Ext and OrderExt classes of certain automorphisms of C^*-algebras arising from Cantor minimal systems *

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Abstract

Giordano, Putnam and Skau showed that the transformation group C^*-algebra arising from a Cantor minimal system is an AT-algebra, and classified it by its K-theory. For approximately inner automorphisms that preserve $C(X)$, we will determine their classes in the Ext and OrderExt groups, and introduce a new invariant for the closure of the topological full group. We will also prove that every automorphism in the kernel of the homomorphism into the Ext group is homotopic to an inner automorphism, which extends Kishimoto's result.

1 Introduction

Let (X, ϕ) be a topological dynamical system on a compact Hausdorff space X with a homeomorphism ϕ. The transformation group C^*-algebra $C^*(X, \phi)$ is the universal C^*-algebra generated by $C(X)$, the algebra of continuous functions on X, and a unitary element that implements the action of ϕ. One of the long standing problems is to find a dynamical condition which characterizes the isomorphism class of the transformation group C^*-algebra. Giordano, Putnam and Skau solved this problem for the Cantor minimal systems ([5]). They introduced the notion of strong orbit equivalence and showed that two Cantor minimal systems are strong orbit equivalent if and only if their transformation group C^*-algebras are isomorphic. Here, a Cantor minimal system means a minimal system on the Cantor set, where the Cantor set is a compact, metrizable, totally disconnected and perfect space, and a minimal system means there is no nontrivial closed invariant subset, which implies the C^*-algebra is simple. Their theory was the first example where K-theory gave a classification for minimal topological dynamical systems. In their subsequent paper [6], they defined several kinds of full groups and proved that these groups are complete invariants for orbit equivalence, strong orbit equivalence and flip conjugacy respectively. These results are analogous to the results in the measurable dynamical setting.

The C^*-algebra theoretic aspect of the above results is indebted to Elliott's celebrated classification theorem of simple real rank zero AT-algebras using K-theory ([3]). Since his classification was accomplished, structure of the automorphism groups of algebras in this

*AMS subject classification:46L40,46L80,54H20 class has been studied by several authors ([4] [10] [11] [12] [13]). Among others, Kishimoto and Kumjian introduced two homomorphisms from the closure of the inner automorphisms to the Ext and OrderExt groups of K-groups, and they investigated the ranges and kernels of these homomorphisms in [10], [11] and [13]. One of purposes of this paper is to compute these homomorphisms for concrete automorphisms arising from Cantor minimal systems, and to introduce a new invariant for Cantor minimal systems using these homomorphisms. There are two kinds of natural automorphisms of the transformation group C^*-algebra of a Cantor minimal system: those arising from the normalizers of the topological full group and those arising from T-valued cocycles of the transformation ([6]). In [6], the notion of the mod map was introduced, which is a homomorphism from the normalizers to the automorphism group of the K-group, and it is shown that the kernel of the mod map is the closure of the topological full group. We will give a concrete formula for a homomorphism from the kernel of the mod map to the Ext group of K-group in terms of the dynamical system. It turns out that this invariant may differ within a strong orbit equivalence class. We will also investigate the structure of T-valued cocycles using the OrderExt group. As a byproduct, we will extend Kishimoto's theorem about the kernel of the homomorphism into the Ext group ([13]) to the case of C^*-algebras arising from Cantor minimal systems.

In the measurable dynamical setting, the importance of cocycles was pointed out by Mackey in [14] and a systematic study was initiated by Zimmer in [21]. Extensions of Cantor minimal systems are investigated in [7] when the factor maps are almost one-to-one. The other purpose of this paper is to examine p to one factor maps between Cantor minimal systems and describe them using the cocycle theory which will be introduced in Section 3. From an element in the dimension group of a Cantor minimal system, we can form an extension system and a shift map on the sheets. We state the K-theoretical condition which ensures that the extension system becomes a minimal system in Lemma 3.6. In Section 7 this construction gives many examples of p to one factor maps and homeomorphisms of order p which commute with the minimal homeomorphism. We will show that the kernel of the homomorphism to the Ext group may be bigger than the topological full group in the last example.

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2 Preliminaries

At first we must recall the basic facts about Cantor minimal systems from [5] and [6]. Let $C^*(X, \phi)$ be the crossed product C^*-algebra constructed from a topological dynamical system (X, ϕ). We denote by δ the implementing unitary. The reader may refer to [19] for the general theory of transformation group C^*-algebras.

Theorem 2.1 (**[5, Theorem 1.15]**) *Let (X, ϕ) be a Cantor minimal system. Then $C^*(X, \phi)$ is a simple real rank zero AT-algebra with the K_1-group* $\mathbf{Z}$. *Moreover any simple unital dimension group except* $\mathbf{Z}$ *can be realized as the K_0-group of this algebra.*

We recall that a C^*-algebra is an AT-algebra if it is expressible as an inductive limit of T-algebras; a T-algebra is a direct sum of matrix algebras over $C(T)$. Let

$$B_\phi = \{f - f \circ \phi^{-1}; f \in C(X, \mathbf{Z})\}$$

be the subgroup of coboundaries of $C(X, \mathbf{Z})$. By the Pimsner-Voiculescu exact sequence, we get that $K^0(X, \phi)$ is order isomorphic to $C(X, \mathbf{Z})/B_\phi$. We denote by $[f]$ the equivalence class of f in $K^0(X, \phi)$. According to Elliott's theorem [3], $C^*(X, \phi)$'s are classified by the simple dimension groups $K^0(X, \phi)$'s.

The following is one of the main results in [5].

Theorem 2.2 ([5, Theorem 2.1]) *Let $(X_i, \phi_i), i = 1, 2,$ be Cantor minimal systems. The following are equivalent:*

(i) *The C^*-algebras $C^*(X_i, \phi_i), i = 1, 2,$ are isomorphic,*

(ii) *The dimension groups $K^0(X_i, \phi_i), i = 1, 2,$ are order isomorphic by a map preserving distinguished order units,*

(iii) *$(X_i, \phi_i), i = 1, 2,$ are strong orbit equivalent.*

We need the notion of topological full groups and mod maps. If (X, ϕ) is a Cantor minimal system, we define the topological full group $\tau[\phi]$ to be

$$\{\gamma \in \mathrm{Homeo}(X) \, ; \, \exists \, n \in C(X, \mathbf{Z}) \textit{ such that } \gamma(x) = \phi^{n(x)}(x) \textit{ for all } x \in X\},$$

and $N(\tau[\phi])$ the normalizer of $\tau[\phi]$ in $\mathrm{Homeo}(X)$. Let $UN(C(X), C^*(X, \phi))$ be the unitary normalizer group of $C(X)$ in $C^*(X, \phi)$. In [17, Lemma 5.1], it was shown that for $\psi \in \tau[\phi]$ there exists v_ψ in $UN(C(X), C^*(X, \phi))$ such that $v_\psi f v_\psi^* = f \circ \psi^{-1}$ for all $f \in C(X)$, and v_ψ is uniquely determined up to $U(C(X))$. The general case is well studied in [20].

When (X, ϕ) is a Cantor minimal system,we use the following notation:

$$\begin{aligned}
\mathrm{Aut}_{C(X)}(C^*(X, \phi)) &= \{ \, \alpha \in \mathrm{Aut}(C^*(X, \phi)) \, ; \, \alpha(C(X)) = C(X)\}, \\
\mathrm{Inn}_{C(X)}(C^*(X, \phi)) &= \mathrm{Aut}_{C(X)}(C^*(X, \phi)) \cap \mathrm{Inn}(C^*(X, \phi)), \\
U_\phi &= \{ \, \overline{f}(f \circ \phi) \, ; \, f \in U(C(X) \, \}.
\end{aligned}$$

For $f \in U(C(X))$ we define the element $\iota(f)$ in $\mathrm{Aut}_{C(X)}(C^*(X, \phi))$ by

$$\iota(f)(g) = g \textit{ for all } g \in C(X) \, , \, \iota(f)(\delta) = \delta f.$$

The automorphism $\alpha \in \mathrm{Aut}_{C(X)}(C^*(X, \phi))$ determines a homeomorphism $\pi(\alpha)$ on X such that

$$\alpha(g) = g \circ \pi(\alpha)^{-1} \textit{ for all } g \in C(X).$$

Then we have the following.

Proposition 2.3 ([6, Proposition 2.4]) *If (X, ϕ) is a Cantor minimal system, we have the following two short exact sequences:*

(i) $1 \to U_\phi \xrightarrow{\ \iota\ } \mathrm{Inn}_{C(X)}(C^*(X, \phi)) \xrightarrow{\ \pi\ } \tau[\phi] \to 1 \,,$

(ii) $1 \to U(C(X)) \xrightarrow{\iota} \mathrm{Aut}_{C(X)}(C^*(X,\phi)) \xrightarrow{\pi} N(\tau[\phi]) \to 1\,,$

Moreover these exact sequences split.

Let $s : N(\tau[\phi]) \to \mathrm{Aut}_{C(X)}(C^*(X,\phi))$ denote the homomorphism defined for $\gamma \in N(\tau[\phi])$ by

$$s(\gamma)(f) = f \circ \gamma^{-1}\ ,\ s(\gamma)(\delta) = v_\psi,$$

where $\psi \in \tau[\phi]$ satisfies $\gamma \circ \phi \circ \gamma^{-1} = \psi$ and v_ψ is the unitary normalizer corresponding to ψ. Then we have $\pi \circ s(\gamma) = \gamma$ for all $\gamma \in N(\tau[\phi])$. In Section 5 and 6 we will discuss the above automorphisms $\iota(f)$ and $s(\gamma)$.

