. The topological space Ω is called a suspension space of (X, φ). There exists a natural $\mathbb{R}^d$-action on Ω induced by the translation $(x,p) \mapsto (x, p+q)$ in $X \times \mathbb{R}^d$. We denote this action by $\tilde{\varphi}$. Clearly $\tilde{\varphi}$ is free and minimal and it is easy to see that $X \times \{0\}$ is a flat Cantor transversal for $(\Omega, \tilde{\varphi})$. In addition, by identifying $X \times \{0\}$ with X, the étale equivalence relation $R_{\tilde{\varphi}}$ agrees with the étale equivalence relation arising from (X, φ). We also remark that the $\mathbb{R}^d$-action $\tilde{\varphi}$ is a special case of the so-called Mackey action (see Section 4 of [Mac]).

Example 2.3. We would like to explain equivalence relations arising from tiling spaces briefly. The reader should see [KP] and the references given there for more details and [BBG, BG] for further developments. We follow the notation used in [KP]. Let T be a tiling of $\mathbb{R}^d$ satisfying the finite pattern condition and let Ω_T be the continuous hull of T. The topological space Ω_T is compact and metrizable. There exists a natural $\mathbb{R}^d$-action φ on Ω_T given by translation. Assume further that T is aperiodic and repetitive. Then the action φ is free and minimal. Let $\Omega_{\mathrm{punc}} \subset \Omega_T$ be as in Definition 5.1 of [KP]. It is easy to check that Ω_{punc} is a flat Cantor transversal. In addition, the étale equivalence relation R_φ defined above agrees with R_{punc} of [KP].

The following is the main theorem of this paper.

Theorem 2.4. *Let $X \subset \Omega$ be a flat Cantor transversal for a free minimal action φ of $\mathbb{R}^d$ on Ω and let $R_\varphi \subset X \times X$ be the minimal equivalence relation induced from (Ω, φ). Then R_φ is affable.*

The proof is quite long and we defer it until the last section of the paper.

Let us recall the algebraic invariant associated to an equivalence relation on the Cantor set X and add a remark on minimal actions of finitely generated abelian groups on X.

Let R be an equivalence relation on a Cantor set X. We say that a Borel probability measure μ on X is R-invariant, if μ is γ-invariant for any $\gamma \in \mathrm{Homeo}(X)$ satisfying $(x, \gamma(x)) \in R$ for all $x \in X$. We let $M(X, R)$ denote the set of R-invariant probability measures on X. This is a weak$*$ compact convex set, in fact, a non-empty Choquet simplex whenever R arises from a free action of an amenable group. We say that (X, R) is uniquely ergodic, if the set $M(X, R)$ has exactly one element. We let $C(X, \mathbb{Z})$ denote the set of all continuous integer-valued functions on X. It is an abelian group with the operation of pointwise addition. The quotient group of $C(X, \mathbb{Z})$ by the subgroup

$$\left\{ f \in C(X, \mathbb{Z}) \mid \int_X f \, d\mu = 0 \text{ for all } \mu \in M(X, R) \right\}$$

is denoted by $D_m(X, R)$. For a function f in $C(X, \mathbb{Z})$, we denote its class in $D_m(X, R)$ by $[f]$. Of course, this is a countable abelian group, but it is also given an order structure [GPS1] by defining the positive cone $D_m(X, R)^+$ as the set of all $[f]$, where $f \geq 0$. It also has a distinguished positive element $[1]$, where 1 denotes the constant function with value 1. Our invariant is the triple $(D_m(X, R), D_m(X, R)^+, [1])$. This ordered abelian group first appeared in [GPS1] in the case of minimal $\mathbb{Z}$-actions as the quotient of a K-group by its subgroup of infinitesimal elements and was shown to be a simple dimension group. If a homeomorphism $h : X_1 \to X_2$ induces an orbit equivalence between R_1 on X_1 and R_2 on X_2, then clearly we have $h_*(M(X_1, R_1)) = M(X_2, R_2)$. Hence h induces an isomorphism between the two triples $(D_m(X_1, R_1), D_m(X_1, R_1)^+, [1])$ and $(D_m(X_2, R_2), D_m(X_2, R_2)^+, [1])$.

Let us add a remark about actions of finitely generated abelian groups. Let G be a finitely generated abelian group and let φ be a minimal action of G on a Cantor set X. It is easy to see that, for any $g \in G$, $X_g = \{x \in X \mid \varphi^g(x) = x\}$ is a G-invariant closed subset. By the minimality of φ, if X_g is not empty, it must be the whole of X. Letting H denote the set of all elements g for which $X_g = X$, the orbits of φ can be realized as the orbits of a free action of the quotient group G/H, which is also a finitely generated abelian group. Since finite groups cannot act minimally on an infinite space, G/H is isomorphic to $\mathbb{Z}^d \oplus K$, where $d \geq 1$ and K is a finite abelian group. In the Appendix, generalizing a result of O. Johansen ([J]), we show that any free minimal action of $\mathbb{Z}^d \oplus K$ on a Cantor set X is orbit equivalent to a free minimal action of $\mathbb{Z}^d$ (see Theorem A.1).

Then as an immediate consequence of Theorem 2.4 and [GPS1, Theorem 2.3], we have the following extension of [GMPS2, Theorem 1.6].

Theorem 2.5. *Let (X, R) and (X', R') be two minimal equivalence relations on Cantor sets which are either AF relations or arise from actions of finitely generated* *abelian groups or tiling spaces on $\mathbb{R}^d$. Then they are orbit equivalent if and only if*

$$(D_m(X,R), D_m(X,R)^+, [1]) \cong (D_m(X',R'), D_m(X',R')^+, [1]),$$

meaning that there is a group isomorphism between $D_m(X,R)$ and $D_m(X',R')$ which is a bijection between positive cones and preserves the class of 1.

As explained in [GMPS1], the range of the invariant D_m for minimal AF relations is precisely the collection of simple, acyclic dimension groups with no non-trivial infinitesimal elements, and exactly the same holds for minimal $\mathbb{Z}$-actions on Cantor sets (see [HPS] and [GPS1]). It follows from the theorem above that every minimal free $\mathbb{Z}^d$-action on a Cantor set is orbit equivalent to a $\mathbb{Z}$-action. But, we do not have an exact description of the range for minimal $\mathbb{Z}^d$-actions, when d is greater than one.

The following two corollaries are immediate consequences of the main theorem and the definitions.

Corollary 2.6. *Let (X,R) and (X',R') be two minimal equivalence relations on Cantor sets which are either AF relations or arise from actions of finitely generated abelian groups or tiling spaces on $\mathbb{R}^d$. Then they are orbit equivalent if and only if there exists a homeomorphism $h : X \to X'$ which implements a bijection between the sets $M(X,R)$ and $M(X',R')$.*

Corollary 2.7. *Let (X,R) and (X',R') be two minimal, uniquely ergodic equivalence relations on Cantor sets which are either AF relations or arise from actions of finitely generated abelian groups or tiling spaces on $\mathbb{R}^d$. Suppose that $M(X,R) = \{\mu\}$ and $M(X',R') = \{\mu'\}$. Then the two systems are orbit equivalent if and only if*

$$\{\mu(U) \mid U \text{ is a clopen subset of } X\} = \{\mu'(U') \mid U' \text{ is a clopen subset of } X'\}.$$

3 Weighted Delaunay triangulations

We will be constructing various tessellations of $\mathbb{R}^d$ in order to obtain a minimal AF subrelation to which the absorption theorem can be applied. A similar construction for Cantor minimal dynamical systems was first made by Forrest [F]. Using Voronoi tessellations, he showed that the equivalence relation arising from a free minimal $\mathbb{Z}^d$-action has a 'large' AF subrelation. As mentioned in the introduction, for our purposes, however, Voronoi tessellations have a serious drawback. We would like to know that disjoint cells should be separated in some controlled manner. In the case of $\mathbb{Z}^2$-actions, we could do this by moving the vertices of the Voronoi tessellation to the incentres of the triangles of its dual tessellation. But, as noted in [GMPS1], this argument does not work in the situation of $\mathbb{R}^d$ for $d > 2$.

The difficulty in the high dimensional space is to get rid of badly shaped tetrahedrons in the dual of the Voronoi tessellations. Such bad tetrahedrons are called

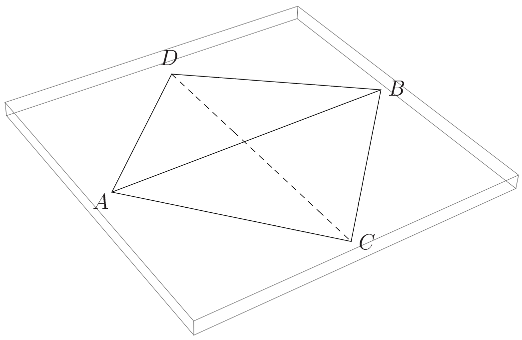

Figure 1: sliver in $\mathbb{R}^3$

slivers. For convenience, let us restrict our attention to $\mathbb{R}^3$ for a moment. A tetrahedron in $\mathbb{R}^3$ is called a sliver, if its four vertices lie close to a plane and its projection to the plane is a convex quadrilateral with no short edge (see Figure 1). In [CDEFT], using the idea of weighted Delaunay triangulations, they studied how to get rid of slivers. More precisely, it was shown that there exists an assignment of weights so that the weighted Delaunay triangulation contains no slivers. In this section we generalize their argument to triangulations of $\mathbb{R}^d$ for any $d > 2$. Note that the hypothesis used in [CDEFT] is weaker than the one needed in our setting, and so the result of [CDEFT] is not recovered from our result. The reader may refer to [Sch] for weighted Delaunay triangulations. Actually, in [Sch], the notions of weighted Voronoi and Delaunay tilings were unified by Laguerre tilings.

Let $d(\cdot, \cdot)$ denote the usual Euclidean metric on $\mathbb{R}^d$. For any non-empty set $A \subset \mathbb{R}^d$ and $p \in \mathbb{R}^d$, we let $d(p, A) = \inf\{d(p, q) \mid q \in A\}$ and $A+p = \{q+p \mid q \in A\}$. For a subset $A \subset \mathbb{R}^d$, the closed convex hull of A is written by $\mathrm{Conv}(A)$. A k-dimensional affine subspace of $\mathbb{R}^d$ is a translation of a k-dimensional vector subspace of $\mathbb{R}^d$. A tessellation of $\mathbb{R}^d$ is a collection of compact subsets of $\mathbb{R}^d$ which cover $\mathbb{R}^d$ with pairwise disjoint interiors. An element of a tessellation is called a cell. A tessellation is called a triangulation, if every cell of it is a d-simplex.

Let $P \subset \mathbb{R}^d$ be a countable subset. For a real number M, we say that P is M-separated if $d(p, q) \geq M$ for all $p \neq q \in P$ and we say that P is M-syndetic if $\bigcup_{p \in P} B(p, M) = \mathbb{R}^d$. The following lemma is an easy geometric exercise, but we include the proof for completeness.

Lemma 3.1. (1) *For any M-separated set $P \subset \mathbb{R}^d$, $p \in \mathbb{R}^d$ and $R > 0$,*

$$\#(P \cap B(p, R)) \leq (2R + M)^d / M^d.$$

(2) *For any M-syndetic set $P \subset \mathbb{R}^d$, $p \in \mathbb{R}^d$ and $R > M$,*

$$\#(P \cap B(p, R)) \geq (R - M)^d / M^d.$$

Proof. Let V_d be the d-dimensional volume of the unit ball in $\mathbb{R}^d$.

(1). Consider the balls $B(q, M/2)$ for all $q \in P \cap B(p, R)$. They are mutually disjoint and contained in $B(p, R + M/2)$. Hence one has

$$\#(P \cap B(p, R)) \times (M/2)^d V_d \leq (R + M/2)^d V_d,$$

and so $\#(P \cap B(p, R))$ is not greater than $(2R + M)^d / M^d$.

(2). Consider the balls $B(q, M)$ for all $q \in P \cap B(p, R)$. Clearly their union covers $B(p, R - M)$. Therefore we get

$$\#(P \cap B(p, R)) \times M^d V_d \geq (R - M)^d V_d,$$

and so $\#(P \cap B(p, R))$ is not less than $(R - M)^d / M^d$. $\square$

Let P be an M-separated and $2M$-syndetic subset of $\mathbb{R}^d$. For each p in P, let

$$V(p) = \{q \in \mathbb{R}^d \mid d(q, p) = d(q, P)\},$$

which is a d-polytope with p in its interior. The collection $\{V(p) \mid p \in P\}$ is a tessellation of $\mathbb{R}^d$ and called the Voronoi tessellation.

For $k = 1, 2, \ldots, d$, we denote by $\Delta_k(P)$ the set of all $k + 1$ distinct points in P which do not lie on a $(k - 1)$-dimensional affine subspace. In other words, $\{p_0, p_1, \ldots, p_k\} \subset P$ belongs to $\Delta_k(P)$ if and only if $\{p_1 - p_0, p_2 - p_0, \ldots, p_k - p_0\}$ is linearly independent. For $\tau \in \Delta_d(P)$, there exists a unique closed ball whose boundary contains τ. We call this closed ball the circumsphere of τ. Let $\mathrm{Del}(P)$ be the set of all $\tau \in \Delta_d(P)$ such that their circumsphere do not intersect with $P \setminus \tau$. Generically, $\{\mathrm{Conv}(\tau) \mid \tau \in \mathrm{Del}(P)\}$ gives a tessellation of $\mathbb{R}^d$, which is called the Delaunay triangulation. It is well known that the Delaunay triangulation is the dual of the Voronoi tessellation.

Next, we have to introduce the notion of weighted Delaunay triangulations. A pair of $p \in \mathbb{R}^d$ and a non-negative real number w is called a weighted point. A weighted point (p, w) may be thought as a sphere centred at p with radius $\sqrt{w}$. We say that two weighted points (p_1, w_1) and (p_2, w_2) are orthogonal if $d(p_1, p_2)^2 = w_1 + w_2$. Let $p_0, p_1, \cdots, p_k$ be $k + 1$ distinct points in $\mathbb{R}^d$ which do not lie on a $(k - 1)$-dimensional affine subspace, where $1 \leq k \leq d$. Let H be the k-dimensional affine subspace containing p_i's. Let $w_0, w_1, \cdots, w_k$ be non-negative real numbers such that the balls $B(p_i, \sqrt{w_i})$ are mutually disjoint. We can see that there exists a unique weighted point (z, u) such that $z \in H$ and (z, u) is orthogonal to (p_i, w_i) for all i in the following way. Let $\langle \cdot, \cdot \rangle$ denote the inner product of $\mathbb{R}^d$. There exist linearly independent vectors $q_1, q_2, \ldots, q_{d-k}$ satisfying $\langle p_i - p_0, q_j \rangle = 0$ for all i, j. Since $\{p_1 - p_0, p_2 - p_0, \ldots, p_k - p_0\} \cup \{q_1, q_2, \ldots, q_{d-k}\}$ are linearly independent, the system of linear equations

$$\langle p_i - p_0, z \rangle = \frac{1}{2}(\langle p_i, p_i \rangle - w_i - \langle p_0, p_0 \rangle + w_0), \ i = 1, 2, \ldots, k,$$

$$\langle q_j, z - p_0 \rangle = 0,\ \ j = 1, 2, \ldots, d - k,$$

has the unique solution z. Put $u = d(p_0, z)^2 - w_0$. Then (z, u) is orthogonal to (p_i, w_i) for all i and z is in H. Conversely, if $z \in H$ and (z, u) is orthogonal to (p_i, w_i), then z must satisfy the system of linear equations above. We call z and (z, u) the orthocentre and the orthosphere of (p_i, w_i), respectively. Usually the intersection of the three altitudes of a triangle is called the orthocentre, but we follow the terminology used in [CDEFT].

Suppose that P is an M-separated and $2M$-syndetic subset of $\mathbb{R}^d$ for some $M > 0$. Put

$$\mathcal{W} = \{w : P \to \mathbb{R} \mid 0 \leq w(p) \leq (M/3)^2\}.$$

An element $w \in \mathcal{W}$ is called a weight function on P.

Definition 3.2. (1) For $\tau = \{p_0, p_1, \ldots, p_k\} \in \Delta_k(P)$, we say that (z, u) is the orthosphere of τ with respect to w if (z, u) is the orthosphere of $(p_i, w(p_i))$.

(2) We say that (z, u) is empty in (P, w) if $d(z, p)^2 - u - w(p) \geq 0$ for all $p \in P$.

(3) We let $\mathrm{Del}(P, w)$ denote the set of all $\tau \in \Delta_d(P)$ such that the orthosphere of τ with respect to w is empty in (P, w).

(4) We let $\mathcal{Z}(P, w)$ be the set of all empty orthospheres of some τ in $\mathrm{Del}(P, w)$.

(5) For $(z, u) \in \mathcal{Z}(P, w)$, let $\delta(z, u)$ be the set of all $p \in P$ such that $d(z, p)^2 = u + w(p)$.

Notice that if the weight function w is constantly zero, $\mathrm{Del}(P, w)$ agrees with $\mathrm{Del}(P)$. The proof of the following can be found in [Sch]. We remark that the condition R1 in [Sch] is satisfied because P is separated and w is bounded, and that the condition R2 in [Sch] is satisfied because P is syndetic.

Lemma 3.3 ([Sch]). (1) *The collection* $\{\mathrm{Conv}(\delta(z, u)) \mid (z, u) \in \mathcal{Z}(P, w)\}$ *is a tessellation of* $\mathbb{R}^d$.

(2) *For each orthosphere* $(z, u) \in \mathcal{Z}(P, w)$, $\delta(z, u)$ *coincides with the extremal points of* $\mathrm{Conv}(\delta(z, u))$.

(3) *For any orthospheres* $(z, u), (z', u') \in \mathcal{Z}(P, w)$, *we have*

$$\mathrm{Conv}(\delta(z, u)) \cap \mathrm{Conv}(\delta(z', u')) = \mathrm{Conv}(\delta(z, u) \cap \delta(z', u')).$$

In other words, $\mathrm{Conv}(\delta(z, u))$ *and* $\mathrm{Conv}(\delta(z', u'))$ *meet face to face.*

If $\#\delta(z, u)$ is greater than $d + 1$, we can partition $\mathrm{Conv}(\delta(z, u))$ into d-simplices whose vertices are in $\delta(z, u)$ and whose interiors are pairwise disjoint. In other words, we can find a subset $\mathcal{D} \subset \mathrm{Del}(P, w)$ such that $\{\mathrm{Conv}(\tau) \mid \tau \in \mathcal{D}\}$ is a tessellation of $\mathbb{R}^d$. We call it the weighted Delaunay triangulation.

We begin with some basic properties of orthospheres. For $\tau \in \Delta_k(P)$, we write the k-dimensional affine subspace which contains τ by $H(\tau)$.

Lemma 3.4. *Let $w \in \mathcal{W}$ be a weight function on P. Suppose that $\tau \in \Delta_l(P)$ is contained in $\tau' \in \Delta_k(P)$, where $2 \leq l < k$. Let (z, u) and (z', u') be the orthospheres of τ and τ' with respect to w, respectively. Let π be the orthogonal projection from $H(\tau')$ to $H(\tau)$. Then we have $\pi(z') = z$ and $d(z', z)^2 = u' - u$. In particular, u is not greater than u'.*

Proof. We have $d(z', \pi(z'))^2 = d(z', p)^2 - d(\pi(z'), p)^2$ for all $p \in \tau$, because $z' - \pi(z')$ is orthogonal to $H(\tau)$. Combining this with $d(z', p)^2 = u' + w(p)$, we get $d(\pi(z'), p)^2 = u' - d(z', \pi(z'))^2 + w(p)$ for all $p \in \tau$. We claim that $u' - d(z', \pi(z'))^2$ is non-negative. If it were negative, then $d(\pi(z'), p) < \sqrt{w(p)}$ for all $p \in \tau$. Thus $\pi(z') \in B(p, \sqrt{w(p)})$. Since there are at least two such p, it does not happen because of $w \in \mathcal{W}$. It follows that $(\pi(z'), u' - d(z', \pi(z'))^2)$ is an orthosphere of $(p, w(p))$ for $p \in \tau$. By the uniqueness of the orthosphere, we get $\pi(z') = z$ and $u' - d(z', \pi(z'))^2 = u$, which completes the proof. □

Lemma 3.5. *Let $w \in \mathcal{W}$ and $\tau = \{p_0, p_1, \ldots, p_d\} \in \mathrm{Del}(P, w)$. Let (z, u) be the orthosphere of τ with respect to w. Then u is less than $4M^2$ and $d(z, p_i)^2 < 5M^2$. In particular, $d(p_i, p_j)$ is less than $2\sqrt{5}M$ for any $i, j = 0, 1, \ldots, d$.*

Proof. Since (z, u) is empty in (P, w), $d(z, p)^2 - u - w(p) \geq 0$ for all $p \in P$. It follows that $B(z, \sqrt{u})$ does not meet P, and so $\sqrt{u}$ is less than $2M$, because P is $2M$-syndetic. For any $i = 0, 1, \ldots, d$, (z, u) is orthogonal to $(p_i, w(p_i))$, that is, $d(z, p_i)^2 = u + w(p_i)$. Therefore $d(z, p_i)^2 < 5M^2$. From this we have $d(p_i, p_j) \leq d(p_i, z) + d(z, p_j) < 2\sqrt{5}M$. □

For each $k = 1, 2, \ldots, d$, we let D_k denote the set of all $\tau \in \Delta_k(P)$ satisfying that there exist $w \in \mathcal{W}$ and $\tau' \in \mathrm{Del}(P, w)$ such that $\tau \subset \tau'$. From the lemma above, we can see that, for any $\tau \in D_k$ and $p, q \in \tau$, $d(p, q)$ is less than $2\sqrt{5}M$.

Lemma 3.6. *For each $k = 1, 2, \ldots, d$, D_k is locally derived from P in the following sense.*

For any $p_1, p_2 \in P$ satisfying

$$(P - p_1) \cap B(0, 2\sqrt{5}M) = (P - p_2) \cap B(0, 2\sqrt{5}M),$$

if $\tau \in D_k$ contains p_1, then $\tau - p_1 + p_2 \in D_k$.

Proof. Suppose that $p_1, p_2 \in P$ satisfy

$$(P - p_1) \cap B(0, 2\sqrt{5}M) = (P - p_2) \cap B(0, 2\sqrt{5}M)$$

and that $\tau \in D_k$ contains p_1. There exists $w \in \mathcal{W}$ and $\tau' \in \mathrm{Del}(P, w)$ such that $\tau \subset \tau'$. From Lemma 3.5, for any $q \in \tau'$, $d(q, p_1) < 2\sqrt{5}M$. It follows that $q - p_1 + p_2$ belongs to P, and so $\tau' - p_1 + p_2 \in \Delta_d(P)$. It suffices to show $\tau' - p_1 + p_2 \in \mathrm{Del}(P, w')$

for some $w' \in \mathcal{W}$. Let (z, u) be the orthosphere of τ' with respect to w. As $\tau' \in \mathrm{Del}(P, w)$, we have $d(z, p)^2 - u - w(p) \geq 0$ for all $p \in P$. Define $w' \in \mathcal{W}$ by

$$w'(p) = \begin{cases} w(p + p_1 - p_2) & p \in B(p_2, 2\sqrt{5}M) \\ 0 & \text{otherwise.} \end{cases}$$

for $p \in P$. Note that if p is in $P \cap B(p_2, 2\sqrt{5}M)$, then $p+p_1-p_2$ is in $P \cap B(p_1, 2\sqrt{5}M)$. Clearly $(z - p_1 + p_2, u)$ is the orthosphere of $\tau' - p_1 + p_2$ with respect to w'.

We would like to show $d(z - p_1 + p_2, p)^2 - u - w'(p) \geq 0$ for all $p \in P$. If $p \notin B(p_2, 2\sqrt{5}M)$, then Lemma 3.5 implies

$$\begin{aligned} d(z - p_1 + p_2, p) &\geq d(p, p_2) - d(z - p_1 + p_2, p_2) = d(p, p_2) - d(z, p_1) \\ &> 2\sqrt{5}M - \sqrt{5}M = \sqrt{5}M. \end{aligned}$$

It follows that $d(z - p_1 + p_2, p)^2 - u - w'(p) > 5M^2 - 4M^2 > 0$. When p is in $B(p_2, 2\sqrt{5}M)$, $p + p_1 - p_2$ is in $P \cap B(p_1, 2\sqrt{5}M)$ and

$$d(z - p_1 + p_2, p)^2 - u - w'(p) = d(z, p + p_1 - p_2)^2 - u - w(p + p_1 - p_2) \geq 0.$$

Hence $(z - p_1 + p_2, u)$ is empty in (P, w'), and so $\tau' - p_1 + p_2$ is in $\mathrm{Del}(P, w')$. $\square$

Lemma 3.7 and Proposition 3.8 are the interpretation in our context of Claim 14 and the Sliver Theorem of [CDEFT]. The following lemma claims that if the weight on p is not in $I(p, \tau, \lambda)$, then the face $\tau \cup \{p\}$ does not appear in $\mathrm{Del}(P, w)$ and that the length of $I(p, \tau, \lambda)$ is small when $\tau \cup \{p\}$ is a 'thin' simplex.

Lemma 3.7. *For any $p \in P$, $\tau \in D_k$ with $\tau \cup \{p\} \in D_{k+1}$ and a map λ from τ to $[0, (M/3)^2]$, there exists an open interval $I(p, \tau, \lambda) \subset \mathbb{R}$ such that the following are satisfied.*

(1) *If there exist $w \in \mathcal{W}$ and $\tilde{\tau} \in \mathrm{Del}(P, w)$ such that $\lambda = w|\tau$ and $\tau \cup \{p\} \subset \tilde{\tau}$, then $w(p)$ is in $I(p, \tau, \lambda)$.*

(2) *The length of $I(p, \tau, \lambda)$ is $8Md(p, H(\tau))$.*

Moreover, the following also holds.

(3) *Let $p_i \in P$, $\tau_i \in D_k$ with $\tau_i \cup \{p_i\} \in D_{k+1}$ and $\lambda_i : \tau_i \to [0, (M/3)^2]$ for $i = 1, 2$. If there exists $t \in \mathbb{R}^d$ such that $\tau_1 + t = \tau_2$, $p_1 + t = p_2$ and $\lambda_1(q) = \lambda_2(q + t)$ for all $q \in \tau_1$, then $I(p_1, \tau_1, \lambda_1)$ is equal to $I(p_2, \tau_2, \lambda_2)$.*

Proof. Let $p \in P$, $\tau \in D_k$ with $\tau \cup \{p\} \in D_{k+1}$ and $\lambda : \tau \to [0, (M/3)^2]$. Put $\tau' = \tau \cup \{p\}$. Let π be the orthogonal projection from $H(\tau')$ to $H(\tau)$. Note that $d(p, H(\tau))$ equals $d(p, \pi(p))$.

Let (z, u) be the orthosphere of the weighted points $(q, \lambda(q))$ for $q \in \tau$. Put

$$a = d(p, H(\tau))^2 + d(z, \pi(p))^2 - u \in \mathbb{R}.$$

We define

$$I(p,\tau,\lambda) = \{x \in \mathbb{R} \mid |x-a| < 4Md(p,H(\tau))\}.$$

Then it is easy to see that (2) and (3) hold.

Let us show (1). Let $w \in \mathcal{W}$ with $\lambda = w|\tau$. Suppose that there exists $\tilde{\tau} \in \mathrm{Del}(P,w)$ which contains $\tau' = \tau \cup \{p\}$. Let (z', u') be the orthosphere of τ' with respect to w. From Lemma 3.4 and 3.5, we have $u' < 4M^2$.

By Lemma 3.4, $\pi(z') = z$. Let b be the signed distance from z to z': b is positive if and only if z' and p lie on the same side of $H(\tau)$ in $H(\tau')$, and b is negative if and only if z' and p lie on different sides of $H(\tau)$ in $H(\tau')$. Then we have

$$\begin{aligned} d(z',p)^2 &= d(z,\pi(p))^2 + (d(p,\pi(p)) - b)^2 \\ &= d(z,\pi(p))^2 + (d(p,H(\tau)) - b)^2. \end{aligned}$$

By Lemma 3.4, we also have $d(z',z)^2 = u' - u$. Combining these equations with $d(z',p)^2 = u' + w(p)$, we get

$$\begin{aligned} w(p) &= d(z,\pi(p))^2 + (d(p,H(\tau)) - b)^2 - d(z',z)^2 - u \\ &= a - 2bd(p,H(\tau)). \end{aligned}$$

Moreover, $|b| = d(z',z) = \sqrt{u'-u} \le \sqrt{u'} < 2M$, and so $w(p)$ belongs to $I(p,\tau,\lambda)$. $\square$

The following proposition, which will be used in the proof of Proposition 4.5, enables us to get rid of slivers. Indeed, condition (2) means that if the weight on p is not in $I(p,w)$, then the resultant weighted Delaunay triangulation does not contain the face $\tau \cup \{p\}$ with $d(p,H(\tau))$ small.

Proposition 3.8. *There exists a constant $c_1 > 0$ depending only on d such that the following holds.*

For any $p \in P$ and $w \in \mathcal{W}$, there exists an open subset $I(p,w) \subset \mathbb{R}$ such that the following are satisfied.

(1) *$[0,(M/3)^2] \setminus I(p,w)$ is not empty.*

(2) *For $1 \le k < d$, suppose that $\tau \in D_k$ with $\tau \cup \{p\} \in D_{k+1}$ and $d(p,H(\tau)) \le c_1 M$. If $w' \in \mathcal{W}$ satisfies $w'(q) = w(q)$ for all $q \in \tau$ and $w'(p) \notin I(p,w)$, then there does not exist $\tilde{\tau} \in \mathrm{Del}(P,w')$ such that $\tau \cup \{p\} \subset \tilde{\tau}$.*

Moreover, $I(p,w)$ is locally derived in the following sense.

(3) *If $p_1, p_2 \in P$ and $w_1, w_2 \in \mathcal{W}$ satisfy*

$$(P - p_1) \cap B(0, 2\sqrt{5}M) = (P - p_2) \cap B(0, 2\sqrt{5}M)$$

and $w_1(q) = w_2(q - p_1 + p_2)$ for all $q \in (P \setminus \{p_1\}) \cap B(p_1, 2\sqrt{5}M)$, then $I(p_1,w_1) = I(p_2,w_2)$.

Proof. We establish the claim for the constant $c_1 = (72 \cdot 10^{d^2} d)^{-1}$.

For $p \in P$ and $w \in \mathcal{W}$, we define

$$I(p, w) = \bigcup_{k=1}^{d-1} \bigcup_{\tau} I(p, \tau, w|\tau),$$

where the second union runs over all $\tau \in D_k$ such that $\tau \cup \{p\} \in D_{k+1}$ and $d(p, H(\tau)) \leq c_1 M$.

Let us check (1). For any $\tau \in D_k$ such that $\tau \cup \{p\} \in D_{k+1}$, we have $\tau \subset P \cap B(p, 2\sqrt{5}M)$. By Lemma 3.1,

$$\#(P \cap B(p, 2\sqrt{5}M)) \leq (4\sqrt{5} + 1)^d < 10^d.$$

It follows that

$$\#\{\tau \in D_k \mid \tau \cup \{p\} \in D_{k+1}\} < 10^{(k+1)d}$$

for each $k = 1, 2, \dots, d-1$. Hence

$$\# \left(\bigcup_{k=1}^{d-1} \{\tau \in D_k \mid \tau \cup \{p\} \in D_{k+1}\} \right) < 10^{2d} + 10^{3d} + \cdots + 10^{d^2} < d10^{d^2}.$$

If $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$ and if $d(p, H(\tau)) \leq c_1 M$, then by Lemma 3.7 (2)

$$|I(p, \tau, w|\tau)| \leq 8c_1 M^2.$$

Hence

$$|I(p, w)| < 8d10^{d^2} c_1 M^2 = M^2/9,$$

and therefore $[0, (M/3)^2] \setminus I(p, w)$ is not empty.

We now verify (2). Suppose that τ belongs to D_k, $\tau \cup \{p\}$ belongs to D_{k+1} and $w' \in \mathcal{W}$ satisfies $w'(q) = w(q)$ for all $q \in \tau$ and $w'(p) \notin I(p, w)$. If $d(p, H(\tau)) \leq c_1 M$, then, by the definition of $I(p, w)$, it contains $I(p, \tau, w|\tau)$, which is equal to $I(p, \tau, w'|\tau)$. It follows that $w'(p)$ does not belong to $I(p, \tau, w'|\tau)$. Hence, by Lemma 3.7 (1), there does not exist $\tilde{\tau} \in \mathrm{Del}(P, w')$ which contains $\tau \cup \{p\}$.

Finally, let us consider (3). Suppose that $p_1, p_2 \in P$ and $w_1, w_2 \in \mathcal{W}$ satisfy

$$(P - p_1) \cap B(0, 2\sqrt{5}M) = (P - p_2) \cap B(0, 2\sqrt{5}M)$$

and $w_1(q) = w_2(q - p_1 + p_2)$ for all $q \in (P \setminus \{p_1\}) \cap B(p_1, 2\sqrt{5}M)$. By symmetry, it suffices to show $I(p_1, w_1) \subset I(p_2, w_2)$. If $x \in I(p_1, w_1)$, then by definition there exists $\tau \in D_k$ such that $\tau \cup \{p_1\} \in D_{k+1}$, $d(p_1, H(\tau)) \leq c_1 M$ and $x \in I(p_1, \tau, w_1|\tau)$. By Lemma 3.6, $\tau' = \tau - p_1 + p_2$ belongs to D_k and $\tau' \cup \{p_2\}$ belongs to D_{k+1}. It follows from Lemma 3.7 (3) that $I(p_1, \tau, w_1|\tau)$ is equal to $I(p_2, \tau', w_2|\tau')$. We still have $d(p_2, H(\tau')) \leq c_1 M$, because τ' is just a translation of τ. Therefore $I(p_2, \tau', w_2|\tau')$ is contained in $I(p_2, w_2)$, which implies $x \in I(p_2, w_2)$. This completes the proof. $\square$

4 φ-regular triangulations of $\mathbb{R}^d$

In this section, we would like to apply the argument of the last section to countable subsets of $\mathbb{R}^d$ arising from a minimal dynamical system on a Cantor set. First, we would like to recall the notion of φ-regularity and local derivedness ([GMPS1, Section 4]). Let $X \subset \Omega$ be a flat Cantor transversal of a free minimal action φ of $\mathbb{R}^d$ on Ω and let $R_\varphi \subset X \times X$ be the étale equivalence relation induced from (Ω, φ).

Suppose that for each x in X, we have a subset $P(x)$ of $\mathbb{R}^d$. We say that this collection is φ-regular if the following hold.

(1) For any $x, \varphi^p(x) \in X$, $P(\varphi^p(x)) = P(x) - p$.

(2) If x is in X and $K \subset \mathbb{R}^d$ is compact, then there exists a neighbourhood U of x in X such that
$$P(x') \cap K = P(x) \cap K$$
for all x' in U.

For a clopen subset $U \subset X$, we define
$$P_U(x) = \{p \in \mathbb{R}^d \mid \varphi^p(x) \in U\}.$$

In what follows, to simplify notation, we will often denote the family $\{P_U(x)\}_{x \in X}$ by P_U. We remark that, from the definition of flat Cantor transversals, there exist positive real numbers $M_0, M_1 > 0$ such that $P_X(x)$ is M_0-separated and M_1-syndetic for every $x \in X$. The following result is an easy consequence of the definition and we omit the proof.

Lemma 4.1. *Let U be a clopen subset of X. The family of sets $P_U(x)$ for $x \in X$ is φ-regular. Conversely, if $\{P(x)\}_{x \in X}$ is a φ-regular family, then*
$$U = \{x \in X \mid 0 \in P(x)\}$$
is clopen.

We consider a family $\{\mathcal{T}(x)\}_{x \in X}$ of tessellations of $\mathbb{R}^d$ which are indexed by the points of X. We say that this collection is φ-regular if the following hold.

(1) For any $x, \varphi^p(x) \in X$, $\mathcal{T}(\varphi^p(x)) = \mathcal{T}(x) - p$.

(2) If x is in X and t is in $\mathcal{T}(x)$, then there is a neighbourhood U of x such that t is in $\mathcal{T}(x')$ for all x' in U.

If P is a φ-regular family, we say that it is M-syndetic, (M-separated, respectively) if $P(x)$ is M-syndetic (M-separated, respectively) for each x in X.

Let $P(x), P'(x)$, $x \in X$ be two families of subsets of $\mathbb{R}^d$. We say that P' is locally derived from P if there is a constant $R > 0$ such that, for any $x_1, x_2 \in X$ and $u_1, u_2 \in \mathbb{R}^d$, if u_1 is in $P'(x_1)$ and

$$(P(x_1) - u_1) \cap B(0, R) = (P(x_2) - u_2) \cap B(0, R),$$

then u_2 is in $P'(x_2)$. In a similar way, we extend this definition replacing either P, P' or both with families of tessellations. For example, a family of tessellations $\{\mathcal{T}(x)\}_{x \in X}$ is said to be locally derived from another family of tessellations $\{\mathcal{T}'(x)\}_{x \in X}$, if there exists $R > 0$ such that for any $x_1, x_2 \in X$ and $u_1, u_2 \in \mathbb{R}^d$, if $t \in \mathcal{T}'(x_1) - u_1$ contains the origin and

$$\{s \in \mathcal{T}(x_1) - u_1 \mid s \cap B(0, R) \neq \emptyset\} = \{s \in \mathcal{T}(x_2) - u_2 \mid s \cap B(0, R) \neq \emptyset\},$$

then t is in $\mathcal{T}'(x_2) - u_2$. The following result is easily derived from the definitions and we omit the proof.