We would like to review the definition of the mod map on the topological full group. When a homeomorphism γ preserves the coboundary subgroup B_ϕ, we can define an automorphism $\mathrm{mod}(\gamma)$ on $K^0(X,\phi)$ as

$$\mathrm{mod}(\gamma)([f]) = [f \circ \gamma^{-1}]$$

for all $f \in C(X,\mathbf{Z})$. It is easily checked that the normalizers of the topological full group preserve B_ϕ, and the automorphism $s(\gamma)$ on $C^*(X,\phi)$ induces the automorphism $\mathrm{mod}(\gamma)$ on the K_0-group when γ is in $N(\tau[\phi])$.

Equip $\mathrm{Homeo}(X)$ with the topology of pointwise convergence in norm on $C(X)$.

Proposition 2.4 ([6, Proposition 2.11]) *In the above setting we have* $\ker(\mathrm{mod}) = \overline{\tau[\phi]}$.

Although $\gamma \in N(\tau[\phi])$ is in the kernel of the mod map, $s(\gamma)$ may induce the non-trivial automorphism on the K_1-group, which is isomorphic to the integer group. We define

$$T(\phi) = \{\gamma \in N(\tau[\phi])\ ;\ \gamma \text{ induces identity maps on } K\text{-groups}\}.$$

The set $T(\phi)$ is a subgroup of $N(\tau[\phi])$ and we investigate this group in Section 5.

Let us now recall the description of the normalizer $N(\tau[\phi])$ of $\tau[\phi]$ in terms of a semi-direct product group. When (X,ϕ) is a Cantor minimal system, we set

$$\begin{aligned} C^\epsilon(\phi) &= \{\gamma \in \mathrm{Homeo}(X); \gamma\circ\phi\circ\gamma^{-1} = \phi \text{ or } \gamma\circ\phi\circ\gamma^{-1} = \phi^{-1}\}, \\ C^+(\phi) &= \{\gamma \in \mathrm{Homeo}(X); \gamma\circ\phi\circ\gamma^{-1} = \phi\}. \end{aligned}$$

Note that $C^\epsilon(\phi)$ is contained in $N(\tau[\phi])$.

Proposition 2.5 ([6, Proposition 5.11]) *If $\tau[\phi] \rtimes C^\epsilon(\phi)$ denotes the semi-direct product of the topological full group of ϕ by $C^\epsilon(\phi)$, then we get the following short exact sequences:*

$$0 \to \mathbf{Z} \xrightarrow{i} \tau[\phi] \rtimes C^\epsilon(\phi) \xrightarrow{\Phi} N(\tau[\phi]) \to 0,$$

where i and Φ are defined by $i(n) = (\phi^n, \phi^{-n})$ and $\Phi(\gamma,\eta) = \gamma\circ\eta$.

We also obtain

$$\Phi^{-1}(T(\phi)) = \tau[\phi] \rtimes (C^+(\phi) \cap \overline{\tau[\phi]})$$

from the above exact sequence.

3 Extensions and Cocycles

In this section we study extensions of Cantor minimal systems. Note that similar results can be found in [15]. If (X, ϕ) and (Y, ψ) are topological dynamical systems, (Y, ψ) is called an extension of (X, ϕ), and (X, ϕ) a factor of (Y, ψ) when there is a continuous map π from Y to X such that $\phi \circ \pi = \pi \circ \psi$. We denote it by $\pi : (Y, \psi) \to (X, \phi)$. The map π is called a factor map. If (X, ϕ) is a minimal system, π is surjective.

We will examine the case that a factor map π is a p to one surjective local homeomorphism for a fixed natural number p. Here the local homeomorphism means that for each $y \in Y$ there is an open neighborhood U of y such that $\pi(U)$ is open in X and $\pi|U$ is a homeomorphism. A local homeomorphism is an open map, and a p to one surjective local homeomorphism is called a p to one covering map. We remark that almost one to one factors are investigated in [7].

Lemma 3.1 *When $\pi : Y \to X$ is a p to one covering map between topological spaces and X is compact and 0-dimensional, the space Y is homeomorphic to the Cartesian product of X and $\{0, 1, \cdots, p-1\}$, with π corresponding to the projection onto X.*

Proof. For each $x \in X$, we can choose a preimage y of x and a clopen neighborhood U_x of y on which the covering map π is a homeomorphism. Since the family $\{\pi(U_x)\}_{x \in X}$ is an open covering of X, there exists a finite set of clopen subsets $U_1, U_2, \cdots, U_n$ such that on each U_k the map π is a homeomorphism and $\{\pi(U_k)\}_k$ covers X. Set

$$\begin{aligned} V_1 &= U_1, \\ V_2 &= V_1 \cup (U_2 \setminus \pi^{-1}(\pi(V_1))), \\ &\vdots \\ V_n &= V_{n-1} \cup (U_n \setminus \pi^{-1}(\pi(V_{n-1}))), \end{aligned}$$

inductively. Then V_n is a clopen set on which the map π is a homeomorphism, and $\pi(V_n)$ equals X. We can regard the clopen set V_n of Y as a copy of X, and the map π as a $p-1$ to one covering map on the complement of V_n. By repeating this argument, we can separate the space Y into p copies of X, say p sheets, and the map π is a homeomorphism on each sheet. □

Let (X, ϕ) be a Cantor minimal system, and $\pi : (Y, \psi) \to (X, \phi)$ a p to one covering map. We would like to describe this extension by a symmetric group valued cocycle on (X, ϕ). When Y is identified with $X \times \{0, 1, \cdots, p-1\}$, there exists $c : X \to S_p$ such that

$$\psi(x, k) = (\phi(x), c(x)(k))$$

for all $(x, k) \in Y$. The p to one extension system (Y, ψ) is completely determined by this map c.

A continuous map $c : X \to S_p$ is called a S_p-valued cocycle on X. We can associate the p to one extension system with the cocycle c in the above way.

Definition 3.2 *Let (X, ϕ) be a dynamical system, and c, c' cocycles on (X, ϕ). We say that c and c' are cohomologous if there is a continuous map $b : X \to S_p$ such that*

$$b(\phi(x))c(x)b(x)^{-1} = c'(x)$$

for all $x \in X$. A cocycle c is a coboundary if there is a map $b : X \to S_p$

$$c(x) = b(\phi(x))b(x)^{-1}$$

for all $x \in X$.

The following lemma shows that the cohomologous cocycles represent equivalent extensions.

Lemma 3.3 *Let c_i $(i = 1, 2)$ be cocycles on a dynamical system (X, ϕ), and $\pi_i : (Y_i, \psi_i) \to (X, \phi)$ the associated extensions. Then the following are equivalent:*

(i) *c_i $(i = 1, 2)$ are cohomologous,*

(ii) *The two extensions are equivalent; i.e, there exists a homeomorphism Ψ from Y_1 to Y_2 such that $\psi_2 \circ \Psi = \Psi \circ \psi_1$ and $\pi_2 \circ \Psi = \pi_1$.*

We would like to consider the minimal extension systems, and describe the cocycle condition that guarantees that the associated extension system becomes minimal.

Lemma 3.4 *Let (X, ϕ) be a Cantor minimal system, and $\pi : (Y, \psi) \to (X, \phi)$ a p to one extension. When $c : X \to S_p$ represents this extension, the following are equivalent:*

(i) *(Y, ψ) is minimal,*

(ii) *(Y, ψ) is topologically transitive, which means that there exists a point whose orbit is dense in Y,*

(iii) *If a cocycle c' is cohomologous to c, the subgroup generated by $\{c'(x)\ ;\ x \in X\}$ acts transitively on $\{0, 1, \cdots, p-1\}$.*

Proof. We only show (iii) $\Rightarrow$ (i). Let F be a non-empty closed subset of Y which is invariant under ψ. The minimality of the system (X, ϕ) implies $\pi(F) = X$. For arbitrary points $x_1, x_2 \in X$, let q_i be the cardinality of the set $\pi^{-1}(x_i) \cap F$. From the minimality of (X, ϕ) and the compactness of the space Y, we can choose a sequence $\{a_n\}$ such that $\phi^{a_n}(x_1)$ converges to x_2 and $\psi^{a_n}(y)$ converges for every $y \in \pi^{-1}(x_1) \cap F$. Since the map π is a local homeomorphism, $y \mapsto \lim_{n\to\infty} \psi^{a_n}(y)$ is an injective map from $\pi^{-1}(x_1) \cap F$ to $\pi^{-1}(x_2) \cap F$, and so $q_1 \le q_2$. In the same way we can show $q_2 \le q_1$. Hence there exists a natural number q such that the cardinality of the set $\pi^{-1}(x) \cap F$ equals q for every $x \in X$.

By Lemma 3.1, we may assume Y is divided into p clopen sheets, namely $Y_1, Y_2, \cdots, Y_p$. For every point $x \in X$, there exists a unique collections of q sheets $Y_{i(1)}, Y_{i(2)}, \cdots, Y_{i(q)}$ uniquely such that x is included in $\pi(F \cap Y_{i(k)})$ for $k = 1, 2, \cdots, q$. Thus X is the disjoint union of closed sets

$$\bigcap_{i \in Q} \pi(F \cap Y_i),$$

where Q runs over all subsets of $\{1, 2, \cdots, p\}$ with cardinality q. This means that X is partitioned into $\begin{pmatrix} p \\ q \end{pmatrix}$ clopen sets. Therefore F is a clopen set. By changing the cocycle, we can assume that F is exactly the union of q sheets, which contradicts (iii). $\square$

In the rest of this section, we consider $\mathbf{Z_p}$-valued cocycles. If c is a $\mathbf{Z_p}$-valued cocycle, we may regard c as an element of $C(X,\mathbf{Z})/pC(X,\mathbf{Z})$. Then c and c' are cohomologous if and only if there exists a function $b \in C(X,\mathbf{Z})$ such that

$$c(x) - c'(x) = b(\phi(x)) - b(x)$$

for all $x \in X$, where the equation is understood modulo p. The set of cocycles which are cohomologous to zero is called the coboundary subgroup. We define the $\mathbf{Z_p}$-valued cohomology group $H^1_\phi(X,\mathbf{Z_p})$ as the quotient of the group of $\mathbf{Z_p}$-valued cocycles by the coboundary subgroup.

Lemma 3.5 *If (X,ϕ) is a Cantor minimal system, the $\mathbf{Z_p}$-valued cohomology group $H^1_\phi(X,\mathbf{Z_p})$ is isomorphic to $K^0(X,\phi)/pK^0(X,\phi)$.*

Proof. There are natural quotient maps from $C(X,\mathbf{Z})$ to $H^1_\phi(X,\mathbf{Z_p})$ and $K^0(X,\phi)/pK^0(X,\phi)$. The image of $f \in C(X,\mathbf{Z})$ is zero in $H^1_\phi(X,\mathbf{Z_p})$ if and only if there exists a function $g \in C(X,\mathbf{Z})$ such that $f(x) = g(\phi(x)) - g(x)$ for all $x \in X$, where the equation is understood modulo p. On the other hand, the image of f is zero in $K^0(X,\phi)/pK^0(X,\phi)$ if and only if there exists functions $g, h \in C(X,\mathbf{Z})$ such that $f - ph = g \circ \phi - g$. So the kernels of these two homomorphisms coincide. □

We also denote the equivalence class in $K^0(X,\phi)/pK^0(X,\phi)$ by $[f]$.

In the next lemma, we describe the K-theoretical condition which completely determines when the extension system associated with a $\mathbf{Z_p}$-valued cocycle becomes a minimal system.

Lemma 3.6 *Let (X,ϕ) be a Cantor minimal system, $[h]$ in $K^0(X,\phi)/pK^0(X,\phi)$, and $\pi : (Y,\psi) \to (X,\phi)$ a corresponding p to one covering map. Then (Y,ψ) is a minimal system if and only if*

$$k[h] \neq 0 \text{ in } K^0(X,\phi)/pK^0(X,\phi)$$

for $k = 1, 2, \cdots, p-1$.

Proof. Let $k[h]$ be zero in $K^0(X,\phi)/pK^0(X,\phi)$. We would like to prove that the system (Y,ψ) is not minimal. It suffices to consider the case that p is a multiple of k, say lk. We may assume

$$Y = X \times \{0, 1, \cdots, p-1\},$$
$$\psi(x,k) = (\phi(x), k + h(x)),$$
$$\pi(x,k) = x$$

for all $(x,k) \in Y$, where $k+h(x)$ is understood modulo p. Let us define a homeomorphism γ on Y as $\gamma(x,k) = (x, k+1)$. The map γ commutes with ψ. We call γ the shift map. We set $Y_0 = X \times \{0, 1, \cdots, l-1\}$ and a homeomorphism ψ_0 on Y_0 such as

$$\psi_0(x,m) = (\phi(x), m + h(x)),$$

where $m + h(x)$ is understood modulo l. Then we can easily see that the factor map π factors through (Y_0,ψ_0), and (Y_0,ψ_0) is an extension of (X,ϕ) which corresponds to $[h]$ in $K^0(X,\phi)/lK^0(X,\phi)$. Since $K^0(X,\phi)$ is torsion free, $k[h] = 0$ in $K^0(X,\phi)/pK^0(X,\phi)$

implies that $[h]$ is zero in $K^0(X,\phi)/lK^0(X,\phi)$ and the system (Y_0,ψ_0) is not minimal. As a factor of a minimal system is also minimal, we can conclude that (Y,ψ) is not minimal.

To prove the other direction, we at first consider the case $h=1$. We would like to show that (Y,ψ) is minimal under the assumption that $k[1]$ is not zero in $K^0(X,\phi)/pK^0(X,\phi)$ for $k=1,2,\cdots,p-1$. Let F be the closure of the orbit $\{\psi^n(x,0); n\in\mathbf{Z}\}$ for a point $x\in X$. By the proof of Lemma 3.4, F is a clopen set. If (x,k) is included in F, there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\psi^{a_n}(x,0)\to(x,k)$ as n goes to infinity. Then

$$\begin{aligned}(x,2k) &= \gamma^k(x,k)\\ &= \gamma^k(\lim_{n\to\infty}\psi^{a_n}(x,0))\\ &= \lim_{n\to\infty}\psi^{a_n}(\gamma^k(x,0))\\ &= \lim_{n\to\infty}\psi^{a_n}(x,k)\end{aligned}$$

exists in F. Hence we can find a natural number l, which satisfies that $(x,k)\in F$ if and only if k is a multiple of l. The number l is a divisor of p. Since F is a clopen set, we can choose a clopen neighborhood E of x in X, which satisfies that $E\times\{k\}\subset F$ if and only if k is a multiple of l. Thus, whenever $\phi^n(x)$ is in E, n is a multiple of l, because the extension system (Y,ψ) is associated with [1]. When one consider the first return map on E and the associated clopen partition of X, one sees that this implies that the height of each tower is a multiple of l. So [1] in $K^0(X,\phi)$ is divisible by l. From the assumption l must be 1, and F equals to Y.

In the general case we need the tower construction of [6]. We may assume the function h is strictly positive. Let $\tilde{X}$ be the space $\{(x,j)\ ;\ x\in X,\ j=1,2,\cdots,h(x)\}$ with the topology induced from $X\times\mathbf{N}$, and $\tilde{\phi}$ the homeomorphism on $\tilde{X}$ satisfying

$$\tilde{\phi}(x,j) = \begin{cases}(x,j+1) & j<h(x)\\ (\phi(x),1) & j=h(x)\end{cases}.$$

Then, the system $(\tilde{X},\tilde{\phi})$ is a Cantor minimal system and there is an order isomorphism from the dimension group $K^0(X,\phi)$ to $K^0(\tilde{X},\tilde{\phi})$, sending the element $[h]$ to the order unit [1]. Let us denote by $(\tilde{Y},\tilde{\psi})$ the extension system of $(\tilde{X},\tilde{\phi})$ associated with $[1]\in K^0(\tilde{X},\tilde{\phi})/pK^0(\tilde{X},\tilde{\phi})$. Thus,

$$\tilde{Y}=\{(x,j,k)\ ;\ x\in X,\ 1\le j\le h(x),\ 0\le k\le p-1\},$$

and

$$\tilde{\psi}(x,j,k) = \begin{cases}(x,j+1,k+1) & j<h(x)\\ (\phi(x),1,k+1) & j=h(x)\end{cases},$$

where $k+1$ is understood modulo p. From the preceding paragraph, the system $(\tilde{Y},\tilde{\psi})$ becomes a Cantor minimal system. The first return map on the clopen set $\{(x,1,k); x\in X\ 0\le k\le p-1\}$ is given by

$$(x,1,k)\mapsto(\phi(x),1,k+h(x)),$$

and this system is obviously isomorphic to (Y,ψ). Therefore (Y,ψ) is minimal. □

As a corollary of the above lemma, we can discuss the minimality of the product system of a Cantor minimal system and the odometer system. The odometer systems are basic examples of Cantor minimal systems. We refer the reader to Section VIII.4 of [1] for details of the odometer systems. Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of natural numbers such that m_{n+1} is a multiple of m_n for each n, and (Y,ψ) the odometer system of type $\{m_n\}_n$.

Proposition 3.7 *When (X,ϕ) is a Cantor minimal system and (Y,ψ) is as above, the product system $(X\times Y,\phi\times\psi)$ is again a Cantor minimal system if and only if $[1]$ in the dimension group $K^0(X,\phi)$ is not divisible by any natural numbers which are divisors of some m_n.*

Proof. Let Y_n be $\{0,1,\cdots,m_n-1\}$, and ψ_n the shift map on Y_n sending k to $k+1$. The odometer system (Y,ψ) can be written as the projective limit system of $\{(Y_n,\psi_n)\}_n$. Hence the product system $(X\times Y,\phi\times\psi)$ is the projective limit system of $\{(X\times Y_n,\phi\times\psi_n)\}_n$. The system $(X\times Y_n,\phi\times\psi_n)$ is the m_n to one extension system of (X,ϕ) associated with $[1]$ in $K^0(X,\phi)/m_nK^0(X,\phi)$.

If $(X\times Y,\phi\times\psi)$ is a minimal system, $(X\times Y_n,\phi\times\psi_n)$ becomes a minimal system for every n, because this system is a factor of $(X\times Y,\phi\times\psi)$. By virtue of Lemma 3.6, $[1]$ in the dimension group cannot have a common divisor with any m_n's.

In order to prove the other implication, it suffices to check that the projective limit system of minimal systems is also minimal. This is trivial. □

Let $[h]$ be an element of $K^0(X,\phi)/pK^0(X,\phi)$, and (Y,ψ) the associated extension system. We use the same notation of Lemma 3.6. The shift map γ on Y commutes with ψ and $\gamma^p=id$. Let us denote by β the automorphism of $C^*(Y,\psi)$ such that

$$\beta(\delta)=\delta\ ,\ \beta(f)=f\circ\gamma^{-1}\ \textit{for all}\ f\in C(Y).$$

We also define an automorphism α of $C^*(X,\phi)$ by

$$\alpha(\delta)=(\exp\frac{-2\pi ih}{p})\delta\ ,\ \alpha(f)=f\ \textit{for all}\ f\in C(X),$$

where $C^*(X,\phi)$ is regarded as a sub-C^*-algebra of $C^*(Y,\psi)$ with the same implementing unitary δ.