Lemma 4.2. *If P is a φ-regular family and P' is locally derived from P, then P' is also φ-regular. Analogous statements hold replacing P, P' or both with families of tessellations.*

Next, we turn to the issue of the existence of φ-regular, separated and syndetic sets for flat Cantor transversals.

Lemma 4.3. *For any $M > 0$, there exists a clopen partition $\{V_1, V_2, \ldots, V_n\}$ of X such that $V_i \cap \varphi^p(V_i) = \emptyset$ for all nonzero $p \in B(0, M)$ and $i = 1, 2, \ldots, n$.*

Proof. For each $x \in X$, let $E_x = \{p \in B(0, M) \mid \varphi^p(x) \in X\}$. By Definition 2.1 (4), there exists a clopen neighbourhood $U_x \subset X$ of x such that $E_x = E_y$ for all $y \in U_x$. By Definition 2.1 (3), E_x is a finite set, and so we may assume that U_x is chosen sufficiently small so that $U_x \cap \varphi^p(U_x) = \emptyset$ for every nonzero $p \in E_x$. Therefore we have $U_x \cap \varphi^p(U_x) = \emptyset$ for all nonzero $p \in B(0, M)$.

By the compactness of X, we can select a finite set $x_1, x_2, \ldots, x_n \in X$ such that U_{x_i}'s cover X. Put

$$V_k = U_{x_k} \setminus \bigcup_{i=1}^{k-1} U_{x_i}.$$

Then $\{V_1, V_2, \ldots, V_n\}$ is a clopen partition of X and $V_k \cap \varphi^p(V_k) = \emptyset$ for all nonzero $p \in B(0, M)$. $\square$

By using the lemma above, we can prove the following in a similar way to [GMPS1, Proposition 4.4]. We omit the proof.

Lemma 4.4. *Suppose that P_X is M_1-syndetic. For any $M > M_1$, there exists a clopen subset $U \subset X$ such that P_U is M-separated and $2M$-syndetic.*

Now we are ready to prove Proposition 4.5, which claims the existence of a weight function w such that any d-simplex τ in the resultant $\mathrm{Del}(\cdot,\cdot)$ is not so 'thin' (i.e. for any $\tau \in \mathrm{Del}(P,w)$ and $p \in \tau$, the distance from p to the $(d-1)$-dimensional affine subspace containing $\tau \setminus \{p\}$ is bounded below). We define a positive constant c_2 by

$$c_2 = \frac{c_1^{d-1}}{(d-1)! V_{d-1} (\sqrt{5})^{d-1}},$$

where c_1 is the constant obtained in Proposition 3.8 and V_{d-1} is the volume of $(d-1)$-dimensional unit ball. We remark that c_2 depends only on d.

Proposition 4.5. *Suppose that P_X is M_1-syndetic. For any $M > M_1$, there exist a clopen subset $U \subset X$ and a locally constant function $w : U \to [0,(M/3)^2]$ such that the following hold, where $w_x : P_U(x) \to [0,(M/3)^2]$ is defined by $w_x(p) = w(\varphi^p(x))$.*

1. *P_U is M-separated and $2M$-syndetic.*
2. *For any $x \in X$ and $\tau \in \mathrm{Del}(P_U(x), w_x)$, τ is contained in a closed ball with radius less than $\sqrt{5}M$.*
3. *For any $x \in X$ and $\tau \in \mathrm{Del}(P_U(x), w_x)$, the volume of $\mathrm{Conv}(\tau)$ is greater than $c_1^{d-1} M^d / d!$.*
4. *For any $x \in X$, $\tau \in \mathrm{Del}(P_U(x), w_x)$ and $p \in \tau$, $d(p, H(\tau \setminus \{p\}))$ is greater than $c_2 M$.*

Proof. By Lemma 4.4, there exists a clopen subset $U \subset X$ such that P_U is M-separated and $2M$-syndetic. We would like to construct $w : U \to [0,(M/3)^2]$ inductively. By Lemma 4.3, we can find a clopen partition $\{V_1, V_2, \ldots, V_n\}$ of X such that $V_i \cap \varphi^p(V_i) = \emptyset$ for all nonzero $p \in B(0, 2\sqrt{5}M)$ and $i = 1, 2, \ldots, n$. Let w_0 be the constant function on U with value zero.

Suppose that we have fixed a locally constant function $w_{i-1} : U \to [0,(M/3)^2]$. Take $x \in X$ and consider the M-separated and $2M$-syndetic subset $P_U(x)$ with the weight function $w_{i-1,x}(p) = w_{i-1}(\varphi^p(x))$. Let us define the map $w_{i,x} : P_U(x) \to [0,(M/3)^2]$. For $p \in P_U(x)$ with $\varphi^p(x) \notin U \cap V_i$, we put $w_{i,x}(p) = w_{i-1,x}(p)$. Suppose that $p \in P_U(x)$ satisfies $\varphi^p(x) \in U \cap V_i$. We define $w_{i,x}(p)$ to be the minimum value of the closed set $[0,(M/3)^2] \setminus I(p, w_{i-1,x})$, which is not empty from Proposition 3.8 (1). Let ω_i denote the function given for $x \in U$, by $\omega_i(x) = \omega_{i,x}(0)$. Then $\omega_i = \omega_{i-1}$ on $U \setminus V_i$ and therefore ω_i is locally constant by assumption. If $x, y \in U \cap V_i$ are close enough, then we have $P_U(x) \cap B(0, 2\sqrt{5}M) = P_U(y) \cap B(0, 2\sqrt{5}M)$. Moreover, for any non-zero $q \in P_U(x) \cap B(0, 2\sqrt{5}M)$, by the choice of V_i, $\varphi^q(x)$ is not in V_i. Likewise $\varphi^q(y)$ is not in V_i. Hence $w_{i,x}(q) = w_{i-1,x}(q)$ equals $w_{i,y}(q) = w_{i-1,y}(q)$, if x and y are sufficiently close. By Proposition 3.8 (3), for such x, y, we can conclude that $I(0, w_{i-1,x})$ equals $I(0, w_{i-1,y})$, which implies $w_{i,x}(0) = w_{i,y}(0)$. Thus, the function w_i is locally constant. Repeating this procedure, we get w_n. Put $w = w_n$.

Take $x \in X$ and let $\tau = \{p_0, p_1, \ldots, p_d\} \in \mathrm{Del}(P_U(x), w_x)$. Let us show (2). Let z be the orthocentre of τ with respect to w_x. By Lemma 3.5, $d(z, p_i)$ is less than $\sqrt{5}M$. It follows that τ is contained in $B(z, \sqrt{5}M)$.

We next verify (3). By Lemma 3.5, for any $p_i, p_j \in \tau$, $d(p_i, p_j)$ is less than $2\sqrt{5}M$. Hence, from the choice of clopen sets V_l's, $\varphi^{p_i}(x)$ and $\varphi^{p_j}(x)$ are not contained in the same V_l for $i \neq j$. Without loss of generality, we may assume $\varphi^{p_i}(x) \in V_{l_i}$ and $l_0 < l_1 < \cdots < l_d$. Let $\tau_k = \{p_0, p_1, \ldots, p_k\}$ for each $k = 1, 2, \ldots, d-1$. Since P_U is M-separated, the one-dimensional volume, that is, the length of $\mathrm{Conv}(\tau_1)$ is not less than M.

For each $k = 2, 3, \ldots, d$, we would like to see $d(p_k, H(\tau_{k-1})) > c_1 M$. Suppose that $d(p_k, H(\tau_{k-1}))$ is not greater than $c_1 M$. By definition of w, for all $q \in \tau_{k-1}$, $w_x(q)$ is equal to $w_{l_k - 1, x}(q)$. In addition, $w_x(p_k) = w_{l_k, x}(p_k)$ does not belong to $I(p_k, w_{l_k - 1, x})$. By applying Proposition 3.8 (2) to p_k, $w_{l_k-1,x}$, τ_{k-1} and w_x, we can conclude that there does not exist $\tilde{\tau} \in \mathrm{Del}(P_U(x), w_x)$ such that $\tau_{k-1} \cup \{p_k\} \subset \tilde{\tau}$. This is clearly a contradiction, and so $d(p_k, H(\tau_{k-1})) > c_1 M$ for all $k = 2, 3, \ldots, d$. It follows that the d-dimensional volume of $\mathrm{Conv}(\tau_d)$ is greater than

$$M \times \frac{1}{2 \cdot 3 \cdot \cdots \cdot d}(c_1 M)^{d-1} = \frac{c_1^{d-1} M^d}{d!}.$$

We next verify (4). By the argument above, for any $p \in \tau$, $\tau \setminus \{p\}$ is contained in a $(d-1)$-dimensional closed disk with radius less than $\sqrt{5}M$. It follows that $(d-1)$-dimensional volume of $\mathrm{Conv}(\tau \setminus \{p\})$ is less than $V_{d-1}(\sqrt{5}M)^{d-1}$, where V_{d-1} is the volume of $(d-1)$-dimensional unit ball. Hence

$$d(p, H(\tau \setminus \{p\})) > \frac{d c_1^{d-1} M^d}{d! V_{d-1}(\sqrt{5}M)^{d-1}} = c_2 M$$

for all $p \in \tau$. □

By the proposition above, we have obtained $\mathrm{Del}(P_U(x), w_x)$ containing no slivers. However, some extra work is necessary to obtain triangulations of $\mathbb{R}^d$, because $\{\mathrm{Conv}(\tau) \mid \tau \in \mathrm{Del}(P_U(x), w_x)\}$ may not be a triangulation in general, or equivalently $\#\delta(z, u)$ may not equal $d+1$ for $(z, u) \in \mathcal{Z}(P_U(x), w_x)$. We would like to triangulate $\mathrm{Conv}(\delta(z, u))$ (for (z, u) with $\#\delta(z, u) > d+1$) in a locally derived manner. We need an elementary fact about triangulations of convex polytopes. In what follows, we use the notation in [St, Lemma 1.1]. But the reader should be warned that Lemma 1.1 of [St] contains only the statement. In page 160 of [Mat], one can find a construction called bottom-vertex triangulation, which is the same as that used here.

Equip $\mathbb{R}^d$ with the lexicographic ordering. Namely, for $p = (p_1, p_2, \ldots, p_d)$ and $q = (q_1, q_2, \ldots, q_d)$ in $\mathbb{R}^d$, p is less than q, if there exists i such that $p_i < q_i$ and $p_j = q_j$ for all $j < i$. Note that the lexicographic ordering is invariant under translation in the sense that $p < q$ implies $p + r < q + r$ for any $r \in \mathbb{R}^d$.

Let $K \subset \mathbb{R}^d$ be a convex polytope. For each non-empty face F of K, we define $m(F)$ to be the minimum element in the set of vertices (i.e. extremal points) of F. A $(d+1)$-tuple $\Phi = (F_0, F_1, \dots, F_d)$ is called a flag of faces, if F_i is an i-dimensional face of K and $F_0 \subset F_1 \subset \cdots \subset F_d = K$. Call Φ a full flag, if $m(F_i)$ is not in F_{i-1} for $1 \leq i \leq d$. For a full flag Φ, we put $\tau_\Phi = \{F_0, m(F_1), \dots, m(F_d)\}$. Then

$$\{\mathrm{Conv}(\tau_\Phi) \mid \Phi \text{ is a full flag of } K\}$$

is a triangulation of K, in which the cells meet face to face. Thus, the union of those $\mathrm{Conv}(\tau_\Phi)$'s is equal to K and for any full flags Φ and Φ', we have

$$\mathrm{Conv}(\tau_\Phi) \cap \mathrm{Conv}(\tau_{\Phi'}) = \mathrm{Conv}(\tau_\Phi \cap \tau_{\Phi'}).$$

We are able to apply this argument to tessellations of $\mathbb{R}^d$ by convex polytopes. Let $\mathcal{T}$ be a tessellation of $\mathbb{R}^d$ by convex polytopes. Suppose that cells in $\mathcal{T}$ meet face to face. Let V be the set of all vertices of all cells in $\mathcal{T}$. For each cell in $\mathcal{T}$, by considering all full flags of it, we obtain its triangulation. Let $\mathcal{D}$ be the union of them, i.e.

$$\mathcal{D} = \{\mathrm{Conv}(\tau_\Phi) \mid \Phi \text{ is a full flag of some cell } t \in \mathcal{T}\}.$$

Notice that τ_Φ is in $\Delta_d(V)$. It is not so hard to see the following.

- Each $\mathrm{Conv}(\tau)$ in $\mathcal{D}$ is contained in a cell of $\mathcal{T}$.
- $\mathcal{D}$ is a triangulation of $\mathbb{R}^d$.
- For any $\mathrm{Conv}(\tau), \mathrm{Conv}(\tau') \in \mathcal{D}$,

  $$\mathrm{Conv}(\tau) \cap \mathrm{Conv}(\tau') = \mathrm{Conv}(\tau \cap \tau'),$$

  that is, cells of $\mathcal{D}$ meet face to face.

Moreover, we can make the following remark. Since the lexicographic ordering on $\mathbb{R}^d$ is invariant by translation, $\mathcal{D}$ is locally derived from $\mathcal{T}$. Thus, if $\mathcal{T}$ is a φ-regular family of tessellations, then $\mathcal{D}$ also becomes φ-regular by Lemma 4.2.

From Lemma 3.3, Proposition 4.5 and the discussion above, we then prove the following proposition.

Proposition 4.6. *Suppose that P_X is M_1-syndetic. For any $M > M_1$, there exist a clopen subset $U \subset X$ and a family of subsets $\mathcal{D}(x) \subset \Delta_d(P_U(x))$ for $x \in X$ satisfying each of the following conditions for all $x \in X$.*

(1) *$P_U(x)$ is M-separated and $2M$-syndetic.*

(2) *The collection $\{\mathrm{Conv}(\tau) \mid \tau \in \mathcal{D}(x)\}$ is a tessellation of $\mathbb{R}^d$.*

(3) *For any $\tau, \tau' \in \mathcal{D}(x)$, we have $\mathrm{Conv}(\tau) \cap \mathrm{Conv}(\tau') = \mathrm{Conv}(\tau \cap \tau')$.*

(4) *For any* $p \in P_U(x)$*, there exists* $\tau \in \mathcal{D}(x)$ *such that* $p \in \tau$*.*

(5) *Any* $\tau \in \mathcal{D}(x)$ *is contained in a closed ball with radius less than* $\sqrt{5}M$*.*

(6) *For any* $\tau \in \mathcal{D}(x)$ *and* $p \in \tau$*,* $d(p, H(\tau \setminus \{p\}))$ *is greater than* $c_2 M$*.*

(7) *For any* $p \in P_X(x)$*,* τ *is in* $\mathcal{D}(x)$ *if and only if* $\tau - p$ *is in* $\mathcal{D}(\varphi^p(x))$*.*

(8) *For any* $\tau \in \mathcal{D}(x)$*, there exists an open neighbourhood* $V \subset X$ *of* $x \in X$ *such that* $\tau \in \mathcal{D}(y)$ *for all* $y \in V$*.*

Proof. From Proposition 4.5, we get a clopen subset $U \subset X$ and a locally constant function $w : U \to [0, (M/3)^2]$ satisfying the conditions given there. The first condition follows at once from Proposition 4.5 (1). For each $x \in X$, we let w_x denote the map from $P_U(x)$ to $[0, (M/3)^2]$ defined by $w_x(p) = w(\varphi^p(x))$.

By Lemma 3.3, for each $x \in X$,

$$\{\mathrm{Conv}(\delta(z, u)) \mid (z, u) \in \mathcal{Z}(P_U(x), w_x)\}$$

is a tessellation of $\mathbb{R}^d$ by convex polytopes, in which cells meet face to face. It follows from the discussion above that there exists $\mathcal{D}(x) \subset \Delta_d(P_U(x))$ such that

(i) Each τ in $\mathcal{D}(x)$ is contained in some $\delta(z, u)$ in $\mathcal{Z}(P_U(x), w_x)$.

(ii) The collection $\{\mathrm{Conv}(\tau) \mid \tau \in \mathcal{D}(x)\}$ is a tessellation of $\mathbb{R}^d$.

(iii) For any $\tau, \tau' \in \mathcal{D}(x)$, we have $\mathrm{Conv}(\tau) \cap \mathrm{Conv}(\tau') = \mathrm{Conv}(\tau \cap \tau')$.

From (ii) and (iii), the conditions (2) and (3) are immediate. The condition (4) is clear. From (i), we see that τ belongs to $\mathrm{Del}(P_U(x), w_x)$, and so the conditions (5) and (6) follow from Proposition 4.5.

The conditions (7) and (8) easily follow from the continuity of w and the local derivedness of the construction of $\mathcal{D}(x)$. □

5 Well-separated tessellations of $\mathbb{R}^d$

In the last section, we constructed a triangulation of $\mathbb{R}^d$ with several nice properties. In this section, we study how to construct a tessellation which is 'dual' to the triangulation, that is, the vertices in the triangulation become cells and cells in the triangulation become vertices. The tessellation will possess various nice properties. Before beginning, we make the following remark. In the actual application, we will begin with a φ-regular collection $P(x)$, find weighted Delaunay triangulations without slivers, and construct a collection of tessellations $\mathcal{T}(x)$ on $\mathbb{R}^d$. It is worth noting as we proceed, that all of our construction are 'locally derived' in the appropriate sense, and so the resulting collection $\mathcal{T}(x)$'s is φ-regular by application of Lemma 4.2.

Definition 5.1. Let $\mathcal{T}$ be a tessellation of $\mathbb{R}^d$.

(1) We say that $\mathcal{T}$ has capacity $C > 0$, if each element of $\mathcal{T}$ contains an open ball of radius C.

(2) We say that $\mathcal{T}$ is K-separated for $K > 0$, if the following hold.

  (a) For any $n \in \mathbb{N}$ and $t_0, t_1, \dots, t_n \in \mathcal{T}$, if there exists $p \in \mathbb{R}^d$ such that $d(t_i, p) < K$ for all $i = 0, 1, \dots, n$, then $t_0 \cap t_1 \cap \dots \cap t_n$ is not empty.

  (b) Any distinct $d + 2$ elements of $\mathcal{T}$ have trivial intersection.

(3) We let the diameter of $\mathcal{T}$ to be the supremum of the diameter of its elements and denote it by $\operatorname{diam}(\mathcal{T})$. When $\{\mathcal{T}(x) \mid x \in X\}$ is a φ-regular family of tessellations, then for any $x, y \in X$ and $t \in \mathcal{T}(x)$, it follows from the minimality of φ that there exists $p \in P_X(y)$ such that $t \in \mathcal{T}(\varphi^p(y))$. Therefore $\operatorname{diam}(\mathcal{T}(x))$ does not depend on $x \in X$ and we denote this value by $\operatorname{diam}(\mathcal{T})$.