Both the automorphisms α and β can be viewed as $\mathbf{Z_p}$-actions. By checking Landstad's condition ([16, 7.8.2]), one verifies the following result.

Proposition 3.8 *In the above setting, the C^*-dynamical systems $(C^*(X,\phi)\rtimes_\alpha\mathbf{Z_p},\hat\alpha,\mathbf{Z_p})$ and $(C^*(Y,\psi),\beta,\mathbf{Z_p})$ are covariantly isomorphic, where $\hat\alpha$ denotes the dual action of α.*

We also remark that the group homomorphism from $K^0(X,\phi)$ to $K^0(Y,\psi)$ which is induced by the inclusion map is injective if the systems are minimal ([8, Proposition 3.1]). When γ denotes the shift map, we have

$$K^0(X,\phi)=\{\sum_{k=0}^{p-1}\gamma_*^k(a)\ ;\ a\in K^0(Y,\psi)\}.$$

4 Ext and OrderExt groups

We would like to recall the notion of Ext and OrderExt groups for K-groups of C^*-algebras. The readers may refer to [10],[11] for details.

When A is a unital C^*-algebra and α is an automorphism of A, the mapping torus of α is the C^*-algebra

$$M_\alpha = \{x \in C([0,1], A); \alpha(x(0)) = x(1)\},$$

and the suspension of A is given by

$$SA = \{x \in C([0,1], A); x(0) = x(1) = 0\}.$$

If α is an approximately inner automorphism, then from the short exact sequence

$$0 \to SA \to M_\alpha \to A \to 0,$$

one obtains a pair of short exact sequences:

$$0 \to K_{1-i}(A) \to K_i(M_\alpha) \to K_i(A) \to 0,$$

for $i = 0, 1$. Let $\eta_i(\alpha)$ denote the class of this sequence in $\mathrm{Ext}(K_i(A), K_{1-i}(A))$.

Since $K_1(A)$ is always the integer group if A arises from a Cantor minimal system, $\eta_1(\alpha)$ is trivial for all α in $\overline{\mathrm{Inn}}(A)$. So we may consider only $\eta_0(\alpha)$.

There are two distinguished normal subgroups of $\overline{\mathrm{Inn}}(A)$ containing $\mathrm{Inn}(A)$. One is the group $\mathrm{HInn}(A)$ of automorphisms that are homotopic to elements of $\mathrm{Inn}(A)$ and the other is the group $\mathrm{AInn}(A)$ of asymptotically inner automorphisms. It is easy to see that

$$\mathrm{Inn}(A) \subset \mathrm{AInn}(A) \subset \mathrm{HInn}(A) \subset \overline{\mathrm{Inn}}(A).$$

Theorem 4.1 ([10, Corollary 3.10]) *If A is a unital simple AT-algebra with real rank zero, the map $\eta_0 \oplus \eta_1$ from $\overline{\mathrm{Inn}}(A)$ to $\bigoplus_{i=0}^1 \mathrm{Ext}(K_i(A), K_{1-i}(A))$ is a surjective homomorphism. Moreover the kernel of this map contains the group $\mathrm{HInn}(A)$.*

Under some additional conditions, Kishimoto proved that the kernel of $\eta_0 \oplus \eta_1$ is precisely $\mathrm{HInn}(A)$.

Theorem 4.2 ([13, Corollary 2.3]) *Let A be a simple unital AT-algebra of real rank zero with a unique tracial state τ such that $K_1(A) \neq \mathbf{Z}$. Then the quotient $\overline{\mathrm{Inn}}(A)/\mathrm{HInn}(A)$ is isomorphic to $\bigoplus_{i=0}^1 \mathrm{Ext}(K_i(A), K_{1-i}(A))$ with the isomorphism induced by $\eta_0 \oplus \eta_1$.*

Let T_A be the set of tracial states on A. When A is $C^*(X, \phi)$, there is a one-to-one correspondence between T_A and M_ϕ, the set of invariant measures on X. For a unitary $u \in M_\alpha$ such that $t \mapsto u(t)$ is piecewise C^1 and for $\mu \in T_A$, we define

$$\tilde{\mu}(u) = \frac{1}{2\pi i} \int_0^1 \mu(\dot{u}(t) u(t)^*) dt.$$

By extending this definition to the matrix algebra over M_α, we obtain a homomorphism $\tilde{\mu} : K_1(M_\alpha) \to \mathbf{R}$. Since $\mu \mapsto \tilde{\mu}(u)$ is continuous and affine, we get a homomorphism $R_\alpha : K_1(M_\alpha) \to \mathrm{Aff}(T_A)$ such that

$$R_\alpha([u])(\mu) = \tilde{\mu}(u)$$

We call R_α the rotation map.

Lemma 4.3 ([11, Lemma 2.2]) *For an approximately inner automorphism α of a simple unital C^*-algebra A, the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{i} & K_1(M_\alpha) \\ & D \searrow \quad \swarrow R_\alpha & \\ & \mathrm{Aff}(T_A), & \end{array}$$

where D is the canonical pairing of $K_0(A)$ with T_A.

Let

$$0 \to K_0(A) \xrightarrow{i} E \to K_1(A) \to 0$$

be an exact sequence with a map $R : E \to \mathrm{Aff}(T_A)$ satisfying $R \circ i = D$. We call the set of all pairs (E,R) the order-extensions of $(K_1(A), K_0(A))$ and denote it by $\mathrm{OrderExt}((K_1(A), K_0(A))$. We can define an abelian group structure on $\mathrm{OrderExt}((K_1(A), K_0(A))$ in the same fashion as for the usual Ext groups.

Theorem 4.4 ([11, Theorem 4.4]) *Let A be a simple unital AT-algebra with real rank zero. Then the natural map*

$$\tilde{\eta_1} \oplus \eta_0 : \overline{\mathrm{Inn}}(A) \to \mathrm{OrderExt}((K_1(A), K_0(A)) \oplus \mathrm{Ext}(K_0(A), K_1(A))$$

is a surjective homomorphism. Moreover the kernel of this map coincides with $\mathrm{AInn}(A)$.

In the case of $C^*(X, \phi)$, η_1 was always trivial. That is, if α is in $\overline{\mathrm{Inn}}(A)$, $K_1(M_\alpha)$ is isomorphic to the group $K_0(A) \oplus \mathbf{Z}$. Therefore we only need the value of the generator of $\mathbf{Z}$ by the rotation map.

Lemma 4.5 *When the C^*-algebra A satisfies $K_1(A) \cong \mathbf{Z}$, the order-extension group* $\mathrm{OrderExt}(K_1(A), K_0(A))$ *is isomorphic to the group* $\mathrm{Aff}(T_A)/K_0(A)$.

Proof. Let (E, R) be an order-extension of $(K_1(A), K_0(A))$. That is, E is an extension of $K_1(A)$ by $K_0(A)$ and R is a homomorphism to $\mathrm{Aff}(T_A)$ with commutativity in the following diagram:

$$\begin{array}{c} 0 \to K_0(A) \xrightarrow{i} E \xrightarrow{q} K_1(A) \to 0 \\ D \searrow \ \downarrow R \\ \mathrm{Aff}(T_A), \end{array}$$

where D is the canonical pairing of $K_0(A)$ with T_A. Since $K_1(A)$ is the integer group, this sequence splits by a homomorphism $r : K_1(A) \to E$. We can define a map π from $\mathrm{OrderExt}(K_1(A), K_0(A))$ to $\mathrm{Aff}(T_A)/K_0(A)$, sending (E, R) to $R(r(u))$ where u represents the generator of $K_1(A)$. If r' is any other splitting, then $R(r(u)) - R(r'(u)) \in D(K_0(A))$, and so the map π is well defined. The surjectivity of π is obvious. To prove injectivity, let us assume $R(r(u))$ is in $K_0(A)$. We can check condition 2 of Proposition 2.5 in [11] easily, hence (E, R) is zero in $\mathrm{OrderExt}(K_1(A), K_0(A))$. □

5 Computation of Ext classes

In this section we introduce a new invariant for the subgroup $T(\phi)$ of $N(\tau[\phi])$, which acts trivially on the K-groups, by calculating the class $\eta_0(\alpha) \in \mathrm{Ext}(K^0(X,\phi), \mathbf{Z})$ when α is in

$$\overline{\mathrm{Inn}}_{C(X)}(C^*(X,\phi)) = \mathrm{Aut}_{C(X)}(C^*(X,\phi)) \cap \overline{\mathrm{Inn}}(C^*(X,\phi)).$$

When the automorphism α of $C^*(X,\phi)$ preserves the maximal abelian C^*-algebra $C(X)$, α can be written as the product of two kinds of automorphisms (Proposition 2.3(ii)). In Section 2, we defined the automorphisms $\iota(g)$ for $g \in U(C(X))$ and $s(\gamma)$ for $\gamma \in N(\tau[\phi])$. At first, we examine $\iota(g)$ and see that this automorphism determines the zero element in the Ext group.

Thanks to (ii) of Proposition 2.3 and the next lemma, we have $\iota(g) \in \overline{\mathrm{Inn}}_{C(X)}(C^*(X,\phi))$.