Suppose that we are given an M-separated and $2M$-syndetic subset P of $\mathbb{R}^d$ and a subset $\mathcal{D} \subset \Delta_d(P)$ satisfying the following.

- The collection $\{\operatorname{Conv}(\tau) \mid \tau \in \mathcal{D}\}$ is a tessellation of $\mathbb{R}^d$.
- For any $\tau, \tau' \in \mathcal{D}$, we have $\operatorname{Conv}(\tau) \cap \operatorname{Conv}(\tau') = \operatorname{Conv}(\tau \cap \tau')$.
- For any $p \in P$, there exists $\tau \in \mathcal{D}$ such that $p \in \tau$.
- Any $\tau \in \mathcal{D}$ is contained in a closed ball with radius less than $\sqrt{5}M$.
- For any $\tau \in \mathcal{D}$ and $p \in \tau$, the distance from p to $H(\tau \setminus \{p\})$ is greater than $c_2 M$.

For $p \in P$, we denote $\{\tau \in \mathcal{D} \mid p \in \tau\}$ by $\mathcal{D}_p$. For every $p \in P$ and $\tau \in \mathcal{D}_p$, we define

$$T(p, \tau) = \left\{ \sum_{q \in \tau} \lambda_q q \mid \sum_q \lambda_q = 1, \lambda_p \geq \lambda_q \geq 0 \text{ for all } q \in \tau \right\}.$$

It is easily seen that the collection $\{T(p, \tau) \mid p \in P, \tau \in \mathcal{D}_p\}$ is a tessellation of $\mathbb{R}^d$. For any $p \in P$, we set

$$T(p) = \bigcup_{\tau \in \mathcal{D}_p} T(p, \tau).$$

Again the collection $\mathcal{T} = \{T(p) \mid p \in P\}$ is a tessellation of $\mathbb{R}^d$. We would like to show that this tessellation $\mathcal{T}$ has various nice properties. Notice that $T(p)$ may not be convex, but is a union of convex polytopes.

For every $p \in P$, we set

$$S(p) = \bigcup_{\tau \in \mathcal{D}_p} \operatorname{Conv}(\tau).$$

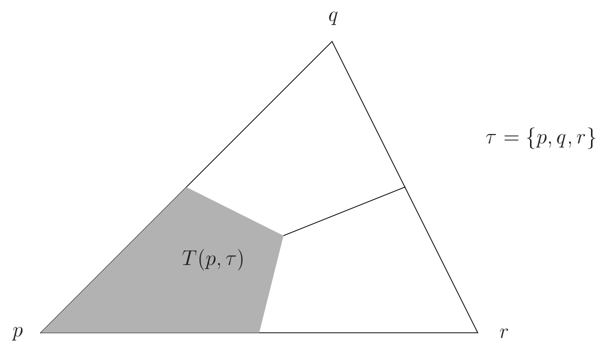

Figure 2: $T(p, \tau)$

Note that $S(p)$ is homeomorphic to a closed unit ball of $\mathbb{R}^d$ and, for any $q \in S(p)$ and $0 \leq \lambda \leq 1$, $\lambda(q-p)+p$ is in $S(p)$. It is also easy to see that the interior of $S(p)$ is equal to

$$\bigcup_{\tau \in \mathcal{D}_p} \mathrm{Conv}(\tau) \setminus \mathrm{Conv}(\tau \setminus \{p\}).$$

Besides, $B(p, c_2M)$ is contained in $S(p)$, because the distance from p to $H(\tau \setminus \{p\})$ is greater than c_2M.

Lemma 5.2. *For every $p \in P$, $B(p, c_2M/(d+1))$ is contained in $T(p)$. In particular, $\mathcal{T}$ has capacity $c_2M/(d+1)$.*

Proof. Clearly $T(p,\tau)$ contains $(d+1)^{-1}(\mathrm{Conv}(\tau) - p) + p$. It follows that $T(p)$ contains $(d+1)^{-1}(S(p) - p) + p$. Hence $T(p)$ contains $B(p, c_2M/(d+1))$. □

Next, we would like to show that $\mathcal{T}$ is well-separated.

Lemma 5.3. *For every $p \in P$ and $0 < \lambda < 1$, the distance from $\lambda(S(p) - p) + p$ to the boundary of $S(p)$ is greater than $(1-\lambda)c_2M$.*

Proof. When $d = 1, 2$, the assertion is an easy geometric observation. Let us consider the case of $d > 2$. Suppose that the minimum distance is achieved by q_1 in $\lambda(S(p) - p)+p$ and q_2 in the boundary of $S(p)$. Let H be a two-dimensional plane containing q_1, q_2 and p. For each $\tau \in \mathcal{D}_p$, $H \cap \mathrm{Conv}(\tau)$ is either $\{p\}$, a line segment with an endpoint p or a triangle with a vertex p. Therefore $H \cap S(p)$ is exactly the same shape as in the case of $d = 2$. It follows that $d(q_1, q_2)$ is greater than $(1-\lambda)c_2M$. □

For a subset $A \subset \mathbb{R}^d$ and $r > 0$, we let $B(A, r)$ denote the r-neighbourhood of A, namely, $B(A, r) = \{p \in \mathbb{R}^d \mid d(p, A) < r\}$.

Lemma 5.4. *For every $p \in P$, $B(T(p), c_2M/(d+1))$ is contained in the interior of $S(p)$.*

Proof. For each $\tau \in \mathcal{D}_p$, we consider $d(d+1)^{-1}(\mathrm{Conv}(\tau) - p) + p$, which is equal to

$$\left\{ \sum_{q \in \tau} \lambda_q q \mid \sum_{q \in \tau} \lambda_q = 1, \lambda_p \geq (d+1)^{-1}, 0 \leq \lambda_q \leq 1 \text{ for all } q \in \tau \right\}.$$

Clearly this set contains $T(p, \tau)$. It follows that $T(p)$ is contained in $d(d+1)^{-1}(S(p) - p) + p$. By the lemma above, the $(d+1)^{-1}c_2 M$-neighbourhood of $d(d+1)^{-1}(S(p)-p)+p$ is contained in the interior of $S(p)$. Therefore we can conclude that $B(T(p), (d+1)^{-1}c_2 M)$ is contained in the interior of $S(p)$. □

Lemma 5.5. *The tessellation $\mathcal{T}$ is $c_2 M/(d+1)$-separated.*

Proof. Suppose that we are given $n \in \mathbb{N}$, $p_0, p_1, \ldots, p_n \in P$ and $p \in \mathbb{R}^d$ such that $d(T(p_i), p) < c_2 M/(d+1)$ for all $i = 0, 1, \ldots, n$. From Lemma 5.4, p is in the interior of $S(p_i)$. It follows that there exists $\tau_i \in \mathcal{D}_{p_i}$ such that p is in $\mathrm{Conv}(\tau_i) \setminus \mathrm{Conv}(\tau_i \setminus \{p_i\})$.

For each $i = 0, 1, \ldots, n$, p is written as

$$p = \sum_{q \in \tau_i} \lambda_{i,q} q,$$

where $0 \leq \lambda_{i,q} \leq 1$ and $\sum_{q \in \tau_i} \lambda_{i,q} = 1$, and this expression is unique. For each $i = 1, 2, \ldots, n$, p belongs to $\mathrm{Conv}(\tau_0) \cap \mathrm{Conv}(\tau_i)$, which is equal to $\mathrm{Conv}(\tau_0 \cap \tau_i)$. If τ_0 does not contain p_i, then $\tau_0 \cap \tau_i$ does not, neither. Hence p should belong to $\mathrm{Conv}(\tau_i \setminus \{p_i\})$, which is a contradiction. Thus, we can conclude that τ_0 contains p_i for all $i = 0, 1, \ldots, n$. By symmetry, each τ_i contains all the points $\{p_0, p_1, \ldots, p_n\}$. Therefore, each $T(p_i, \tau_i)$ contains the barycentre $z = (n+1)^{-1} \sum_{i=0}^{n} p_i$. It follows that each $T(p_i)$ contains z.

In addition, since the cardinality of τ_0 is $d+1$, n must be less than $d+1$. It follows that any $d+2$ distinct elements have trivial intersection. □

The following lemma will be necessary to show that 'boundary points' have measure zero in a later section. For $A \subset \mathbb{R}^d$, we let ∂A denote the boundary of A.

Lemma 5.6. *For every $p \in P$ and $L > 0$, the d-dimensional volume of $B(\partial T(p), L)$ is less than $10^{d^2} 2 L V_{d-1} (\sqrt{5}M + L)^{d-1}$, where V_{d-1} is the $(d-1)$-dimensional volume of the unit ball of $\mathbb{R}^{d-1}$.*

Proof. For each $\tau \in \mathcal{D}_p$ and a bijection $f : \{1, 2, \ldots, d\} \to \tau \setminus \{p\}$, we define

$$\kappa(\tau, f) = \left\{ \frac{1}{k+1} \left(p + \sum_{i=1}^{k} f(i) \right) \mid k = 1, 2, \ldots, d \right\}.$$

Then $\kappa(\tau, f)$ consists of d distinct points in $\mathrm{Conv}(\tau)$. In particular, $\kappa(\tau, f)$ is contained in a $(d-1)$-dimensional disk of radius $\sqrt{5}M$, because τ is contained in a closed ball with radius less than $\sqrt{5}M$.

It is easy to see

$$T(p,\tau)=\bigcup \operatorname{Conv}(\{p\}\cup\kappa(\tau,f)),$$

where the union runs over all bijections $f:\{1,2,\dots,d\}\to\tau\setminus\{p\}$. Moreover, we have

$$\partial T(p)\cap T(p,\tau)=\bigcup \operatorname{Conv}(\kappa(\tau,f)),$$

and so

$$B(\partial T(p),L)=\bigcup B(\operatorname{Conv}(\kappa(\tau,f)),L),$$

where the union runs over all the pairs of $\tau\in\mathcal{D}_p$ and bijections $f:\{1,2,\dots,d\}\to\tau\setminus\{p\}$. Since τ is contained in a closed ball with radius less than $\sqrt{5}M$, for any $q\in\tau$, one has $d(p,q)<2\sqrt{5}M$. Since P is M-separated, by Lemma 3.1 (1), we have $\#(P\cap B(p,2\sqrt{5}M))\le(4\sqrt{5}+1)^d<10^d$. The map sending (τ,f) to $(f(1),f(2),\dots,f(d))$ is injective, and so the number of pairs (τ,f) is less than $\#(P\cap B(p,2\sqrt{5}M))^d<(10^d)^d=10^{d^2}$.

It remains for us to estimate the d-dimensional volume of $B(\operatorname{Conv}(\kappa(\tau,f)),L)$. As mentioned above, $\kappa(\tau,f)$ is contained in a $(d-1)$-dimensional disk of radius $\sqrt{5}M$. Hence the d-dimensional volume of $B(\operatorname{Conv}(\kappa(\tau,f)),L)$ is not greater than

$$2LV_{d-1}(\sqrt{5}M+L)^{d-1},$$

where V_{d-1} is the $(d-1)$-dimensional volume of the unit ball of $\mathbb{R}^{d-1}$. This completes the proof. $\square$

Now we would like to apply the construction of $\mathcal{T}$ to the family $\mathcal{D}(x)$ obtained in the last section. Let $X\subset\Omega$ be a flat Cantor transversal of a free minimal action φ of $\mathbb{R}^d$ on Ω and let $R_\varphi\subset X\times X$ be the étale equivalence relation induced from (Ω,φ).

Proposition 5.7. *Suppose that P_X is M_1-syndetic. For any $M>M_1$, there exist a clopen subset $U\subset X$, a φ-regular family of tessellations $\mathcal{T}(x)$ and bijections $T_x:P_U(x)\to\mathcal{T}(x)$ for $x\in X$ such that the following conditions hold for each $x\in X$.*

1. *For each $p\in P_U(x)$, $B(p,c_2M/(d+1))$ is contained in $T_x(p)$. In particular, $\mathcal{T}(x)$ has capacity $c_2M/(d+1)$.*
2. *$\mathcal{T}(x)$ is $c_2M/(d+1)$-separated.*
3. *For each $p\in P_U(x)$ and $L>0$, the d-dimensional volume of $B(\partial T_x(p),L)$ is less than $10^{d^2}2LV_{d-1}(\sqrt{5}M+L)^{d-1}$.*
4. *Each element of $T_x(p)$ meets at most 10^d other elements.*

(5) *Let $0 \leq n < d$. If $n+1$ distinct elements $t_0, t_1, \ldots, t_n$ in $\mathcal{T}(x)$ meet, then there exist $t_{n+1}, \ldots, t_d \in \mathcal{T}(x)$ such that $t_0, t_1, \ldots, t_d$ are all distinct and $t_0 \cap t_1 \cap \cdots \cap t_d$ is not empty.*

(6) *Let $\tau = \{p_0, p_1, \ldots, p_d\}$ be $d+1$ distinct points in $P_U(x)$. If $T_x(p_0) \cap T_x(p_1) \cap \cdots \cap T_x(p_d)$ is not empty, then $B(\mathrm{Conv}(\tau), c_2 M/(d+1))$ is contained in $\bigcup_i T_x(p_i)$. In addition, $B(T_x(p_0), c_2 M/(d+1))$ does not meet $\mathrm{Conv}(\tau \setminus \{p_0\})$.*

Proof. From Proposition 4.6, we get a clopen set $U \subset X$ and a family $\{\mathcal{D}(x) \mid x \in X\}$. By applying the construction in this section to each $\mathcal{D}(x)$, a tessellation $\mathcal{T}(x) = \{T_x(p) \mid p \in P_U(x)\}$ is obtained. By the conditions (7) and (8) of Proposition 4.6 and the definition of $\mathcal{T}(x)$, it is easy to see that the family $\mathcal{T}(x)$ for $x \in X$ is φ-regular.

The conditions (1), (2) and (3) are immediate consequences from Lemma 5.2, 5.5 and 5.6. The condition (4) is clear from the proof of Lemma 5.6. The condition (5) is also clear from the proof of Lemma 5.5.

Let us check the condition (6). Let $\tau = \{p_0, p_1, \ldots, p_d\}$ be $d+1$ distinct points in $P_U(x)$ and suppose that $T_x(p_0) \cap T_x(p_1) \cap \cdots \cap T_x(p_d)$ is not empty. From the proof of Lemma 5.5, we see that τ is in $\mathcal{D}(x)$. Suppose that $B(\mathrm{Conv}(\tau), c_2 M/(d+1))$ meets $T_x(q)$. By Lemma 5.4, $\mathrm{Conv}(\tau)$ meets $S(q)$. It follows that q is in τ. Therefore $B(\mathrm{Conv}(\tau), c_2 M/(d+1))$ is covered by $T_x(p_i)$'s. Since $B(T_x(p_0), c_2 M/(d+1))$ is contained in the interior of $S(p_0)$ and $\mathrm{Conv}(\tau \setminus \{p_0\})$ is in the boundary of $S(p_0)$, the last statement also follows easily. $\square$

6 Refining tessellations

In the last section, we gave a method of producing φ-regular tessellations. The next step is to show how we may produce a sequence having larger and larger elements (more and more separated) in such a way that each element of one is the union of elements from the previous. At the same time, we will need several extra technical conditions which will be used later in the proof of the main result. While we will provide rigorous and fairly complete arguments, most of these properties can be seen fairly easily by drawing some pictures. Moreover, most of our arguments are similar to those in [GMPS1, Theorem 5.1], and so the reader may refer to it.

Before stating the result, we will need some notation. We are considering a tessellation $\mathcal{T}$ of $\mathbb{R}^d$ by polyhedral regions with non-overlapping interiors. Given a point p in $\mathbb{R}^d$, we would like to say that this point belongs to a unique element of $\mathcal{T}$. Of course, this is false since the elements overlap on their boundaries. To resolve this difficulty in an arbitrary, but consistent way, we define, for any t in $\mathcal{T}$,

$$t^* = \{p \in \mathbb{R}^d \mid p + (\varepsilon, \varepsilon^2, \ldots, \varepsilon^d) \in t \text{ for all sufficiently small } \varepsilon > 0\},$$

where ε^i is the i-th power of ε. It is not so hard to see the following.

- t contains t^* and t^* contains the interior of t.
- If $\mathcal{T}$ is a tessellation by polyhedral regions, then the collection $\{t^* \mid t \in \mathcal{T}\}$ is a partition of $\mathbb{R}^d$.
- $(t+p)^* = t^* + p$ for any $p \in \mathbb{R}^d$.

We remark that any other definition of t^* would work, as long as the above properties are satisfied.

A comment is in order regarding tessellations. In the process we are about to undertake, we will take unions of polyhedra, which may be disconnected. In addition, a vertex in some polyhedron may only belong to one other element of the tessellation. So we do not use the terms 'vertex', 'edge' and 'face'. Instead, we would like to regard these objects in a combinatorial way as the (non-empty) intersection of several polyhedra.

Let $X \subset \Omega$ be a flat Cantor transversal of a free minimal action φ of $\mathbb{R}^d$ on Ω and let $R_\varphi \subset X \times X$ be the étale equivalence relation induced from (Ω, φ). We suppose that P_X is M_0-separated and M_1-syndetic.

Theorem 6.1. *There exist a sequence of clopen subsets $U_0, U_1, U_2, \dots$ of X and a sequence of φ-regular tessellations $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \dots$ satisfying each of the following conditions for all $l \geq 0$ and x in X.*

1. *$\mathcal{T}_l(x)$ has capacity $(10^d + l + 1)M_1$.*
2. *$\mathcal{T}_{l+1}(x)$ is $\operatorname{diam}(\mathcal{T}_l)$-separated.*
3. *Each element of $\mathcal{T}_l(x)$ meets at most 10^d other elements.*
4. *If $t_1, t_2, \dots, t_n \in \mathcal{T}_l(x)$ are mutually distinct and $t_1 \cap t_2 \cap \cdots \cap t_n$ is non-empty, then there exist $t_{n+1}, \dots, t_{d+1}$ in $\mathcal{T}_l(x)$ such that $t_i \neq t_j$ for $i \neq j$ and $t_1 \cap t_2 \cap \cdots \cap t_{d+1}$ is non-empty.*
5. *Each element of $\mathcal{T}_l(x)$ is contained in an element of $\mathcal{T}_{l+1}(x)$.*
6. *For any $s \in \mathcal{T}_{l+1}(x)$, $\#\{t \in \mathcal{T}_l(x) \mid t' \subset s$ for any $t' \in \mathcal{T}_l(x)$ with $t \cap t' \neq \emptyset\}$ is not less than $10^{d^2} \#(P_X(x) \cap B(\partial s, \operatorname{diam}(\mathcal{T}_0)))$.*
7. *There exists a bijection $\pi_{l,x} : \mathcal{T}_l(x) \to P_{U_l}(x)$ such that $\pi_{l,x}(t)$ is in the interior of t for each $t \in \mathcal{T}_l(x)$.*

Proof. By Proposition 5.7, we may choose a clopen subset U_0 and a φ-regular family of tessellations $\mathcal{T}_0$ such that $\operatorname{diam}(\mathcal{T}_0(x))$ has capacity $(10^d + 1)M_1$ for each $x \in X$. The other properties required for $\mathcal{T}_0$ easily follow from Proposition 5.7.

Next, we suppose that we have found a clopen subset U_l and a φ-regular family of tessellations $\mathcal{T}_l$ satisfying the desired conditions for some $l \geq 0$. There exists a constant $K > 0$ such that P_{U_l} is K-syndetic, because U_l is clopen. Put

$$L = \operatorname{diam}(\mathcal{T}_0) + M_0/2 + \operatorname{diam}(\mathcal{T}_l).$$

Let $\pi_{l,x} : \mathcal{T}_l(x) \to P_{U_l}(x)$ be the bijection described in the condition (7).

We would like to construct U_{l+1} and $\mathcal{T}_{l+1}$. We find a constant $M > 0$ satisfying each of the following:

$$c_2(d+1)^{-1}M \geq (10^d + l + 2)M_1 + \operatorname{diam}(\mathcal{T}_l), \tag{6.1}$$

$$c_2(d+1)^{-1}M \geq 2\operatorname{diam}(\mathcal{T}_l) \tag{6.2}$$

and

$$\left(\frac{c_2(d+1)^{-1}M - 2\operatorname{diam}(\mathcal{T}_l) - K}{K}\right)^d \geq 10^{d^2} \times \frac{10^{d^2} 2LV_{d-1}(\sqrt{5}M + L)^{d-1}}{V_d(M_0/2)^d}, \tag{6.3}$$

where V_k denotes the k-dimensional volume of the unit ball in $\mathbb{R}^k$. We note that in the third inequality, the left hand side is a polynomial of degree d with variable M and the right hand side is a polynomial of degree $d-1$ with variable M.

By applying Proposition 5.7 to the constant M, we may find a clopen subset $U \subset X$, a φ-regular family of tessellations $\mathcal{T}$ and bijections $\pi_x : \mathcal{T}(x) \to P_U(x)$ for x in X satisfying the conditions given there. For each x in X and s in $\mathcal{T}(x)$, we define

$$\tilde{s} = \bigcup t,$$

where the union runs over all the cells $t \in \mathcal{T}_l(x)$ such that $\pi_{l,x}(t)$ is in s^*. We now define a new tessellation $\mathcal{T}_{l+1}(x)$ to be the collection of all $\tilde{s}$, where s is in $\mathcal{T}(x)$. Since $\mathcal{T}_l$ and P_{U_l} are φ-regular, this new family of tessellations $\mathcal{T}_{l+1}$ is also φ-regular. Clearly, condition (5) is satisfied.

We first observe that, for any s in $\mathcal{T}(x)$, each point in $\tilde{s}$ is in some element t of $\mathcal{T}_l(x)$ with $\pi_{l,x}(t) \in s^*$. As the diameter of t is at most $\operatorname{diam}(\mathcal{T}_l)$, it follows that every point of $\tilde{s}$ is within distance $\operatorname{diam}(\mathcal{T}_l)$ of s.

We next verify that $\mathcal{T}_{l+1}(x)$ has capacity $(10^d + l + 2)M_1$. Each element s in $\mathcal{T}(x)$ has capacity $c_2(d+1)^{-1}M$ and hence contains an open ball $B(p, c_2(d+1)^{-1}M)$. It follows that the open ball $B(p, c_2(d+1)^{-1}M - \operatorname{diam}(\mathcal{T}_l))$ is contained in $\tilde{s}$. Therefore $\tilde{s}$ has capacity $(10^d + l + 2)M_1$ by (6.1).

We will show that the map sending s in $\mathcal{T}(x)$ to $\tilde{s}$ in $\mathcal{T}_{l+1}(x)$ is a bijection which preserves non-trivial (multiple) intersections. The first step in this is to observe that if $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ have a non-trivial intersection, then so do $s_1, s_2, \ldots, s_n$. Let p be in the intersection of $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$. Since every point of $\tilde{s}_i$ is within distance $\operatorname{diam}(\mathcal{T}_l)$ of s_i, we obtain $d(p, s_i) \leq \operatorname{diam}(\mathcal{T}_l)$ for each $i = 1, 2, \ldots, n$. By Proposition 5.7 (2), $\mathcal{T}(x)$ is $c_2(d+1)^{-1}M$-separated. It follows from (6.2) that $s_1, s_2, \ldots, s_n$ have a non-trivial intersection. In particular, any distinct $d+2$ elements in $\mathcal{T}_{l+1}(x)$ do not meet.

Now, we want to consider the situation that $s_1, s_2, \ldots, s_n$ are n distinct elements of $\mathcal{T}(x)$ with a non-trivial intersection. We will show that $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ also have a non-trivial intersection. From Proposition 5.7 (5), we may find $s_{n+1}, \ldots, s_{d+1}$ in $\mathcal{T}(x)$ such that $s_1, s_2, \ldots, s_{d+1}$ are all distinct and have a non-trivial intersection. It suffices to show that $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{d+1}$ have a non-trivial intersection. The proof is by contradiction. We assume that they do not meet. For each i, we put $p_i = \pi_x(s_i) \in P_U(x)$. Let $\tau = \{p_1, p_2, \ldots, p_{d+1}\}$. From the observation above, for each i, $\tilde{s}_i$ is contained in $B(s_i, \mathrm{diam}(\mathcal{T}_l))$. By Proposition 5.7 (6), $B(s_i, c_2(d+1)^{-1}M)$ does not meet $\mathrm{Conv}(\tau \setminus \{p_i\})$. It follows from (6.2) that $\tilde{s}_i$ does not meet $\mathrm{Conv}(\tau \setminus \{p_i\})$. Besides, by Proposition 5.7 (6), $B(\mathrm{Conv}(\tau), c_2(d+1)^{-1}M)$ is covered by s_i's. If $\mathrm{Conv}(\tau)$ meets $\tilde{s} \in \mathcal{T}_{l+1}(x)$ which is distinct from $\tilde{s}_i$'s, then $B(\mathrm{Conv}(\tau), c_2(d+1)^{-1}M)$ meets s, which is a contradiction. Hence $\mathrm{Conv}(\tau)$ is covered by $\tilde{s}_i$'s. We define a continuous function $f : \mathrm{Conv}(\tau) \to \mathbb{R}$ by

$$f(q) = \sum_{i=1}^{d+1} d(q, \tilde{s}_i).$$

Since we have assumed that $\tilde{s}_i$'s have trivial intersections, f is strictly positive. Define a continuous map $g : \mathrm{Conv}(\tau) \to \mathrm{Conv}(\tau)$ by

$$g(q) = f(q)^{-1} \sum_{i=1}^{d+1} d(q, \tilde{s}_i) p_i$$

for $q \in \mathrm{Conv}(\tau)$. It is easy to see that $g(q)$ is in $\mathrm{Conv}(\tau)$. By the Brouwer fixed point theorem, we can find a fixed point q of the map g. There exists i such that q is in $\tilde{s}_i$, because $\mathrm{Conv}(\tau)$ is contained in the union of $\tilde{s}_i$'s. By the definition of g, one has $g(q) \in \mathrm{Conv}(\tau \setminus \{p_i\})$. But, we have observed that $\tilde{s}_i$ does not meet $\mathrm{Conv}(\tau \setminus \{p_i\})$. This contradiction establishes the desired result.

Therefore, by conditions (4) and (5) of Proposition 5.7, we obtain conditions (3) and (4) of Theorem 6.1.

We next consider condition (2). Let $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ be elements in $\mathcal{T}_{l+1}(x)$ and let p be in $\mathbb{R}^d$. Suppose that $d(p, \tilde{s}_i)$ is less than $\mathrm{diam}(\mathcal{T}_l)$ for each $i = 1, 2, \ldots, n$. Since every point of $\tilde{s}_i$ is within distance $\mathrm{diam}(\mathcal{T}_l)$ of s_i, we get $d(p, s_i) < 2\,\mathrm{diam}(\mathcal{T}_l)$. From Proposition 5.7 (2), $\mathcal{T}(x)$ is $c_2(d+1)^{-1}M$-separated. It follows from (6.2) that $s_1, s_2, \ldots, s_n$ have a non-trivial intersection. Hence $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ also have a non-trivial intersection.

We now consider condition (6). Let $\tilde{s}$ be an element in $\mathcal{T}_{l+1}(x)$. From Proposition 5.7 (1), s contains an open ball $B(p, c_2(d+1)^{-1}M)$. We first claim

$$\begin{aligned}&\{t \in \mathcal{T}_l(x) \mid d(\pi_{l,x}(t), p) < c_2(d+1)^{-1}M - 2\,\mathrm{diam}(\mathcal{T}_l)\}\\ &\subset \{t \in \mathcal{T}_l(x) \mid t' \subset \tilde{s} \text{ for any } t' \in \mathcal{T}_l(x) \text{ with } t \cap t' \neq \emptyset\}.\end{aligned}$$

Indeed, if $d(\pi_{l,x}(t), p)$ is less than $c_2(d+1)^{-1}M - 2\,\mathrm{diam}(\mathcal{T}_l)$, then $d(\pi_{l,x}(t'), p)$ is less than $c_2(d+1)^{-1}M$ for any $t' \in \mathcal{T}_l(x)$ such that $t \cap t' \neq \emptyset$. It follows that t' is contained in $\tilde{s}$. Since $P_{U_l}(x)$ is K-syndetic, by Lemma 3.1 (2), one has

$$\begin{aligned}
&\#\{t \in \mathcal{T}_l(x) \mid t' \subset \tilde{s} \text{ for any } t' \in \mathcal{T}_l(x) \text{ with } t \cap t' \neq \emptyset\} \\
&\geq \#(P_{U_l}(x) \cap B(p, c_2(d+1)^{-1}M - 2\operatorname{diam}(\mathcal{T}_l))) \\
&> \left(\frac{c_2(d+1)^{-1}M - 2\operatorname{diam}(\mathcal{T}_l) - K}{K}\right)^d.
\end{aligned}$$

We next consider

$$E = P_X(x) \cap B(\partial\tilde{s}, \operatorname{diam}(\mathcal{T}_0)).$$

If q is in $\partial\tilde{s}$, then there exists $s_0 \in \mathcal{T}(x)$ which is distinct from s and $q \in \tilde{s} \cap \tilde{s}_0$. Therefore there exist $t, t_0 \in \mathcal{T}_l(x)$ such that $q \in t \cap t_0$, $\pi_{l,x}(t) \in s^*$ and $\pi_{l,x}(t_0) \in s_0^*$. If q is in the interior of s, then the line segment from q to $\pi_{l,x}(t_0)$ meets the boundary of s. If q is not in s, then the line segment from q to $\pi_{l,x}(t)$ meets the boundary of s. It follows that $d(q, \partial s)$ is not greater than $\operatorname{diam}(\mathcal{T}_l)$. From this, for any $u \in E$, we get

$$\begin{aligned}
B(u, M_0/2) &\subset B(\partial\tilde{s}, \operatorname{diam}(\mathcal{T}_0) + M_0/2) \\
&\subset B(\partial s, \operatorname{diam}(\mathcal{T}_0) + M_0/2 + \operatorname{diam}(\mathcal{T}_l)) \\
&= B(\partial s, L).
\end{aligned}$$

For any $u \neq u' \in E$, $B(u, M_0/2)$ and $B(u', M_0/2)$ are disjoint, because $P_X(x)$ is M_0-separated. This, together with Proposition 5.7 (3), implies that

$$\#E \times V_d(M_0/2)^d \leq 10^{d^2} 2LV_{d-1}(\sqrt{5}M + L)^{d-1}.$$

It follows from these estimates and (6.3) that

$$\#\{t \in \mathcal{T}_l(x) \mid t' \subset \tilde{s} \text{ for any } t' \in \mathcal{T}_l(x) \text{ with } t \cap t' \neq \emptyset\} \geq 10^{d^2} \times \#E,$$

which completes the proof of (6).

As a final point, we put $U_{l+1} = U$ and define the map $\pi_{l+1,x}(\tilde{s}) = \pi_x(s)$ for all x in X and $s \in \mathcal{T}(x)$. Let us check (7). From Proposition 5.7 (1), the open ball centred at $\pi_x(s)$ with radius $c_2(d+1)^{-1}M$ is contained in s. Since $c_2(d+1)^{-1}M$ is greater than $\operatorname{diam}(\mathcal{T}_l)$, $\pi_x(s)$ is in the interior of $\tilde{s}$ as required. □

7 AF equivalence relations and boundaries

Let $X \subset \Omega$ be a flat Cantor transversal of a free minimal action φ of $\mathbb{R}^d$ on Ω and let $R_\varphi \subset X \times X$ be the étale equivalence relation induced from (Ω, φ). We suppose that P_X is M_0-separated and M_1-syndetic.

In this section, we would like to use our earlier construction of a nested sequence of φ-regular tessellations to construct the data necessary in the application of the absorption theorem to give a proof of the main result. This needs, first of all, an AF relation. The obvious choice is by using the interiors of the cells in the tessellation. These equivalence relations are actually too large. We will refine them by considering all $(d+1)$-tuples $t_1, t_2, \ldots, t_d, t_{d+1}$ which have non-trivial intersection. At the same time, we also keep track of boundary sets B^{n+1} along the $(d-n)$-dimensional faces for each $n = 1, 2, \ldots, d$. We will apply the absorption theorem [M3, Theorem 3.2] d times: the n-th application of the absorption theorem enlarges the equivalence relation along the $(d-n)$-dimensional faces for $n = 1, 2, \ldots, d$.

We need some notation. Let $\{\mathcal{T}(x) \mid x \in X\}$ be a φ-regular tessellation. For x in X and any t in $\mathcal{T}(x)$, we let $N(x,t)$ denote the set of all t' in $\mathcal{T}(x)$, including t, which intersect t. For x in X and a $(d+1)$-tuple $\xi = (t_1, t_2, \ldots, t_{d+1})$ in $\mathcal{T}(x)$, the i-th coordinate of ξ is denoted by $\xi(i)$. For $n = 2, 3, \ldots, d+1$, let S_n denote the permutation group on $\{1, 2, \ldots, n\}$. We regard S_n as a subgroup of S_{d+1} in a obvious fashion. For $\sigma \in S_{d+1}$ and $\xi = (t_1, t_2, \ldots, t_{d+1})$, we define $\sigma(\xi)$ by $\sigma(\xi)(i) = \xi(\sigma^{-1}(i))$.

We begin with our refining sequence of φ-regular tessellations $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ provided by Theorem 6.1 of the last section. For any x in X, we let

$$i(x, \cdot) : \mathcal{T}_l(x) \to \mathcal{T}_{l+1}(x)$$

be the unique function such that $i(x,t) \supset t$ for every $t \in \mathcal{T}_l(x)$. If $k \geq 1$, we let i^k denote the composition of k functions i mapping $\mathcal{T}_l$ to $\mathcal{T}_{l+k}$ for any $l \geq 0$. By the φ-regularity, for any $x \in X$, $l \geq 0$, $k \geq 1$ and $t \in \mathcal{T}_l(x)$, there exists an open neighbourhood U of x such that $t \in \mathcal{T}_l(y)$ and $i^k(y,t) = i^k(x,t)$ for every $y \in U$.

For each $x \in X$ and $l \in \mathbb{N}$, we let $\mathcal{T}_l^{(1)}(x)$ denote the set of all $(d+1)$-tuples $\xi = (t_1, t_2, \ldots, t_{d+1})$ in $\mathcal{T}_l(x)$ such that $\bigcap_{i=1}^{d+1} t_i$ is non-empty and $t_i \neq t_j$ for $i \neq j$. Further, for each $n = 2, 3, \ldots, d+1$, we let

$$\mathcal{T}_l^{(n)}(x) = \left\{ \xi \in \mathcal{T}_l^{(1)}(x) \mid \#\{i^k(x, \xi(1)), \ldots, i^k(x, \xi(n))\} = n \text{ for all } k \geq 1 \right\}.$$

In other words, ξ is in $\mathcal{T}_l^{(n)}(x)$ if and only if $i^k(x,\xi(1)), i^k(x,\xi(2)), \ldots, i^k(x,\xi(n))$ are all distinct for any $k \geq 1$. Clearly we have

$$\mathcal{T}_l^{(1)}(x) \supset \mathcal{T}_l^{(2)}(x) \supset \cdots \supset \mathcal{T}_l^{(d+1)}(x).$$

First, we would like to define a surjective map $\theta_x : P_X(x) \to \mathcal{T}_0^{(1)}(x)$ for each $x \in X$ in a φ-regular fashion as follows. Consider all possible triples

$$(t, t^* \cap P_X(x), N(x,t)),$$

where x is in X and t is in $\mathcal{T}_0(x)$. We consider $(t_1, t_1^* \cap P_X(x_1), N(x_1, t_1))$ and $(t_2, t_2^* \cap P_X(x_2), N(x_2, t_2))$ to be equivalent if they are translates of one another, namely that there exists $p \in \mathbb{R}^d$ such that

$$t_2 = t_1 + p, \ t_2^* \cap P_X(x_2) = t_1^* \cap P_X(x_1) + p \text{ and } N(x_2, t_2) = N(x_1, t_1) + p.$$

Since $\mathcal{T}_0$ is φ-regular, there exists a finite number of equivalence classes. We let $\mathcal{P}$ be a finite set containing exactly one representative of each equivalence class.

Let (t, F, N) be in $\mathcal{P}$. For $p \in F$, we let $\theta(t, F, N, p)$ be a $(d+1)$-tuple $\xi = (t, t_1, t_2, \ldots, t_d)$ such that $t_i \in N$, $t \cap t_1 \cap \cdots \cap t_d \neq \emptyset$ and $\#\xi = d+1$. We show that we can choose $\theta(t, F, N, p)$ so that $\theta(t, F, N, \cdot)$ becomes a surjection from F to such $(d+1)$-tuples. From Theorem 6.1 (3), the number of such $(d+1)$-tuples ξ is less than $(10^d)^d$. On the other hand, $\mathcal{T}_0(x)$ has capacity $(10^d+1)M_1$ by Theorem 6.1 (1), and so each element of $\mathcal{T}_0(x)$ contains a ball of radius $(10^d+1)M_1$. It follows from Lemma 3.1 (2) that the cardinality of F is not less than $(10^d)^d$. Hence, we can choose $\theta(t, F, N, p)$ so that $\theta(t, F, N, \cdot)$ becomes a surjection from F to the $(d+1)$-tuples as above.