Lemma 5.1 *If (X,ϕ) is a Cantor minimal system, U_ϕ is dense in $U(C(X))$.*

Proof. Take a unitary $g \in C(X)$ arbitrarily. Since X is 0-dimensional, there exists a self-adjoint element h in $C(X)$ such that $g = \exp 2\pi i h$. The function h determines a continuous affine function on M_ϕ, which is canonically identified with the state space of $K^0(X,\phi)$. As $K^0(X,\phi)$ is a simple dimension group, $K^0(X,\phi)$ is dense in $\mathrm{Aff}(M_\phi)$ in the sup norm ([2, Theorem 4.4]). Thus, for arbitrarily small $\varepsilon > 0$ there exists $f \in C(X,\mathbf{Z})$ such that

$$|\mu(f-h)| < \varepsilon$$

for all $\mu \in M_\phi$, and $g = \exp 2\pi i(h-f)$. Using the same argument as in Lemma 2.4 of [8], one can obtain a Kakutani-Rohlin partition $\{Y(k,j); k = 1,2,\cdots,K\ j = 0,1,\cdots,m_k - 1\}$ ([9, Section 4]), which satisfies

$$\left| \frac{1}{m_k} \sum_{j=0}^{m_k-1} (f-h)(\phi^j(x)) \right| < \varepsilon$$

for $k = 1,2,\cdots,K$ and all $x \in Y(k,0)$. Hence there is a function $l \in C(X,\mathbf{R})$, such that

$$\| \ f - h - (l - l\circ\phi) \ \|_\infty < \varepsilon,$$

which implies $\| \ g - \exp 2\pi i(l - l\circ\phi) \ \|_\infty$ is small. □

Since $\iota(g)$ is an approximately inner automorphism, we can consider $\eta_0(\iota(g))$ for $g \in U(C(X))$.

Lemma 5.2 *Let (X,ϕ) be a Cantor minimal system. The automorphism $\iota(g)$ is in $\mathrm{HInn}(C^*(X,\phi))$, for all $g \in U(C(X))$. So $\eta_0(\iota(g))$ is trivial.*

Proof. One can take the logarithm of a given g. From the above lemma, $\iota_0(g)$ is in $\mathrm{HInn}(C^*(X,\phi))$. By Theorem 4.1, $\eta_0(\iota(g))$ is trivial. □

Now, we examine the class of $s(\gamma)$ in $\mathrm{Ext}(K^0(X,\phi),\mathbf{Z})$, and describe this new invariant with dynamical language. By virtue of the remark after Proposition 2.5, we may assume γ is in $C^+(\phi) \cap \overline{\tau[\phi]}$, and thanks to Elliott's theorem we have $s(\gamma) \in \overline{\mathrm{Inn}}(C^*(X,\phi))$.

If (X,ϕ) is a Cantor minimal system, $A_{\{y\}}$ denotes the AF-subalgebra of $C^*(X,\phi)$ for $y \in X$ ([17, Section 3]). Take a sequence of Kakutani-Rohlin partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$ and an increasing sequence of finite dimensional C^*-subalgebras $\{A_n\}_{n=1}^{\infty}$, such that $C(\mathcal{P}_n)$ equals the diagonal algebra of A_n and $A_{\{y\}} = \overline{\bigcup_{n=1}^{\infty} A_n}$.

Let γ be in $C^+(\phi) \cap \overline{\tau[\phi]}$, and A be the C^*-algebra $C^*(X,\phi)$. Define

$$M_{s(\gamma),n} = \{x \in C([0,1], C^*(X,\phi))\ ;\ s(\gamma)(x(0)) = x(1)\ \ x(0) \in A_n\},$$

$$M_{s(\gamma),\{y\}} = \{x \in C([0,1], C^*(X,\phi))\ ;\ s(\gamma)(x(0)) = x(1)\ \ x(0) \in A_{\{y\}}\}.$$

The Ext class $\eta_0(s(\gamma))$ is obtained from the exact sequence:

$$0 \to SC^*(X,\phi) \to M_{s(\gamma)} \to C^*(X,\phi) \to 0.$$

Since the inclusion map from $A_{\{y\}}$ to $C^*(X,\phi)$ induces an isomorphism on K_0-groups ([17, Theorem 4.1]), the center arrow in the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to & SC^*(X,\phi) & \to & M_{s(\gamma),\{y\}} & \to & A_{\{y\}} & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \to & SC^*(X,\phi) & \to & M_{s(\gamma)} & \to & C^*(X,\phi) & \to 0 \end{array}$$

also induces an isomorphism of the K_0-groups. Thus, the Ext class $\eta_0(s(\gamma))$ is given by the exact sequence:

$$0 \to K_1(C^*(X,\phi)) \to K_0(M_{s(\gamma),\{y\}}) \to K_0(A_{\{y\}}) \to 0.$$

This sequence is obtained by the inductive limit of :

$$\begin{array}{ccccccc} 0 \to K_1(C^*(X,\phi)) & \to & K_0(M_{s(\gamma),n}) & \to & K_0(A_n) \to 0 \\ \| & & i_n \downarrow & & \downarrow \\ 0 \to K_1(C^*(X,\phi)) & \to & K_0(M_{s(\gamma),n+1}) & \to & K_0(A_{n+1}) \to 0 \end{array}$$

for $n = 1, 2, \cdots$. Since $K_0(A_n)$ is the direct sum of $\mathbf{Z}$, the above exact sequences split. In each summand of A_n we fix a minimal projection in $C(\mathcal{P}_n)$, and denote by C_n the set of these projections.

For each n, one can take a unitary v_n in $UN(C(X), C^*(X,\phi))$ satisfying

$$v_n f v_n^* = f \circ \gamma^{-1} \quad for\ f \in C(\mathcal{P}_n),$$

because γ is in the closure of $\tau[\phi]$. Let $v_n(t)$ be a unitary path in $M_2(C^*(X,\phi))$ such that

$$v_n(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ ,\ v_n(1) = \begin{bmatrix} v_n & 0 \\ 0 & v_n^* \end{bmatrix}.$$

Then

$$t \mapsto v_n(t) \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} v_n(t)^*$$

is a well defined projection of $M_2(M_{s(\gamma),n})$ for all $p \in C_n$. We define the map $\rho_n : K_0(A_n) \to K_0(M_{s(\gamma),n})$ by sending $[p]$ to the K_0-class of this projection.

Let us denote by Δ the infinitesimal generator of the dual action of $\mathbf{T}$ on $C^*(X, \phi)$. Δ is a derivation defined on some dense subalgebra, satisfying

$$\Delta\left(\sum_{k=-N}^{N} \delta^k f_k\right) = 2\pi i \sum_{k=-N}^{N} k\delta^k f_k,$$

for $f_k \in C(X)$, $-N \leq k \leq N$ ([6, Section 5]). When $v = \sum_{k=-N}^{N} \delta^k f_k$ is in $UN(C(X), C^*(X, \phi))$, we have

$$v^*\Delta(v) = 2\pi i \sum_{k=-N}^{N} k f_k,$$

and the K_1 class of the unitary v is given by

$$\frac{1}{2\pi i}\mu(v^*\Delta(v)) \in \mathbf{Z}.$$

The following lemma is the main part of the calculation, in which we formulate the embeddings of the n-th step exact sequence into the $n+1$-th step exact sequence, using the tools of the dynamical system.

Lemma 5.3 *Let $\{p_k\}_{k=1}^{l} \subset C_{n+1}$ and $q \in C_n$ be projections such that $[q] = \sum_{k=1}^{l} m_k [p_k]$ for some $m_k \in \mathbf{N}$ in $K_0(A_{n+1})$, and μ an invariant measure. Then we have*

$$i_n \circ \rho_n([q]) \;-\; \sum_{k=1}^{l} m_k \; \rho_{n+1}([p_k]) = \frac{1}{2\pi i}\left(\mu(v_n^*\Delta(v_n)q) - \sum_{k=1}^{l} m_k \mu(v_{n+1}^*\Delta(v_{n+1})p_k)\right),$$

where $K_1(C^*(X, \phi))$ is identified with the integer group.

Proof. We only prove the equality when $l = 1$ and $m_1 = 1$. One can compute the general case in the same way. We put $p = p_1$. Then

$$i_n \circ \rho_n([q]) \;-\; \rho_{n+1}([p])$$

is represented as the difference of the projections

$$e(t) = Ad\begin{bmatrix} v_n(t) & 0 \\ 0 & v_{n+1}(t) \end{bmatrix}\left(\begin{bmatrix} q & & & \\ & 0 & & \\ & & 1-p & \\ & & & 0 \end{bmatrix}\right)$$

and

$$f(t) = Ad v_{n+1}(t)\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

The element $[v_{n+1}]$ in $K_1(C^*(X, \phi))$ equals the difference of the projection f and

$$t \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $q = \delta^k p \delta^{*k}$, the projection e is equal to

$$t \mapsto u(t) \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} u(t)^*,$$

where the unitary u of $M_4(C([0,1], C^*(X,\phi)))$ is

$$u(t) = \begin{bmatrix} v_n(t) & 0 \\ 0 & v_{n+1}(t) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta^{*k} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q & 0 & 1-q & 0 \\ 0 & 1 & 0 & 0 \\ 1-q & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\tilde{u}$ be a unitary of $M_4(M_{s(\gamma)})$ such that

$$\tilde{u}(0) = \begin{bmatrix} q & 0 & (1-q)\delta^k & 0 \\ 0 & 1 & 0 & 0 \\ 1-q & 0 & q\delta^k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then e is equivalent to $\tilde{u}e\tilde{u}^*$ and $\tilde{u}u(0) = 1$. By computing the 1-1 entry of $\tilde{u}u(1)$, we get the unitary $v_n q + \delta^k v_{n+1} \delta^{*k}(1-q)$. So we have

$$\begin{aligned} & i_n \circ \rho_n([q]) \; - \; \rho_{n+1}([p]) \\ = \; & [v_n q + \delta^k v_{n+1} \delta^{*k}(1-q)] \; - \; [v_{n+1}] \\ = \; & \frac{1}{2\pi i}(\mu(v_n^* \Delta(v_n) q) - \mu(v_{n+1}^* \Delta(v_{n+1}) p)). \end{aligned}$$

□

We must review the definition of the Ext group. The exact sequence

$$0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0$$

is an injective resolution of $\mathbf{Z}$. So we can regard the group $\mathrm{Ext}(K^0(X,\phi), \mathbf{Z})$ as the cokernel of the map from $\mathrm{Hom}(K^0(X,\phi), \mathbf{R})$ to $\mathrm{Hom}(K^0(X,\phi), \mathbf{R}/\mathbf{Z})$. Let us denote by κ the quotient map from $\mathrm{Hom}(K^0(X,\phi), \mathbf{R}/\mathbf{Z})$ to $\mathrm{Ext}(K^0(X,\phi), \mathbf{Z})$.

For a homeomorphism $\gamma \in C^+(\phi) \cap \overline{\tau[\phi]}$, we would like to define an element $\Phi(\gamma)$ of $\mathrm{Hom}(K^0(X,\phi), \mathbf{R}/\mathbf{Z})$. Fix an invariant measure μ.

Take a function $f \in C(X, \mathbf{Z})$ arbitrarily. Since γ is in the kernel of the mod map, there exists a function $g \in C(X, \mathbf{Z})$ such that

$$f - f \circ \gamma^{-1} = g - g \circ \phi^{-1},$$

and g is determined uniquely up to constant functions. So we put

$$\Phi(\gamma)([f]) \; = \; \mu(g) \; + \; \mathbf{Z}.$$

Lemma 5.4 *The map* $\Phi(\gamma)$ *is well defined and* Φ *is a homomorphism from* $C^+(\phi)\cap\overline{\tau[\phi]}$ *to* $\mathrm{Hom}(K^0(X,\phi),\mathbf{R}/\mathbf{Z})$.

Proof. For $f\in C(X,\mathbf{Z})$ we have

$$f-f\circ\phi^{-1}-(f-f\circ\phi^{-1})\circ\gamma^{-1}=f-f\circ\gamma^{-1}-(f-f\circ\gamma^{-1})\circ\phi^{-1},$$

which implies

$$\begin{aligned}\Phi(\gamma)([f-f\circ\phi^{-1}]) &= \mu(f-f\circ\gamma^{-1})\ +\ \mathbf{Z}\\ &= 0\ +\ \mathbf{Z}.\end{aligned}$$

Therefore $\Phi(\gamma)$ is a well defined homomorphism from $K^0(X,\phi)$. One can easily check that Φ is a group homomorphism. □

Now we can state the main theorem, which says that the map $\kappa\circ\Phi$ gives us an invariant, taking its value in the Ext group, for a homeomorphism which acts on the K-groups trivially.

Theorem 5.5 *When* (X,ϕ) *is a Cantor minimal system,* $\eta_0(s(\gamma))$ *equals* $\kappa\circ\Phi(\gamma)$ *for* $\gamma\in C^+(\phi)\cap\overline{\tau[\phi]}$.

Proof. To get the element in $\mathrm{Hom}(K^0(X,\phi),\mathbf{R}/\mathbf{Z})$ from the exact sequence

$$0\to K_1(C^*(X,\phi))\to K_0(M_{s(\gamma),\{y\}})\to K^0(X,\phi)\to 0,$$

we must find a map from $K_0(M_{s(\gamma),\{y\}})$ to $\mathbf{R}$ which extends the natural isomorphism $K_1(C^*(X,\phi))\cong\mathbf{Z}$. For each n the group $K_0(M_{s(\gamma),n})$ is isomorphic to

$$\mathbf{Z}\oplus\bigoplus_{p\in C_n}\mathbf{Z},$$

and the generators are $[\delta]$ and $\rho_n([p])$ $(p\in C_n)$. Let us define a homomorphism from $K_0(M_{s(\gamma),n})$ to $\mathbf{R}$ by

$$\begin{aligned}[\delta] &\mapsto 1\\ \rho_n([p]) &\mapsto \frac{1}{2\pi i}\mu(v_n^*\Delta(v_n)p).\end{aligned}$$

Then by Lemma 5.3, we have a well defined homomorphism from $K_0(M_{s(\gamma),\{y\}})$ to $\mathbf{R}$.

Hence we obtain a map which sends $[p]$ to $(2\pi i)^{-1}\mu(v_n^*\Delta(v_n)p)\ +\ \mathbf{Z}$. We would like to see that this map agrees with $\Phi(\gamma)$. Since v_n is a unitary normalizer of $C(X)$, there exist projections $\{p_k\}_{k\in\mathbf{Z}}\subset C(X)$ such that $p=\sum p_k$, which is actually a finite sum, and $v_np_k=\delta^kp_k$ for each k. When one puts

$$g_k=\begin{cases}\sum_{i=0}^{k-1}\delta^ip_k\delta^{*i} & k>0\\ 0 & k=0\\ -\sum_{i=1}^{-k}\delta^{*i}p_k\delta^i & k<0\end{cases},$$

the equation

$$p_k-\delta^kp_k\delta^{*k}=g_k-\delta g_k\delta^*$$

holds. Thus, we have

$$\begin{aligned}
p - p \circ \gamma^{-1} &= p - v_n p v_n^* \\
&= \sum p_k - v_n p_k v_n^* \\
&= \sum p_k - \delta^k p_k \delta^{*k} \\
&= \sum g_k - \delta g_k \delta^*,
\end{aligned}$$

and $\mu(g_k) = k\mu(p_k)$, which implies

$$\begin{aligned}
\Phi(\gamma)([p]) &= \mu(\sum g_k) \ + \ \mathbf{Z} \\
&= \sum k\mu(p_k) \ + \ \mathbf{Z} \\
&= \frac{1}{2\pi i}\mu(v_n^* \Delta(v_n)p) \ + \ \mathbf{Z}.
\end{aligned}$$

This equation holds for every $p \in C_n$, and so we have the conclusion. □

In Section 7, we will look at several examples illustrating the range and the kernel of this invariant.

6 Computation of OrderExt classes

In the preceding section, we proved that the automorphism $\iota(g)$ for $g \in U(C(X))$ is approximately inner but its class $\eta_0(\iota(g))$ is always zero in the Ext group. So we will consider the OrderExt group, in order to obtain more information about $\iota(g)$. In this section, we will compute the OrderExt classes of automorphisms arising from dynamical systems and describe them in the context of dynamical systems.

We must remark that in light of Lemma 4.5 we only consider the value of $[\delta]$ by the associated rotation map.

By the remark after Proposition 2.5, $\gamma \in T(\phi)$ is decomposed as a product of $\sigma \in \tau[\phi]$ and $\tau \in C^+(\phi) \cap \ker(\mathrm{mod})$. The constant valued function $t \mapsto \delta$ is a lift of δ in $M_{s(\tau)}$, which implies that the value of a lift of $[\delta]$ by the rotation map associated with $s(\gamma)$ is always zero. Thus $\tilde{\eta}_1(s(\gamma))$ is trivial for all $\gamma \in T(\phi)$, and we cannot get any information of $s(\gamma)$ from the OrderExt group.

We would like to consider the case of $\iota(g)$ for $g \in U(C(X))$. Take a self-adjoint element $h \in C(X)$ such that $g = \exp 2\pi i h$. Then

$$u : t \mapsto (\exp 2\pi i t h)\delta$$

is a unitary lifting of δ in $M_{\iota(g)}$. By the definition of the rotation map, we have

$$\begin{aligned}
R_{\iota(g)}([u])(\mu) &= \frac{1}{2\pi i}\int_0^1 \mu(\dot{u}(t)u(t)^*)dt \\
&= \frac{1}{2\pi i}\int_0^1 \mu(2\pi i h)dt \\
&= \mu(h),
\end{aligned}$$

for $\mu \in M_\phi$.

From this computation, we can determine the kernel of the map from the unitary group of $C(X)$ to the OrderExt group.

Proposition 6.1 *Let (X, ϕ) be a Cantor minimal system. For a unitary $g \in C(X)$, the following conditions are equivalent:*

(i) *The automorphism $\iota(g)$ is in the kernel of $\tilde{\eta}_1$,*

(ii) *There exists a self-adjoint element $h \in C(X)$ such that $g = \exp 2\pi i h$ and $\mu(h) = 0$ for every $\mu \in M_\phi$,*

(iii) *The unitary g is asymptotically homotopic to the coboundary subgroup U_ϕ. That is there exists a continuous map $[0,1) \ni t \mapsto g_t \in U(C(X))$ such that $\overline{g_t}(g_t \circ \phi)$ goes to g as t goes to one.*

Proof. (i) $\Rightarrow$ (ii). Suppose $g = \exp 2\pi i h$. By assumption there is $f \in C(X, \mathbf{Z})$ which satisfies

$$R_{\iota(g)}([\delta])(\mu) = \mu(f)$$

for all $\mu \in M_\phi$. Hence we get $g = \exp 2\pi i(h - f)$ and $\mu(h - f) = 0$.

(ii) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (iii). Using the same argument as in Lemma 5.1, we can find self-adjoint elements $f_n \in C(X)$ for $n = 1, 2, \cdots$, satisfying

$$\|h + (f - f \circ \phi)\|_\infty \ < \ \frac{1}{n}.$$

Combining the f_n linearly, we obtain a unitary path $\{g_t\}_{t \in [0,1)} \subset U_\phi$ such that

$$\lim_{t \to 1} \overline{g_t}(g_t \circ \phi) = g.$$

Hence g is asymptotically homotopic to U_ϕ.