Haven chosen these items for our representative patterns $\mathcal{P}$, we extend their definition by translation as follows. Let x be in X and p be in $P_X(x)$. Take $t \in \mathcal{T}_0(x)$ such that $p \in t^*$. We find the unique q in $\mathbb{R}^d$ with $(t+q, t^* \cap P_X(x)+q, N(x,t)+q)$ in $\mathcal{P}$ and define

$$\theta_x(p) = \theta(t+q, t^* \cap P_X(x)+q, N(x,t)+q, p+q) - q.$$

The following lemma follows at once from the definitions and we omit the proof.

Lemma 7.1. *For any $x \in X$, we have the following.*

(1) $\theta_x : P_X(x) \to \mathcal{T}_0^{(1)}(x)$ *is surjective.*

(2) *For any $p \in P_X(x)$ and $t = \theta_x(p)(1)$, one has $p \in t^*$.*

(3) *For any $p, q \in P_X(x)$, $\theta_{\varphi^q(x)}(p-q) = \theta_x(p) - q$.*

(4) *For any $p \in P_X(x)$, there exists an open neighbourhood $U \subset X$ of x such that $p \in P_X(y)$ and $\theta_y(p) = \theta_x(p)$ for all $y \in U$.*

For each $n = 1, 2, 3, \ldots, d+1$, we define a subset B^n of X by

$$B^n = \{\varphi^p(x) \in X \mid x \in X, p \in P_X(x), \theta_x(p) \in \mathcal{T}_0^{(n)}(x)\}.$$

It is easy to see

$$X = B^1 \supset B^2 \supset B^3 \supset \cdots \supset B^{d+1}.$$

The subsets B^n are d-dimensional counterparts of minimal and maximal paths in Bratteli-Vershik models for minimal $\mathbb{Z}$-actions ([HPS]). Actually, when $d=1$, one can construct the refining sequence of tessellations so that B^2 consists only of two points, namely the minimal path and the maximal path. For general d, the 'boundary' has a hierarchic structure: B^n corresponds to $(d-n+1)$-dimensional faces of polytopes.

Lemma 7.2. *For every $n = 1, 2, 3, \ldots, d+1$, B^n is a non-empty closed subset of X.*

Proof. Fix $1 \leq n \leq d+1$. For $x \in X$ and $l \in \mathbb{N}$, we let

$$\mathcal{T}_{0,l}^{(n)}(x) = \left\{ \xi \in \mathcal{T}_0^{(1)}(x) \mid \#\{i^l(x,\xi(1)),\dots,i^l(x,\xi(n))\} = n \right\}.$$

Then $\mathcal{T}_{0,l}^{(n)}(x) \supset \mathcal{T}_{0,l+1}^{(n)}(x)$ and $\mathcal{T}_0^{(n)}(x) = \bigcap_l \mathcal{T}_{0,l}^{(n)}(x)$. Define

$$B_l^n = \{\varphi^p(x) \in X \mid x \in X, p \in P_X(x), \theta_x(p) \in \mathcal{T}_{0,l}^{(n)}(x)\}.$$

It is clear that $B_l^n \supset B_{l+1}^n$ and $B^n = \bigcap_l B_l^n$. Thus, it suffices to show that B_l^n is non-empty and closed.

By Theorem 6.1 (4), there exist $t_1, t_2, \dots, t_{d+1}$ in $\mathcal{T}_l(x)$ such that $t_i \neq t_j$ for $i \neq j$ and $t_1 \cap t_2 \cap \cdots \cap t_{d+1} \neq \emptyset$. Since each element of $\mathcal{T}_l(x)$ is a union of elements of $\mathcal{T}_0(x)$, we may find $s_1, s_2, \dots, s_{d+1}$ in $\mathcal{T}_0(x)$ such that $i^l(x, s_j) = t_j$ and $s_1 \cap s_2 \cap \cdots \cap s_{d+1} \neq \emptyset$. It follows that $(s_1, s_2, \dots, s_{d+1})$ belongs to $\mathcal{T}_{0,l}^{(d+1)}(x)$. Hence $\mathcal{T}_{0,l}^{(n)}(x)$ is non-empty for any $n = 1, 2, \dots, d+1$. From this, we see that B_l^n is non-empty.

Let us show that B_l^n is closed. Take $x = \varphi^0(x) \in X \setminus B_l^n$ arbitrarily. Put $\xi = \theta_x(0)$. Since ξ is not in $\mathcal{T}_{0,l}^{(n)}(x)$,

$$\#\{i^l(x,\xi(1)), i^l(x,\xi(2)), \dots, i^l(x,\xi(n))\} < n.$$

By the φ-regularity, there exists an open neighbourhood U of x such that, for any $y \in U$, one has $\theta_y(0) = \theta_x(0) = \xi$ and $i^l(y,\xi(j)) = i^l(x,\xi(j))$ for $j = 1, 2, \dots, n$. It follows that

$$\#\{i^l(y,\xi(1)), i^l(y,\xi(2)), \dots, i^l(y,\xi(n))\} < n.$$

Therefore one has $\theta_y(0) = \xi \notin \mathcal{T}_{0,l}^{(n)}(y)$ and $y = \varphi^0(y)$ is not in B_l^n. This then implies that U is contained in $X \setminus B_l^n$, and so B_l^n is closed. $\square$

Next, we would like to introduce equivalence relations Q^n and R^n on each B^n. First, for each $n = 2, 3, \dots, d+1$, we define an equivalence relation Q^n on B^n as follows. Let $\varphi^p(x)$ and $\varphi^q(x)$ be two points in B^n, where $\theta_x(p)$ and $\theta_x(q)$ are in $\mathcal{T}_0^{(n)}(x)$. Put $\xi = \theta_x(p)$ and $\eta = \theta_x(q)$. For every $l \in \mathbb{N}$, we let the pair $(\varphi^p(x), \varphi^q(x))$ be in Q_l^n, if

$$\{i^l(x,\xi(k)) \mid k = 1, 2, \dots, n\} = \{i^l(x,\eta(k)) \mid k = 1, 2, \dots, n\}.$$

It is easy to see that Q_l^n is an equivalence relation on B^n and $Q_l^n \subset Q_{l+1}^n$. We define

$$Q^n = \bigcup_{l \in \mathbb{N}} Q_l^n.$$

Lemma 7.3. *Equipped with the relative topology from R_φ, the equivalence relation Q^n on B^n is an AF relation for each $n = 2, 3, \dots, d+1$.*

Proof. It suffices to show that Q_l^n is a compact étale relation on B^n for each $l \in \mathbb{N}$. First, let us verify the étaleness of Q_l^n. Take $(\varphi^p(x), \varphi^q(x)) \in Q_l^n$, where $\xi = \theta_x(p)$ and $\eta = \theta_x(q)$ are in $\mathcal{T}_0^{(n)}(x)$ and

$$\{i^l(x, \xi(k)) \mid k = 1, 2, \ldots, n\} = \{i^l(x, \eta(k)) \mid k = 1, 2, \ldots, n\}.$$

There exists a clopen neighbourhood U of x in X such that $p, q \in P_X(y)$ for all $y \in U$. Put

$$O = \{(\varphi^p(y), \varphi^q(y)) \in R_\varphi \mid y \in U\}.$$

Then O is a clopen neighbourhood of $(\varphi^p(x), \varphi^q(x))$ in R_φ and the maps

$$r : (\varphi^p(y), \varphi^q(y)) \mapsto \varphi^p(y) \quad \text{and} \quad s : (\varphi^p(y), \varphi^q(y)) \mapsto \varphi^q(y)$$

are local homeomorphisms from O to $\varphi^p(U)$ and $\varphi^q(U)$, respectively. From Lemma 7.1 (4) and the definition of i, we may also assume that the neighbourhood U of x in X is chosen sufficiently small so that, for any $y \in U$, one has $\xi = \theta_y(p)$, $\eta = \theta_y(q)$ and $i^l(y, \xi(k)) = i^l(x, \xi(k))$, $i^l(y, \eta(k)) = i^l(x, \eta(k))$ for each $k = 1, 2, \ldots, n$.

It is clear that $O \cap Q_l^n$ is a clopen neighbourhood of $(\varphi^p(x), \varphi^q(x))$ in Q_l^n. In order to prove the étaleness of Q_l^n, we would like to show that the restriction of the map r to $O \cap Q_l^n$ is a homeomorphism to $\varphi^p(U) \cap B^n$, and it suffices to show that this map is a surjection. Choose $y \in U$ and suppose that $\varphi^p(y)$ is in B^n. It follows that $\theta_y(p) = \xi$ is in $\mathcal{T}_0^{(n)}(y)$, and so we have

$$\#\{i^m(y, \xi(1)), i^m(y, \xi(2)), \ldots, i^m(y, \xi(n))\} = n$$

for every $m \geq 0$. Since

$$\begin{aligned}
&\{i^l(y, \xi(1)), i^l(y, \xi(2)), \ldots, i^l(y, \xi(n))\} \\
&= \{i^l(x, \xi(1)), i^l(x, \xi(2)), \ldots, i^l(x, \xi(n))\} \\
&= \{i^l(x, \eta(1)), i^l(x, \eta(2)), \ldots, i^l(x, \eta(n))\} \\
&= \{i^l(y, \eta(1)), i^l(y, \eta(2)), \ldots, i^l(y, \eta(n))\},
\end{aligned}$$

we can see that

$$\#\{i^m(y, \eta(1)), i^m(y, \eta(2)), \ldots, i^m(y, \eta(n))\} = n$$

for every $m \geq 0$, that is, $\varphi^q(y)$ is in B^n. In addition, we have $(\varphi^p(y), \varphi^q(y))$ is in Q_l^n. Thus, $(\varphi^p(y), \varphi^q(y))$ is in $O \cap Q_l^n$, and so the map r is a local homeomorphism and Q_l^n is étale.

Similarly, it can be easily shown that Q_l^n is a closed subset of R_φ.

We next verify that Q_l^n is compact. Take $(\varphi^p(x), \varphi^q(x)) \in Q_l^n$, where $\xi = \theta_x(p)$ and $\eta = \theta_x(q)$ are in $\mathcal{T}_0^{(n)}(x)$ and

$$\{i^l(x, \xi(k)) \mid k = 1, 2, \ldots, n\} = \{i^l(x, \eta(k)) \mid k = 1, 2, \ldots, n\}.$$

From Lemma 7.1 (2), we have $p \in \xi(1)^*$ and $q \in \eta(1)^*$. By definition, one has

$$\xi(1)^* \subset \xi(1) \subset i^l(x, \xi(1)) \quad \text{and} \quad \eta(1)^* \subset \eta(1) \subset i^l(x, \eta(1)).$$

Since $i^l(x, \xi(1))$ meets $i^l(x, \eta(1))$, we can conclude that $d(p, q) \leq 2\,\mathrm{diam}(\mathcal{T}_l(x))$. It follows from the φ-regularity of $\mathcal{T}_l$ that $\mathrm{diam}(\mathcal{T}_l(x))$ is bounded uniformly over all $x \in X$. Therefore Q_l^n is compact. $\square$

We need to introduce another equivalence relation R^n on each B^n for $n = 1, 2, \ldots, d+1$. The relation R^n will be defined by using functions

$$j_n(x, \cdot) : \mathcal{T}_l^{(n)}(x) \to \mathcal{T}_{l+1}^{(n)}(x)$$

for every $x \in X$, $l \geq 0$ and $n = 1, 2, \ldots, d+1$. To define the map j_n, we need some notation.

For each $l \geq 0$, let U_l be the clopen set corresponding to the φ-regular tessellation $\mathcal{T}_l$ as described in Theorem 6.1. There exists a bijection between $P_{U_l}(x)$ and $\mathcal{T}_l(x)$ for each x in X. As mentioned in Section 4, we equip $\mathbb{R}^d$ with the lexicographic ordering. By transferring the lexicographic ordering on $P_{U_l}(x) \subset \mathbb{R}^d$, we can equip $\mathcal{T}_l(x)$ with a linear ordering. By the translation invariance of the lexicographic ordering and the φ-regularity of P_{U_l} and $\mathcal{T}_l$, we obtain the φ-regularity of the linear ordering on $\mathcal{T}_l(x)$ in the following sense.

- For any $x \in X$ and $p \in P_X(x)$, if $t, t' \in \mathcal{T}_l(x)$ satisfy $t < t'$, then $t - p, t' - p \in \mathcal{T}_l(\varphi^p(x))$ satisfy $t - p < t' - p$.
- For any $x \in X$ and $p \in P_X(x)$, there exists an open neighbourhood U of x in X such that, if $t, t' \in \mathcal{T}_l(x)$ satisfy $t < t'$, then $t, t' \in \mathcal{T}_l(y)$ satisfy $t < t'$ in $\mathcal{T}_l(y)$ for any $y \in U$.

By using this linear ordering on $\mathcal{T}_l(x)$, we can equip $\mathcal{T}_l^{(1)}(x)$ with the lexicographic ordering. It is clear that the ordering on $\mathcal{T}_l^{(1)}(x)$ is again φ-regular in an obvious sense.

We would like to define a map $a_l(x, \cdot)$ for each $l \geq 0$. Fix $l \geq 0$. For each $x \in X$, let $I_l(x) \subset \mathcal{T}_l(x)$ be the subset consisting of all $t \in \mathcal{T}_l(x)$ such that $i(x, N(x, t)) = \{i(x, t)\}$. Let $a_l(x, \cdot) : I_l(x) \to \mathcal{T}_{l+1}^{(1)}(x)$ be a map which satisfies the following.

- For any $t \in I_l(x)$, we have $a_l(x, t)(1) = i(x, t)$.
- For any $t \in I_l(x)$ and $p \in P_X(x)$, we have $a_l(\varphi^p(x), t - p) = a_l(x, t) - p$.
- For any $t \in I_l(x)$, there exists an open neighbourhood $U \subset X$ of x such that $t \in I_l(y)$ and $a_l(y, t) = a_l(x, t)$ for every $y \in U$.
- For any $\zeta \in \mathcal{T}_{l+1}^{(1)}(x)$ and $s = \zeta(1)$, we have

$$\#\{t \in I_l(x) \mid a_l(x, t) = \zeta\} \geq \#(P_X(x) \cap B(\partial s, \mathrm{diam}(\mathcal{T}_0(x)))).$$

The second and third conditions follow from the φ-regularity of $\mathcal{T}_l$ and $\mathcal{T}_{l+1}$. For $s \in \mathcal{T}_{l+1}(x)$, from Theorem 6.1 (3), the number of $(d+1)$-tuples $\zeta \in \mathcal{T}_{l+1}^{(1)}(x)$ such that $\zeta(1) = s$ is less than 10^{d^2}. This, together with Theorem 6.1 (6), implies the fourth condition.

We now turn to the definition of the map $j_n(x, \cdot) : \mathcal{T}_l^{(n)}(x) \to \mathcal{T}_{l+1}^{(n)}(x)$. Fix $1 \le n \le d+1$, $l \ge 0$ and x in X. Take $\xi \in \mathcal{T}_l^{(n)}(x)$. Put

$$M = \{i(x, \xi(1)), i(x, \xi(2)), \dots, i(x, \xi(d+1))\}$$

and

$$N = \{i(x, t) \mid t \in \mathcal{T}_l(x), t \cap \xi(1) \cap \xi(2) \cap \cdots \cap \xi(n) \neq \emptyset\}.$$

Clearly $M \subset N$. By definition of $\mathcal{T}_l^{(n)}(x)$, one has $n \le \#M$. It follows from Theorem 6.1 (2) that the elements in N have non-trivial intersection and the cardinality of N is not greater than $d+1$.

First, if $\#N = 1$ (this automatically implies $n = \#M = 1$), then we define $j_1(x, \xi) = a_l(x, \xi(1))$.

Let us consider the case of $\#N \ge 2$. For $m = 1, 2, \dots, \#M$, we let

$$k_m = \min\{k \mid \#\{i(x, \xi(1)), \dots, i(x, \xi(k))\} = m\}.$$

It is easily verified that $k_1 = 1, k_2 = 2, \dots, k_n = n$ and $k_n < k_{n+1} < \cdots < k_{\#M} \le d+1$. Consider all the $(d+1)$-tuples $\zeta \in \mathcal{T}_{l+1}^{(1)}(x)$ such that

$$\zeta(m) = i(x, \xi(k_m)) \text{ for all } m = 1, 2, \dots, \#M$$

and

$$\{\zeta(1), \zeta(2), \dots, \zeta(\#N)\} = N.$$

Define $j_n(x, \xi)$ to be the minimum element in the set of such $(d+1)$-tuples ζ with respect to the linear ordering on $\mathcal{T}_{l+1}^{(1)}(x)$. From Theorem 6.1 (4), such a $(d+1)$-tuple exists. Moreover, from Theorem 6.1 (3), there exist only finitely many such $(d+1)$-tuples. Hence $j_n(x, \xi) \in \mathcal{T}_{l+1}^{(1)}(x)$ is well-defined. Note that for any $k > \#N$, $j_n(x, \xi)(k)$ is the minimum element in the set of all tiles in $\mathcal{T}_{l+1}(x)$ which meet $j_n(x, \xi)(1) \cap j_n(x, \xi)(2) \cap \cdots \cap j_n(x, \xi)(k-1)$.

From the definition above, one has $j_n(x, \xi)(k) = i(x, \xi(k))$ for any $\xi \in \mathcal{T}_l^{(n)}(x)$ and $k = 1, 2, \dots, n$. Therefore $j_n(x, \xi)$ belongs to $\mathcal{T}_{l+1}^{(n)}(x)$. In addition, it is also easy to see that $j_n(x, \cdot)$ is φ-regular in an appropriate sense. We remark that $j_n(x, \xi)(k)$ for $k > \#N$ depends only on N.

The following lemma is an easy consequence of the definitions and we omit the proof.

Lemma 7.4. *For $1 \le n \le d+1$, $l \ge 0$ and $x \in X$, we have the following.*

1. *For any $\xi \in \mathcal{T}_l^{(m)}(x)$ and $n \le m$, one has $j_n(x, \xi)(k) = i(x, \xi(k))$ for each $k = 1, 2, \dots, m$.*

(2) *For any* $\xi \in \mathcal{T}_l^{(n)}(x)$ *and* $m = 1, 2, \dots, d+1$, *one has*

$$\{j_n(x,\xi)(k) \mid k = 1, 2, \dots, m'\} = \{i(x, \xi(k)) \mid k = 1, 2, \dots, m\},$$

where $m' = \#\{i(x,\xi(1)), \dots, i(x,\xi(m))\}$.