(iii) $\Rightarrow$ (ii). Let $\{g_t\}_{t \in [0,1)}$ be a unitary path in U_ϕ and $\overline{g_t}(g_t \circ \phi) \longrightarrow g$ as $t \to 1$. We can take a continuous path $\{h_t\}_{t \in [0,1)} \subset C(X)_{s.a.}$ such that $g_t = \exp 2\pi i h_t$. Let f be a self-adjoint element such that $g = \exp 2\pi i f$. Then

$$\begin{aligned} \lim_{t \to 1} \exp 2\pi i(h_t - h_t \circ \phi) &= \lim_{t \to 1} \overline{g_t}(g_t \circ \phi) \\ &= g \ = \ \exp 2\pi i f, \end{aligned}$$

which implies that there exists $h \in C(X, \mathbf{Z})$ such that

$$\lim_{t \to 1} h_t - h_t \circ \phi \ = \ f + h.$$

Therefore $\mu(f + h) = 0$ for $\mu \in M_\phi$. ∎

Next, we would like to examine the range of the homomorphism $\tilde{\eta}_1 \circ \iota$ from $U(C(X))$ to the OrderExt group.

Lemma 6.2 *When (X, ϕ) is a Cantor minimal system, the natural map from $C(X)_{s.a.}$ to* $\mathrm{Aff}(M_\phi)$ *is surjective. Thus, the map $\tilde{\eta}_1 \circ \iota$ is a surjective map.*

Proof. Let h be a continuous affine function on M_ϕ. We define

$$F \ = \ \{\tau \in C(X)^* \ ; \ \tau(f\circ\phi) = \tau(f) \ \textit{for all}\ f \in C(X)\}.$$

Since F is the linear span of M_ϕ, we can extend h to a linear map from F. Moreover we can see that h is weak-$*$ continuous on the unit ball of F. The algebra $C(X)$ is norm closed in the dual of F. So we have that h is weak-$*$ continuous on the whole of F. By the Hahn-Banach theorem, we can extend h to a weak-$*$ continuous map $\tilde{h}$ defined on $C(X)^*$. Then $(\tilde{h}+\tilde{h}^*)/2$ is the desired element. □

From the above lemma we get the following corollary, which is an extension of a theorem of Kishimoto (Theorem 4.2).

Theorem 6.3 *Let A be a unital simple AT-algebra with real rank zero. Suppose $K_1(A)$ is isomorphic to $\mathbf{Z}$. Then the kernel of $\eta_0 \oplus \eta_1$ is* $\mathrm{HInn}(A)$.

Proof. We may assume A is isomorphic to $C^*(X,\phi)$ for some Cantor minimal system (X,ϕ). Let $\alpha \in \overline{\mathrm{Inn}}(A)$ be in the kernel. From the above lemma we can choose a unitary $g \in C(X)$ such as

$$\tilde{\eta_1}(\iota(g)) = \tilde{\eta_1}(\alpha)$$

in $\mathrm{OrderExt}(\mathbf{Z}, K_0(A))$. By Theorem 4.4, $\iota(g)\circ\alpha^{-1}$ is included in $\mathrm{AInn}(A)$. Since $\iota(g)$ is homotopic to the identity, the automorphism α is in $\mathrm{HInn}(A)$. □

7 Examples

(1) The first example is the odometer system. The reader may refer to Section VIII.4 of [1]. Let $\{m_n\}_n$ be a sequence of natural numbers such that m_{n+1} is a multiple of m_n for each n, and (X,ϕ) an odometer system of type $\{m_n\}_n$. The dimension group $K^0(X,\phi)$ is order isomorphic to a subgroup of $\mathbf{Q}$.

We would like to know the range and the kernel of the group homomorphism $\eta_0\circ s$ from $T(\phi)$ to the Ext group. We must remark that the mod map is zero because the dimension group admits no non-trivial automorphisms.

The space X is a topological abelian group with the unit $(0,0,0,\cdots)$ and ϕ acts by adding $(1,0,0,\cdots)$. Hence every element of the commutant group $C^+(\phi)$ is obtained by adding a certain element of X. So we have

$$X \cong C^+(\phi) = \overline{\{\phi^n ; n \in \mathbf{Z}\}}.$$

Let the homeomorphism γ be the element in the commutant group $C^+(\phi)$, which corresponds to $(a_1,a_2,a_3,\cdots)$ in X. When we take the characteristic function f on the clopen set $\{(\overbrace{0,0,\cdots,0}^{n},*,*,\cdots)\}$ as the representative of $1/m_n$ in $K^0(X,\phi)$, there exists a function

$$g = \sum_{i=0}^{N} f\circ\phi^{-i},$$

where $N = a_1 + a_2m_1 + a_3m_2 + \cdots + a_nm_{n-1} - 1$, and it satisfies $f - f\circ\gamma^{-1} = g - g\circ\phi^{-1}$. We have

$$\mu(g) = \frac{1}{m_n}(a_1 + a_2m_1 + a_3m_2 + \cdots + a_nm_{n-1})$$

for the unique invariant measure μ. Hence, by Theorem 5.5, $\eta_0(s(\gamma))$ is represented as the group homomorphism from $K^0(X,\phi)$ to $\mathbf{R}/\mathbf{Z}$ sending

$$\frac{1}{m_n} \mapsto \frac{1}{m_n}(a_1 + a_2 m_1 + a_3 m_2 + \cdots + a_n m_{n-1}) + \mathbf{Z}.$$

We would like to see that the group $\mathrm{Ext}(K^0(X,\phi),\mathbf{Z})$ is isomorphic to $X/\mathbf{Z}$, where the integer group is contained in X as the orbit of the unit by ϕ. We define a group homomorphism F from $\mathrm{Hom}(K^0(X,\phi),\mathbf{R}/\mathbf{Z})$ to $X/\mathbf{Z}$ in the following way. Let q be an element of $\mathrm{Hom}(K^0(X,\phi),\mathbf{R}/\mathbf{Z})$. We can fix real numbers q_n for $n = 0, 1, 2, \cdots$ such that $q(1/m_n) = q_n + \mathbf{Z}$, where m_0 equals 1. Then $m_n q_n - q_0$ is an integer. Hence, we can define the map F_0 from $\mathrm{Hom}(K^0(X,\phi),\mathbf{R}/\mathbf{Z})$ to X by

$$F_0(q) = \{m_n q_n - q_0\}_{n=1}^{\infty},$$

where X is identified with the projective limit of the group $\mathbf{Z}/m_n\mathbf{Z}$ and each summand is understood modulo m_n. Since

$$(m_{n+1}q_{n+1} - q_0) - (m_n q_n - q_0) = m_n(\frac{m_{n+1}}{m_n} q_{n+1} - q_n)$$

and $m_{n+1}q_{n+1}/m_n - q_n$ is an integer, the map F_0 is well defined. Moreover F_0 is surjective. Let F be the composition of F_0 and the quotient map from X to $X/\mathbf{Z}$. The map F is a well defined group homomorphism and the kernel is exactly the set of liftable maps from $K^0(X,\phi)$ to $\mathbf{R}/\mathbf{Z}$. Thus, F induces an isomorphism between $\mathrm{Ext}(K^0(X,\phi),\mathbf{Z})$ and $X/\mathbf{Z}$.

So we see that $\eta_0 \circ s$ is a surjective map from $C^+(\phi)$ to the Ext group, and its kernel is only $\{\phi^n; n \in \mathbf{Z}\}$. In this case we have got a topological version of Theorem 4.1.

Let us now consider the extension system. When the supernatural number $\{m_n\}_n$ and a natural number p do not have a common divisor, we can construct a p to one covering map $\pi : (Y,\psi) \to (X,\phi)$ associated with $[1] \in K^0(X,\phi)/pK^0(X,\phi)$. Then we can easily see that (Y,ψ) is also an odometer system of type $\{pm_n\}_n$. Of course, the shift map on Y acts trivially on $K^0(Y,\psi)$.

(2) The Denjoy system is another basic example of Cantor minimal systems ([18]).

Let $\alpha \in [0,1]$ be an irrational number, and consider the irrational rotation R_α on the circle $S^1 \cong \mathbf{R}/\mathbf{Z}$. When Q is a countable R_α-invariant subset of S^1 which contains 0, Q can be represented as the disjoint union of at most countably many orbits, say $\{R_\alpha^n(\gamma_k) = \gamma_k + n\alpha\ ;\ n \in \mathbf{Z}\}$ for $k = 1, 2, \cdots, K$ where each γ_k is in $[0,1]$, γ_1 equals 0 and K may be infinity. We consider the abelian C^*-algebra $C(X)$, inside the set of Borel functions on S^1, which is generated by $C(S^1)$ and

$$\{\chi_{[n\alpha, n\alpha+\gamma_k)}\ ;\ n \in \mathbf{Z},\ k = 1, 2, \cdots, K\}.$$

The space X is gotten from the circle by cutting at the points $n\alpha + \gamma_k$ for $n \in \mathbf{Z}$ and $k = 1, 2, \cdots, K$. We denote by ϕ the irrational rotation on the space X. Then the system (X,ϕ) becomes a Cantor minimal system, and we call this the Denjoy system of type $(\alpha; 0, \gamma_2, \gamma_3, \cdots, \gamma_K)$. By [18], The dimension group $K^0(X,\phi)$ is isomorphic to

$$\mathbf{Z} \oplus \mathbf{Z} \oplus \bigoplus_{k=2}^{K} \mathbf{Z}$$

with generators $e_0, e_1, e_2, \cdots e_K$, the unit e_0 and $\mu(e_0) = 1, \mu(e_1) = \alpha, \mu(e_j) = \gamma_j$, for $j = 2, 3, \cdots K$, where μ is the unique state.