(3) *If* $\xi, \eta \in \mathcal{T}_l^{(n)}(x)$ *satisfy*

$$\{\xi(k) \mid k = 1, 2, \dots, n\} = \{\eta(k) \mid k = 1, 2, \dots, n\},$$

then there exists $\sigma \in S_{d+1}$ *such that*

$$\sigma(j_n(x,\xi)) = j_n(x,\eta) \text{ and } \sigma(\{1, 2, \dots, n\}) = \{1, 2, \dots, n\}.$$

(4) *For* $\xi \in \mathcal{T}_l^{(n)}(x)$ *and* $\sigma \in S_m$ *with* $\sigma(\{1, 2, \dots, n\}) = \{1, 2, \dots, n\}$, *there exists* $\sigma' \in S_{m'}$ *such that*

$$\sigma'(j_n(x,\xi)) = j_n(x, \sigma(\xi))$$

and

$$\sigma'(k) = \sigma(k) \text{ for all } k = 1, 2, \dots, n,$$

where $m' = \#\{i(x,\xi(1)), \dots, i(x,\xi(m))\}$.

For $k \geq 1$, we let $j_n^k(x, \cdot)$ denote the composition of k functions $j_n(x, \cdot)$ mapping $\mathcal{T}_l^{(n)}(x)$ to $\mathcal{T}_{l+k}^{(n)}(x)$ for any $l \geq 0$.

For each $n = 1, 2, \dots, d+1$, we define an equivalence relation R^n on B^n as follows. Let $\varphi^p(x)$ and $\varphi^q(x)$ be two points in B^n, where $\theta_x(p)$ and $\theta_x(q)$ are in $\mathcal{T}_0^{(n)}(x)$. Put $\xi = \theta_x(p)$ and $\eta = \theta_x(q)$. For every $l \in \mathbb{N}$, we let the pair $(\varphi^p(x), \varphi^q(x))$ be in R_l^n, if there exists $\sigma \in S_n$ such that

$$\sigma(j_n^l(x,\xi)) = j_n^l(x,\eta).$$

It is easy to see that R_l^n is an equivalence relation on B^n. If $\sigma(j_n^l(x,\xi)) = j_n^l(x,\eta)$, then by applying Lemma 7.4 (4) to the case of m=n, we get

$$\begin{aligned}\sigma(j_n^{l+1}(x,\xi)) &= \sigma(j_n(x, j_n^l(x,\xi))) = j_n(x, \sigma(j_n^l(x,\xi))) \\ &= j_n(x, j_n^l(x,\eta)) = j_n^{l+1}(x,\eta).\end{aligned}$$

Hence R_l^n is contained in R_{l+1}^n. Define an equivalence relation R^n on B^n by

$$R^n = \bigcup_{l \in \mathbb{N}} R_l^n.$$

The following lemma can be shown in a similar way to Lemma 7.3. We omit the proof.

Lemma 7.5. *Equipped with the relative topology from* R_φ, *the equivalence relation* R^n *on* B^n *is an AF relation for each* $n = 1, 2, \dots, d+1$.

We wish to establish several facts about R^n and Q^n. Let us collect notation and terminology about equivalence relations. Let R be an equivalence relation on X. For a subset $A \subset X$, we set

$$R[A] = \{x \in X \mid \text{there exists } y \in A \text{ such that } (x, y) \in R\}.$$

For $x \in X$, we denote $R[\{x\}]$ by $R[x]$ and call it the R-orbit of x. For a subset $Y \subset X$, we let $R|Y$ denote $R \cap (Y \times Y)$ and call it the restriction of R to Y. Suppose that R is equipped with a topology in which R is étale ([GPS2, Definition 2.1]). A closed subset $Y \subset X$ is called R-étale or étale with respect to R, if the restriction $R|Y$ with the relative topology from R is étale. A subset $Y \subset X$ is called R-thin, if $\mu(Y)$ is zero for any R-invariant probability measure μ on X.

We begin with the following.

Lemma 7.6. *The AF relation R^1 on $B^1 = X$ is minimal.*

Proof. Take $x \in X$ and a non-empty clopen subset $U \subset X$ arbitrarily. It suffices to show that $R^1[x]$ meets U. There exists $M > 0$ such that $P_U(x)$ is M-syndetic.

Let l be a natural number such that $(10^d + l + 1)M_1 \geq M$. Put $\zeta = j_1^{l+1}(x, \theta_x(0))$. By the definition of $a_l(x, \cdot)$, there exists $t \in I_l(x)$ such that $a_l(x, t) = \zeta$. It follows from Theorem 6.1 (1) that the tile t contains an open ball of radius $(10^d + l + 1)M_1$. Hence $P_U(x)$ meets the interior of t. Thus, there exists $p \in P_U(x)$ such that $i^l(x, \theta_x(p)(1)) = t$. From this, we get $j_1^l(x, \theta_x(p))(1) = t$, which implies $j_1^{l+1}(x, \theta_x(p)) = a_l(x, t) = \zeta$. Therefore $(x, \varphi^p(x))$ is in R^1_{l+1}. Since $\varphi^p(x)$ is in U, $R^1[x]$ meets U as required. □

Lemma 7.7. *The closed subset $B^2 \subset X$ is R^1-thin.*

Proof. For each $l \in \mathbb{N}$, let B^2_l be as in Lemma 7.2. Since $\bigcap_l B^2_l = B^2$, it suffices to show $\mu(B^2_l) \to 0$ as $l \to \infty$ for any R^1-invariant measure μ. Fix $l \in \mathbb{N}$ and $x \in X$. We will compare $\#R^1_{l+1}[x]$ and $\#(R^1_{l+1}[x] \cap B^2_{l+1})$. Let $\zeta = j_1^{l+1}(x, \theta_x(0))$ and $s = \zeta(1)$.

Suppose that $t \in \mathcal{T}_l(x)$ is in $I_l(x)$ and $a_l(x, t) = \zeta$. From the definition of $a_l(x, \cdot)$, we notice that the number of such tiles t is not less than

$$\#(P_X(x) \cap B(\partial s, \operatorname{diam}(\mathcal{T}_0(x)))).$$

From the definition of $j_1(x, \cdot)$, for any $\xi \in \mathcal{T}_l^{(1)}(x)$ such that $\xi(1) = t$, one has $j_1(x, \xi) = \zeta$. If $p \in P_X(x)$ is contained in the interior of t, then $i^l(x, \theta_x(p)(1))$ is equal to t by Lemma 7.1 (2) and the definition of $i(x, \cdot)$. By the repeated use of Lemma 7.4 (1), we get $j_1^l(x, \theta_x(p))(1) = t$. Therefore $j_1^{l+1}(x, \theta_x(p)) = \zeta$. Thus, if $p \in P_X(x)$ is contained in the interior of t, then $(x, \varphi^p(x))$ is in R^1_{l+1}. By Theorem 6.1 (1), t contains an open ball of radius $(10^d + l + 1)M_1$. It follows from Lemma 3.1 (2) that the number of points in the intersection of $P_X(x)$ and the interior of t is not less than $(10^d + l)^d$. Hence we have

$$\begin{aligned}\#R^1_{l+1}[x] &= \#\{p \in P_X(x) \mid (x, \varphi^p(x)) \in R^1_{l+1}\} \\ &\geq (10^d + l)^d \times \#(P_X(x) \cap B(\partial s, \operatorname{diam}(\mathcal{T}_0(x)))).\end{aligned}$$

Next, suppose that $\varphi^p(x)$ is in $R^1_{l+1}[x] \cap B^2_{l+1}$. Put $\eta = \theta_x(p)$. By Lemma 7.1 (2), p belongs to $\eta(1)^*$. From $(x, \varphi^p(x)) \in R^1_{l+1}$, one has $j_1^{l+1}(x,\eta) = \zeta$, and so

$$i^{l+1}(x, \eta(1)) = j_1^{l+1}(x,\eta)(1) = \zeta(1) = s.$$

Since $\varphi^p(x)$ is in B^2_{l+1}, we have $i^{l+1}(x,\eta(1)) \neq i^{l+1}(x,\eta(2))$. By the definition of the map i, $\eta(1) \subset i^{l+1}(x,\eta(1))$ and $\eta(2) \subset i^{l+1}(x,\eta(2))$. Therefore

$$d(p, \partial s) \leq d(p, s \cap i^{l+1}(x,\eta(2))) \leq d(p, \eta(1) \cap \eta(2)) < \operatorname{diam}(\mathcal{T}_0(x)).$$

It follows that

$$\#\{p \in P_X(x) \mid \varphi^p(x) \in R^1_{l+1}[x] \cap B^2_{l+1}\} \leq \#(P_X(x) \cap B(\partial s, \operatorname{diam}(\mathcal{T}_0(x)))).$$

Consequently,

$$\mu(B^2_{l+1}) \leq (10^d + l)^{-d}$$

for any R^1-invariant probability measure μ on X, which completes the proof. $\square$

Lemma 7.8. *For each $n = 2, 3, \ldots, d+1$, R^n is an open subrelation of Q^n.*

Proof. To show $R^n \subset Q^n$, let $(\varphi^p(x), \varphi^q(x))$ be a pair in R^n. Let $\xi = \theta_x(p)$ and $\eta = \theta_x(q)$. There exists a natural number l and $\sigma \in S_n$ such that $\sigma(j_n^l(x,\xi)) = j_n^l(x,\eta)$. By the repeated use of Lemma 7.4 (1), we get

$$j_n^l(x,\xi)(k) = i^l(x,\xi(k)) \quad \text{and} \quad j_n^l(x,\eta)(k) = i^l(x,\eta(k))$$

for any $k = 1, 2, \ldots, n$. Hence

$$\{i^l(x,\xi(k)) \mid k = 1, 2, \ldots, n\} = \{i^l(x,\eta(k)) \mid k = 1, 2, \ldots, n\},$$

which implies that $(\varphi^p(x), \varphi^q(x))$ is in Q^n.

Since both R^n and Q^n are étale with the induced topology from R_φ, we see that R^n is open in Q^n. $\square$

Lemma 7.9. *For each $n = 1, 2, \ldots, d$, the closed subset B^{n+1} is R^n-étale.*

Proof. Let $(\varphi^p(x), \varphi^q(x))$ be a pair in $R^n|B^{n+1}$. It suffices to show that there exists a clopen neighbourhood $U \subset X$ of x such that for any $y \in U$, if $(\varphi^p(y), \varphi^q(y))$ is in R^n and $\varphi^p(y)$ is in B^{n+1}, then $\varphi^q(y)$ is also in B^{n+1}.

Put $\xi = \theta_x(p)$ and $\eta = \theta_x(q)$. Since $(\varphi^p(x), \varphi^q(x))$ is in $R^n|B^{n+1}$, ξ and η are in $\mathcal{T}_0^{(n+1)}(x)$ and there exists $l \in \mathbb{N}$ and $\sigma \in S_n$ such that $\sigma(j_n^l(x,\xi)) = j_n^l(x,\eta)$. By the repeated use of Lemma 7.4 (1), we get

$$j_n^l(x,\xi)(k) = i^l(x,\xi(k)) \quad \text{and} \quad j_n^l(x,\eta)(k) = i^l(x,\eta(k))$$

for any $k = 1, 2, \ldots, n+1$. It follows that

$$\{i^l(x,\xi(k)) \mid k = 1, 2, \ldots, n+1\} = \{i^l(x,\eta(k)) \mid k = 1, 2, \ldots, n+1\}.$$

From Lemma 7.1 (4), there exists a clopen neighbourhood U of x such that for any $y \in U$, $\xi = \theta_y(p)$ and $\eta = \theta_y(q)$. We may also assume that U is chosen sufficiently small so that, for any $y \in U$ and $k = 1, 2, \dots, n+1$,

$$i^l(y, \xi(k)) = i^l(x, \xi(k)) \quad \text{and} \quad i^l(y, \eta(k)) = i^l(x, \eta(k)).$$

Suppose that y is in U, $(\varphi^p(y), \varphi^q(y))$ is in R^n and $\varphi^p(y)$ is in B^{n+1}. We get

$$\begin{aligned}
&\{i^l(y, \eta(k)) \mid k = 1, 2, \dots, n+1\} \\
&= \{i^l(x, \eta(k)) \mid k = 1, 2, \dots, n+1\} \\
&= \{i^l(x, \xi(k)) \mid k = 1, 2, \dots, n+1\} \\
&= \{i^l(y, \xi(k)) \mid k = 1, 2, \dots, n+1\}.
\end{aligned}$$

Combining this with $\varphi^p(y) \in B^{n+1}$, we can conclude that $\varphi^q(y)$ is in B^{n+1}, which completes the proof. □

Lemma 7.10. *For each $n = 1, 2, \dots, d$, $R^n|B^{n+1}$ is an open subrelation of Q^{n+1}.*

Proof. From the proof of the lemma above, it is clear that $R^n|B^{n+1}$ is contained in Q^{n+1}. Since both relations are étale with the induced topology from R_φ, we see that $R^n|B^{n+1}$ is open in Q^{n+1}. □

Lemma 7.11. *For each $n = 1, 2, \dots, d$, we have $R^n[B^{n+1}] = B^n \cap R_\varphi[B^{n+1}]$.*

Proof. Clearly $R^n[B^{n+1}]$ is a subset of $B^n \cap R_\varphi[B^{n+1}]$, and so it suffices to show the other inclusion.

Let x be in $B^n \cap R_\varphi[B^{n+1}]$. Put $\xi = \theta_x(0)$. For any $l \in \mathbb{N}$, we have

$$\#\{i^l(x, \xi(k)) \mid k = 1, 2, \dots, n\} = n.$$

From Lemma 7.4 (1), we also get $j^l_n(x, \xi)(k) = i^l(x, \xi(k))$ for any $l \in \mathbb{N}$ and $k = 1, 2, \dots, n$. There exists $p \in P_X(x)$ such that $\varphi^p(x) \in B^{n+1}$. Put $\eta = \theta_x(p)$. Similarly, we have

$$\#\{i^l(x, \eta(k)) \mid k = 1, 2, \dots, n+1\} = n+1$$

for any $l \in \mathbb{N}$. Choose $L \in \mathbb{N}$ sufficiently large so that

$$\operatorname{diam}(\mathcal{T}_{L-1}) > \max\{d(0, \xi(k)) \mid k = 1, 2, \dots, n\}$$

and

$$\operatorname{diam}(\mathcal{T}_{L-1}) > \max\{d(0, \eta(k)) \mid k = 1, 2, \dots, n+1\}.$$

From Theorem 6.1 (2), $\mathcal{T}_L(x)$ is $\operatorname{diam}(\mathcal{T}_{L-1})$-separated. Hence we see that, for any $k = 1, 2, \dots, n+1$, $i^L(x, \eta(k))$ meets

$$i^L(x, \xi(1)) \cap i^L(x, \xi(2)) \cap \dots \cap i^L(x, \xi(n)).$$

Define

$$M = \{j_n^{L+1}(x,\xi)(k) \mid k = 1,2,\dots,d+1\}.$$

It follows from the definition of the map j_n that $i^{L+1}(x,\eta(k))$ is in M for each $k = 1,2,\dots,n+1$. For each $l > L$, we put

$$m_l = \#\{i^{l-L}(x,t) \mid t \in M\}.$$

It is easily seen that $(m_l)_{l>L}$ is a decreasing sequence of positive integers. Therefore, the limit exists. In addition, from the argument above, m_l is not less than $n+1$, and so the limit is not less than $n+1$. By the repeated use of Lemma 7.4 (2), for all $l > L$, we have

$$\{j_n^l(x,\xi)(k) \mid k = 1,2,\dots,m_l\} = \{i^{l-L}(x,t) \mid t \in M\}.$$

Hence, there exists $L' > L$ such that $j_n^{L'}(x,\xi)$ is in $\mathcal{T}_{L'}^{(n+1)}(x)$. Since we can write each element of $\mathcal{T}_{L'}(x)$ as a union of elements of $\mathcal{T}_0(x)$, there exists $\zeta \in \mathcal{T}_0^{(n+1)}(x)$ such that

$$i^{L'}(x,\zeta(k)) = j_n^{L'}(x,\xi)(k)$$

for all $k = 1,2,\dots,d+1$. From this we have $j_n^{L'}(x,\zeta) = j_n^{L'}(x,\xi)$. By Lemma 7.1 (1), we can find $q \in P_X(x)$ such that $\theta_x(q) = \zeta$. It is easy to see that $\varphi^q(x)$ is in B^{n+1} and $(x,\varphi^q(x))$ is in R^n. $\square$

For each $n = 1,2,\dots,d$, we define a subset C^n of X by

$$C^n = R_\varphi[B^n] \setminus R_\varphi[B^{n+1}].$$

We let $C^{d+1} = R_\varphi[B^{d+1}]$. Clearly $C^1, C^2, \dots, C^{d+1}$ are mutually disjoint and R_φ-invariant. Besides, their union is equal to X.

Lemma 7.12. *For each $n = 1,2,\dots,d+1$, we have $R^n|(B^n \cap C^n) = R_\varphi|(B^n \cap C^n)$.*

Proof. Clearly $R^n|(B^n \cap C^n)$ is contained in $R_\varphi|(B^n \cap C^n)$, and so it suffices to show the other inclusion.

Let $(\varphi^p(x), \varphi^q(x))$ be a pair in $R_\varphi|(B^n \cap C^n)$. Put $\xi = \theta_x(p)$ and $\eta = \theta_x(q)$. From $\varphi^p(x), \varphi^q(x) \in B^n$, for any $l \in \mathbb{N}$, one has

$$\#\{i^l(x,\xi(k)) \mid k = 1,2,\dots,n\} = n$$

and

$$\#\{i^l(x,\eta(k)) \mid k = 1,2,\dots,n\} = n.$$

We first claim that for any finite subset $M \subset \mathcal{T}_0(x)$, there exists $L > 0$ such that

$$\#\{i^L(x,t) \mid t \in M\} \le n.$$

The proof is by contradiction. Suppose that $M \subset \mathcal{T}_0(x)$ is a finite subset satisfying

$$\#\{i^l(x,t) \mid t \in M\} > n$$

for any $l > 0$. By taking a subset of M if necessary, we may assume that $M = \{t_1, t_2, \dots, t_{n+1}\}$ and

$$\#\{i^l(x,t_k) \mid k = 1,2,\dots,n+1\} = n+1$$

for any $l > 0$. Choose $L \in \mathbb{N}$ sufficiently large so that

$$\operatorname{diam}(\mathcal{T}_{L-1}) > \max\{d(0,t_k) \mid k = 1,2,\dots,n+1\}.$$

From Theorem 6.1 (2), $\mathcal{T}_L(x)$ is $\operatorname{diam}(\mathcal{T}_{L-1})$-separated. Since

$$d(0, i^L(x,t_k)) \leq d(0,t_k) < \operatorname{diam}(\mathcal{T}_{L-1}),$$

we can see that

$$i^L(x,t_1) \cap i^L(x,t_2) \cap \cdots \cap i^L(x,t_{n+1}) \neq \emptyset.$$

Since we can write each element of $\mathcal{T}_L(x)$ as a union of elements in $\mathcal{T}_0(x)$, there exist $s_1, s_2, \dots, s_{n+1}$ in $\mathcal{T}_0(x)$ such that

$$s_1 \cap s_2 \cap \cdots \cap s_{n+1} \neq \emptyset$$

and $i^L(x,s_k) = i^L(x,t_k)$ for each $k = 1,2,\dots,n+1$. We note that

$$\#\{i^l(x,s_k) \mid k = 1,2,\dots,n+1\} = n+1$$

for any $l > 0$. By Theorem 6.1 (4), there exists ζ in $\mathcal{T}_0^{(n+1)}(x)$ such that $\zeta(k) = s_k$ for each $k = 1,2,\dots,n+1$. This contradicts $x \notin R_\varphi[B^{n+1}]$, and so the claim follows.

By applying this claim to a finite set

$$\{\xi(k) \mid k = 1,2,\dots,n\} \cup \{\eta(k) \mid k = 1,2,\dots,n\},$$

we can find $L > 0$ such that

$$\{i^L(x,\xi(k)) \mid k = 1,2,\dots,n\} = \{i^L(x,\eta(k)) \mid k = 1,2,\dots,n\}.$$

It follows from Lemma 7.4 (3) that there exists $\sigma \in S_d$ such that

$$\sigma(j_n^{L+1}(x,\xi)) = j_n^{L+1}(x,\eta) \text{ and } \sigma(\{1,2,\dots,n\}) = \{1,2,\dots,n\}.$$

Define

$$N = \{j_n^{L+1}(x,\xi)(k) \mid k = 1,2,\dots,d+1\}.$$

For $l > L$, we let

$$m_l = \#\{i^{l-L}(x,t) \mid t \in N\}.$$

It is easily seen that $(m_l)_{l>L}$ is a decreasing sequence of positive integers. Since x is in B^n, m_l is not less than n for any $l > L$. It follows from exactly the same argument as in the claim above that there exists $L' > L$ such that $m_{L'} = n$. By the repeated use of Lemma 7.4 (4), we can find $\sigma' \in S_n$ such that

$$\sigma'(j_n^{L'}(x, \xi)) = j_n^{L'}(x, \eta).$$

Therefore the pair $(\varphi^p(x), \varphi^q(x))$ belongs to R^n. □

We are now ready to give a proof of the main result. When $(S, \mathcal{O})$ is a topological space and A is a subset of S, to simplify notation, we denote the induced topology on A by $\mathcal{O}$, too.

Proof of Theorem 2.4. We first note that the equivalence relations $R^1, R^2, \dots, R^{d+1}$ and $Q^2, Q^3, \dots, Q^{d+1}$ are subsets of R_φ. The proof will be completed by using the absorption theorem [M3, Theorem 3.2] and the splitting theorem [M3, Theorem 2.1] repeatedly.

Let $\mathcal{O}_1$ be the étale topology on R_φ. By Lemma 7.5 and 7.6, $(R^1, \mathcal{O}_1)$ is a minimal AF relation on $B^1 = X$. By Lemma 7.7, B^2 is a closed R^1-thin subset. By Lemma 7.9, B^2 is étale with respect to $(R^1, \mathcal{O}_1)$. By Lemma 7.3, $(Q^2, \mathcal{O}_1)$ is an AF relation on B^2. By Lemma 7.10, $R^1|B^2$ is an open subrelation of Q^2. Then [M3, Theorem 3.2] applies and yields an étale topology $\mathcal{O}_2$ on $R^1 \vee Q^2$ satisfying the following.

(1-a) $(R^1 \vee Q^2, \mathcal{O}_2)$ is a minimal AF relation on X.

(1-b) B^2 is $(R^1 \vee Q^2)$-thin.

(1-c) B^2 is étale with respect to $(R^1 \vee Q^2, \mathcal{O}_2)$.

(1-d) Two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ agree on Q^2.

By Lemma 7.12, one has $R^1|C^1 = R_\varphi|C^1$. In particular, $(R^1 \vee Q^2)|C^1 = R_\varphi|C^1$. From Lemma 7.11, we have $R^1[B^2] = R_\varphi[B^2]$. By Lemma 7.12, we also have $R^2|(B^2 \cap C^2) = R_\varphi|(B^2 \cap C^2)$. Since R^2 is a subrelation of Q^2 by Lemma 7.8, one gets $Q^2|(B^2 \cap C^2) = R_\varphi|(B^2 \cap C^2)$. Combining these, we have $(R^1 \vee Q^2)|C^2 = R_\varphi|C^2$. Hence we obtain the following.

(1-e) $(R^1 \vee Q^2)|(C^1 \cup C^2) = R_\varphi|(C^1 \cup C^2)$.

Next, we would like to apply the splitting theorem [M3, Theorem 2.1] to $(R^1 \vee Q^2, \mathcal{O}_2)$, B^2 and R^2. By Lemma 7.8, R^2 is an open subrelation of Q^2 with the topology $\mathcal{O}_1$. Evidently $(R^1 \vee Q^2)|B^2$ is equal to Q^2. It follows from (1-d) that R^2 is an open subrelation of $(R^1 \vee Q^2)|B^2$ with the topology $\mathcal{O}_2$. Combining this with (1-a), (1-b) and (1-c), we can apply [M3, Theorem 2.1] and obtain a subrelation $\widetilde{R}^2$ of $R^1 \vee Q^2$ satisfying the following.

(1-f) $(\widetilde{R}^2, \mathcal{O}_2)$ is a minimal AF relation.

(1-g) $\widetilde{R}^2|B^2$ equals R^2.

(1-h) $\widetilde{R}^2[B^2]$ equals $(R^1 \vee Q^2)[B^2]$.

(1-i) If $x \in X$ does not belong to $(R^1 \vee Q^2)[B^2]$, then $\widetilde{R}^2[x]$ equals $(R^1 \vee Q^2)[x]$.

(1-j) Any $\widetilde{R}^2$-invariant measure on X is $(R^1 \vee Q^2)$-invariant.

By Lemma 7.12, $R^2|(B^2 \cap C^2)$ is equal to $R_\varphi|(B^2 \cap C^2)$. This together with (1-g), (1-h) and (1-i) implies that $\widetilde{R}^2|(C^1 \cup C^2)$ is equal to $(R^1 \vee Q^2)|(C^1 \cup C^2)$. Combining this with (1-e), one gets the following.

(1-k) $\widetilde{R}^2|(C^1 \cup C^2) = R_\varphi|(C^1 \cup C^2)$.

Moreover, we claim the following.

(1-l) $\widetilde{R}^2[B^3] = R_\varphi[B^3]$.

To verify this, suppose that (x, y) is in R_φ and $y \in B^3$. By $R^1[B^2] = R_\varphi[B^2]$ and (1-h), there exists $z \in B^2$ such that $(x, z) \in \widetilde{R}^2$. In particular, z is in $B^2 \cap R_\varphi[B^3]$. From Lemma 7.11, z is in $R^2[B^3]$. It follows from (1-g) that there exists $w \in B^3$ such that $(z, w) \in \widetilde{R}^2$. Thus, x belongs to $\widetilde{R}^2[B^3]$.

We next wish to apply the absorption theorem [M3, Theorem 3.2] to $(\widetilde{R}^2, \mathcal{O}_2)$, B^3 and Q^3. Let us check the hypotheses. By (1-f), $(\widetilde{R}^2, \mathcal{O}_2)$ is a minimal AF relation on X. From (1-b) and (1-j), B^3 is $\widetilde{R}^2$-thin. By Lemma 7.9, B^3 is étale with respect to $(R^2, \mathcal{O}_1)$. From (1-g), $\widetilde{R}^2|B^3 = R^2|B^3$. By (1-d), two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ agree on $R^2|B^3$, because R^2 is a subset of Q^2. It follows that B^3 is étale with respect to $(\widetilde{R}^2, \mathcal{O}_2)$. By Lemma 7.10 and (1-d), $\widetilde{R}^2|B^3 = R^2|B^3$ with the topology $\mathcal{O}_2$ is an open subrelation of Q^3 with the topology $\mathcal{O}_1$. Therefore we can apply [M3, Theorem 3.2] and obtain an étale topology $\mathcal{O}_3$ on $\widetilde{R}^2 \vee Q^3$ satisfying the following.

(2-a) $(\widetilde{R}^2 \vee Q^3, \mathcal{O}_3)$ is a minimal AF relation on X.

(2-b) B^3 is $(\widetilde{R}^2 \vee Q^3)$-thin.

(2-c) B^3 is étale with respect to $(\widetilde{R}^2 \vee Q^3, \mathcal{O}_3)$.

(2-d) Two topologies $\mathcal{O}_1$ and $\mathcal{O}_3$ agree on Q^3.

By Lemma 7.12, we have $R^3|(B^3 \cap C^3) = R_\varphi|(B^3 \cap C^3)$. Since R^3 is a subrelation of Q^3 by Lemma 7.8, one gets $Q^3|(B^3 \cap C^3) = R_\varphi|(B^3 \cap C^3)$. Combining this with (1-l), we get $(\widetilde{R}^2 \vee Q^3)|C^3 = R_\varphi|C^3$. This, together with (1-k), implies the following.

(2-e) $(\widetilde{R}^2 \vee Q^3)|(C^1 \cup C^2 \cup C^3) = R_\varphi|(C^1 \cup C^2 \cup C^3)$.

Next, we would like to apply the splitting theorem [M3, Theorem 2.1] to $(\widetilde{R}^2 \vee Q^3, \mathcal{O}_3)$, B^3 and R^3. By Lemma 7.8, R^3 is an open subrelation of Q^3 with the topology $\mathcal{O}_1$. Evidently $(\widetilde{R}^2 \vee Q^3)|B^3$ is equal to Q^3. It follows from (2-d) that R^3 is an open subrelation of $(\widetilde{R}^2 \vee Q^3)|B^3$ with the topology $\mathcal{O}_3$. Combining this with (2-a), (2-b) and (2-c), we can apply [M3, Theorem 2.1] and obtain a subrelation $\widetilde{R}^3$ of $\widetilde{R}^2 \vee Q^3$ satisfying the following.

(2-f) $(\widetilde{R}^3, \mathcal{O}_3)$ is a minimal AF relation.

(2-g) $\widetilde{R}^3|B^3$ equals R^3.

(2-h) $\widetilde{R}^3[B^3]$ equals $(\widetilde{R}^2 \vee Q^3)[B^3]$.

(2-i) If $x \in X$ does not belong to $(\widetilde{R}^2 \vee Q^3)[B^3]$, then $\widetilde{R}^3[x]$ equals $(\widetilde{R}^2 \vee Q^3)[x]$.

(2-j) Any $\widetilde{R}^3$-invariant measure on X is $(\widetilde{R}^2 \vee Q^3)$-invariant.

By Lemma 7.12, $R^3|(B^3 \cap C^3)$ is equal to $R_\varphi|(B^3 \cap C^3)$. This together with (2-g), (2-h) and (2-i) implies that $\widetilde{R}^3|(C^1 \cup C^2 \cup C^3)$ is equal to $(\widetilde{R}^2 \vee Q^3)|(C^1 \cup C^2 \cup C^3)$. Combining this with (2-e), one gets the following.

(2-k) $\widetilde{R}^3|(C^1 \cup C^2 \cup C^3) = R_\varphi|(C^1 \cup C^2 \cup C^3)$.

Moreover, we claim the following.

(2-l) $\widetilde{R}^3[B^4] = R_\varphi[B^4]$.

To verify this, suppose that (x, y) is in R_φ and $y \in B^4$. By (1-l) and (2-h), there exists $z \in B^3$ such that $(x, z) \in \widetilde{R}^3$. In particular, z is in $B^3 \cap R_\varphi[B^4]$. From Lemma 7.11, z is in $R^3[B^4]$. It follows from (2-g) that there exists $w \in B^4$ such that $(z, w) \in \widetilde{R}^3$. Thus, x belongs to $\widetilde{R}^3[B^4]$.

By repeating these arguments, we finally obtain a subrelation $\widetilde{R}^d$ of R_φ and an étale topology $\mathcal{O}_{d+1}$ on $\widetilde{R}^d \vee Q^{d+1}$ satisfying the following.

(d-a) $(\widetilde{R}^d \vee Q^{d+1}, \mathcal{O}_{d+1})$ is a minimal AF relation on X.

(d-e) $(\widetilde{R}^d \vee Q^{d+1})|(C^1 \cup C^2 \cup \cdots \cup C^{d+1}) = R_\varphi|(C^1 \cup C^2 \cup \cdots \cup C^{d+1})$.

Since $C^1 \cup C^2 \cup \cdots \cup C^{d+1}$ equals X, these two conditions imply that $(R_\varphi, \mathcal{O}_{d+1})$ is a minimal AF equivalence relation, thereby completing the proof. □

Appendix A remark on free actions

Theorem A.1 presented below is a generalization of a result proved by O. Johansen in his thesis [J], which has not been published. Since this is relevant for our investigation we will give a proof.

Theorem A.1. *Let K be a finite group and let $d \geq 1$. For any free action φ of $K \oplus \mathbb{Z}^d$ as homeomorphisms on the Cantor set X, there exists a free action ψ of $\mathbb{Z}^d$ on X such that $R_\varphi = R_\psi$.*

To prove the theorem we shall need the following lemma.

Lemma A.2. *Let K be a finite group acting freely on the Cantor set X. There exists a clopen subset E in X such that $\{k(E) \mid k \in K\}$ is a (clopen) partition of X.*

Proof. For each x in X, a clopen set E_x containing x can be chosen such that the sets $k(E_x)$, for k in K, are disjoint. Then $\{E_x \mid x \in X\}$ covers X. Choose a finite subcovering $\{E_{x_j} \mid j \in J\}$. Let $\mathcal{P}$ be the clopen partition of X generated by $\{k(E_{x_j}) \mid k \in K,\ j \in J\}$. Then $\mathcal{P}$ is K-invariant, i.e. $k(M) \in \mathcal{P}$ if $M \in \mathcal{P}$ and $k \in K$. Furthermore, if $M \in \mathcal{P}$ then $M \subset E_{x_j}$ for some $j \in J$, and so the sets $k(M), k \in K$, are disjoint. The relation $M \sim N$ on $\mathcal{P}$ defined by $M = k(N)$ for some $k \in K$ is an equivalence relation. Choose one element M_l from each equivalence class and let $E = \bigcup_l M_l$. Then E will have the desired property. □

Proof of Theorem A.1. It will be convenient to denote $\varphi^{(k,0)}$ and $\varphi^{(0,h)}$ by k and h, respectively, i.e. $\varphi^{(k,0)}(x) = k(x)$ and $\varphi^{(0,h)}(x) = h(x)$, where $k \in K$, $h \in \mathbb{Z}^d$ and $x \in X$. Let $Y \subset X$ be the clopen set of the lemma associated to the action of K. Let $K = \{e{=}k_0, k_1, \ldots, k_{n-1}\}$, where e is the identity element of K, and let $\{Y_i{=}k_i(Y) \mid i = 0, 1, \ldots, n{-}1\}$ be the clopen partition of X according to the lemma.

Let $\pi_Y : X \to Y$ be the continuous map defined by

$$\pi_Y(x) = k_i^{-1}(x) \qquad \text{if } x \in Y_i, \quad i = 0, 1, \ldots, n-1.$$

It is easy to see that $\pi_Y \circ h \circ k$ equals $\pi_Y \circ h$ for any $h \in \mathbb{Z}^d$ and $k \in K$. Let $\{h_1, h_2, \ldots, h_d\}$ be a basis for $\mathbb{Z}^d$. For $l \in \{2, \ldots, d\}$, let $\psi_l : X \to X$ be defined by

$$\psi_l(x) = k_j(\pi_Y(h_l(x))) \qquad \text{if } x \in Y_j$$

and let

$$\psi_1(x) = \begin{cases} \pi_Y(h_1(x)) & \text{if } x \in Y_{n-1} \\ k_{i+1}(k_i^{-1}(x)) & \text{if } x \in Y_i \text{ and } i \neq n-1. \end{cases}$$

Clearly $\psi_1, \psi_2, \ldots, \psi_d$ are homeomorphisms on X.

Now let $2 \leq l_1, l_2 \leq d$. For $x \in Y_i$, we have

$$\begin{aligned}
\psi_{l_1}(\psi_{l_2}(x)) &= (\psi_{l_1} \circ k_i \circ \pi_Y \circ h_{l_2})(x) \\
&= (k_i \circ \pi_Y \circ h_{l_1} \circ k_i \circ \pi_Y \circ h_{l_2})(x) \\
&= (k_i \circ \pi_Y \circ h_{l_1} \circ h_{l_2})(x) \\
&= (k_i \circ \pi_Y \circ h_{l_2} \circ h_{l_1})(x) = \psi_{l_2}(\psi_{l_1}(x)).
\end{aligned}$$

In a similar way one checks that ψ_1 commutes with ψ_l, where $l \in \{2, \ldots, d\}$. So $\psi_1, \psi_2, \ldots, \psi_d$ give rise to an action ψ of $\mathbb{Z}^d$ as homeomorphisms on X.

We prove that $R_\varphi = R_\psi$, or, what is the same, that the φ-orbits and the ψ-orbits coincide. Obviously we have $R_\psi[x] \subset R_\varphi[x]$ for all $x \in X$. To prove the converse, let $x \in Y_i$. Take $k \in K$ and suppose $k(x) \in Y_j$. Then one can see $\psi_1^{j-i}(x) = k(x)$, and so $k(x)$ is in $R_\psi[x]$. Next, to prove $h_l(x) \in R_\psi[x]$, assume that $h_l(x)$ belongs to Y_j. Then, for $l = 2, \ldots, d$,

$$h_l(x) = (k_j \circ \pi_Y \circ h_l)(x) = (k_j \circ k_i^{-1} \circ \psi_l)(x),$$

and

$$h_1(x) = (k_j \circ \pi_Y \circ h_1 \circ \psi_1^{n-1-i})(x) = (k_j \circ \psi_1^{n-i})(x).$$

Therefore $h_l(x) \in R_\psi[x]$.

What remains to be shown is that ψ is a free action of $\mathbb{Z}^d$. This is easy to see. In fact, if $x \in Y_j$, then each $\psi_2, \ldots, \psi_d$ map x into Y_j, and ψ_1 maps x into Y_{j+1}, where $j + 1$ is understood modulo n. Suppose that $(\psi_1^{a_1} \circ \psi_2^{a_2} \circ \cdots \circ \psi_d^{a_d})(x) = x$. Then a_1 is divisible by n. Let $a_1 = nb$. Since K commutes with every $h_1, h_2, \ldots, h_d$, it is easily seen that there exists $k \in K$ such that

$$x = (\psi_1^{a_1} \circ \psi_2^{a_2} \circ \cdots \circ \psi_d^{a_d})(x) = (k \circ h_1^b \circ h_2^{a_2} \circ \cdots \circ h_d^{a_d})(x).$$

It follow from the freeness of φ that $b = a_2 = \cdots = a_d = 0$. Hence ψ is free. $\square$

The following corollary follows from Theorem 2.4 and Theorem A.1.

Corollary A.3. *Any minimal action of a finitely generated abelian group on the Cantor set is affable.*

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