Let $\pi : (X, \phi) \to (S^1, R_\alpha)$ be the obvious factor map. If $\gamma \in \mathrm{Homeo}(X)$ commutes with ϕ, we can choose $\tau \in C^+(R_\alpha)$ such that $R_\alpha \circ \pi(x) = \pi \circ \gamma(x)$ for some $x \in X$. Since (X, ϕ) is a minimal system, $R_\alpha \circ \pi = \pi \circ \gamma$. Therefore the commutant group $C^+(\phi)$ consists of only the rotations, which preserve the points at which the circle is cut. But the dimension group $K^0(X, \phi)$ is a direct sum of the integer group, and the Ext group is zero. So we cannot obtain the invariant for the commutant group in this case.

Now, we consider the p to one extension system (Y, ψ) of (X, ϕ) associated with $[1] \in K^0(X, \phi)/pK^0(X, \phi)$. The first return map on one sheet of Y gives us the system which is conjugate to (X, ϕ^p). The Cantor minimal system (X, ϕ^p) is the Denjoy system of type $(p\alpha; \{j\alpha + \gamma_k; j = 0, 1, \cdots, p-1\ k = 1, 2, \cdots, K\})$. The dimension group $K^0(Y, \psi)$ is order isomorphic to $K^0(X, \phi^p)$ with the map sending the order unit $[1]$ to $[p]$.

Next, we examine the extension (Y, ψ) of (X, ϕ) associated with e_1. The space Y can be identified with the the union of p circles, and $\psi(y)$ goes to the next circle or sheet only when y is in the interval $[0, \alpha)$. So we can see the system (Y, ψ) as the α/p-rotation on one circle with some cuts. Thus, (Y, ψ) is isomorphic to the Denjoy system of type $(\alpha/p; \{j/p + \gamma_k/p; j = 0, 1, \cdots, p-1\ k = 1, 2, \cdots, K\})$. The K-theory of this system has already been mentioned.

Finally, we would like to compute the K-group of the product system of the Denjoy system and the odometer system (see Proposition 3.7). Let (X, ϕ) be the Denjoy system of type $(\alpha; 0)$, (Y, ψ) the odometer system of type $\{m_n\}_n$ and (Y_n, ψ_n) the factor system of (Y, ψ) as in Proposition 3.7. We denote by (Z_n, ω_n) the product system $(X \times Y_n, \phi \times \psi_n)$, which is the m_n to one extension system of (X, ϕ). By the preceding paragraph, the dimension group of the system (Z_n, ω_n) is isomorphic to

$$K^0(X, \phi^{m_n}) \cong \mathbf{Z} \oplus \bigoplus_{k=0}^{m_n-1} \mathbf{Z},$$

with generators $[1]$ and $[\chi_{[0,\alpha)}], [\chi_{[\alpha,2\alpha)}], \cdots, [\chi_{[(m_n-1)\alpha, m_n\alpha)}]$, where these intervals are understood as clopen sets of X, and the distinguished order unit is $m_n \oplus 0$. The inclusion of C^*-algebras $C^*(Z_n, \omega_n) \subset C^*(Z_{n+1}, \omega_{n+1})$ induces an injective map from $K^0(X, \phi^{m_n})$ to $K^0(X, \phi^{m_{n+1}})$, sending

$$[1] \mapsto m[1]$$

and

$$[\chi_{[k\alpha,(k+1)\alpha)}] \mapsto \sum_{j=0}^{m-1} [\chi_{[(jm_n+k)\alpha,(jm_n+k+1)\alpha)}]$$

for $k = 0, 1, \cdots, m_n - 1$, where m is m_{n+1}/m_n. Since $C^*(X \times Y, \phi \times \psi)$ is isomorphic to the inductive limit C^*-algebra of $C^*(Z_n, \omega_n)$, we can conclude that the dimension group $K^0(X \times Y, \phi \times \psi)$ is obtained by the inductive limit of $K^0(X, \phi^{m_n})$. Therefore it is order isomorphic to

$$K^0(Y, \psi) \oplus C(Y, \mathbf{Z})$$

with the order unit $[1] \oplus 0$ and a unique state such that

$$[f] \oplus g \mapsto \mu(f) + \alpha\mu(g)$$

for $f, g \in C(Y, \mathbf{Z})$ and a unique invariant measure μ on (Y, ψ).

When the Denjoy system (X, ϕ) is of type $(\alpha; 0, \gamma_2, \cdots, \gamma_K)$, the computation can be done in the same fashion. The dimension group $K^0(X \times Y, \phi \times \psi)$ is isomorphic to

$$K^0(Y, \psi) \oplus C(Y, \mathbf{Z}) \oplus \bigoplus_{k=2}^{K} C(Y, \mathbf{Z})$$

with the order unit $[1] \oplus 0 \oplus 0$ and a unique state such that

$$[f] \oplus g \oplus (g_k)_{k=2}^{K} \mapsto \mu(f) + \alpha\mu(g) + \sum_{k=2}^{K} \gamma_k \mu(g_k)$$

for functions $f, g, g_2, \cdots, g_K \in C(Y, \mathbf{Z})$.

(3)

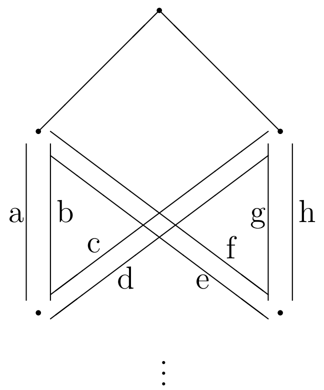

Fig. 1

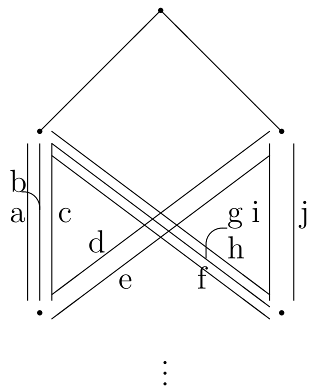

Fig. 2

We give an example where the map $\eta_0 \circ s$ is not surjective. We will also see that the extension systems may not be strong orbit equivalent even if they arise from two Cantor minimal systems, which are strong orbit equivalent, and the same elements of the dimension groups.

Let (X, ϕ) be the Cantor minimal system associated with Fig.1, where the order is determined by $a < b < c < d$ and $e < g < f < h$. Then $K^0(X, \phi)$ is isomorphic to $\mathbf{Z}[1/2]$. So (X, ϕ) is strong orbit equivalent to the odometer system of type 2^∞.

When (Z, ω) is the odometer system of type 2^∞, we can define the factor map $\pi : (X, \phi) \to (Z, \omega)$ by

$$a, e \mapsto 0 \quad b, g \mapsto 1$$

$$c, f \mapsto 2 \quad d, h \mapsto 3,$$

where Z is the infinite product space of $\{0, 1, 2, 3\}$. The map π is two to one on the orbits whose tails consist of 1 and 2, and one to one on the other orbits.

If $\gamma \in \mathrm{Homeo}(X)$ commutes with ϕ, we can choose $\tau \in C^+(\omega)$ by the same argument of Example(2) such that $\tau \circ \pi = \pi \circ \gamma$ holds. The orbits of ω whose tails consist of 1 and

2 must be preserved by τ. Hence we conclude that τ is ω^n for some $n \in \mathbf{Z}$. The map $\gamma \circ \phi^n$ commutes with ϕ and satisfies $\pi = \pi \circ \gamma \circ \phi^n$. But π is one to one on some orbits, thus $\gamma \circ \phi^n$ must be the identity map. So we have

$$C^+(\phi) = \{\phi^n \ ; \ n \in \mathbf{Z}\},$$

and in this case $\eta_0 \circ s$ is the zero map.

We can consider the three to one extension (Y, ψ) of (X, ϕ) associated with $[1] \in \mathbf{Z}[1/2]/3\mathbf{Z}[1/2]$. We would like to compute the dimension group $K^0(Y, \psi)$.

Let x be the point $(h, h, \cdots)$ in X, the infinite path space, and $y_i = (i, h, h, \cdots)$ for $i = 0, 1, 2$ be the points in $Y = \{0, 1, 2\} \times X$ which are the preimages of x. By Theorem 4.1 of [17], we get the exact sequence:

$$0 \to \mathbf{Z}^2 \to K_0(A_{\{y_0,y_1,y_2\}}) \to K^0(Y, \psi) \to 0,$$

where the integer group is contained in the infinitesimal subgroup of $K_0(A_{\{y_0,y_1,y_2\}})$.

We would like to write down the Bratteli diagram of $A_{\{y_0,y_1,y_2\}}$. We can see that the clopen set $\{(2, h, *, *, \cdots)\}$ is transferred by ψ as follows:

$$(2, h, \cdots) \leftarrow (1, f, \cdots) \leftarrow (0, g, \cdots) \leftarrow (2, e, \cdots) \leftarrow (1, h, \cdots) \ \mathit{or} \ (1, d, \cdots),$$

and the clopen set $\{(2, d, *, *, \cdots)\}$ is transferred by ψ as follows:

$$(2, d, \cdots) \leftarrow (1, c, \cdots) \leftarrow (0, b, \cdots) \leftarrow (2, a, \cdots) \leftarrow (1, h, \cdots) \ \mathit{or} \ (1, d, \cdots).$$

We denote these towers by $[2, h]$ and $[2, d]$. The clopen sets $\{(1, h, *, \cdots)\}$, $\{(1, d, *, \cdots)\}$, $\{(0, h, *, \cdots)\}$ and $\{(0, d, *, \cdots)\}$ are transferred in a similar way and their towers are denoted by $[1, h]$, $[1, d]$, $[0, h]$, and $[0, d]$.

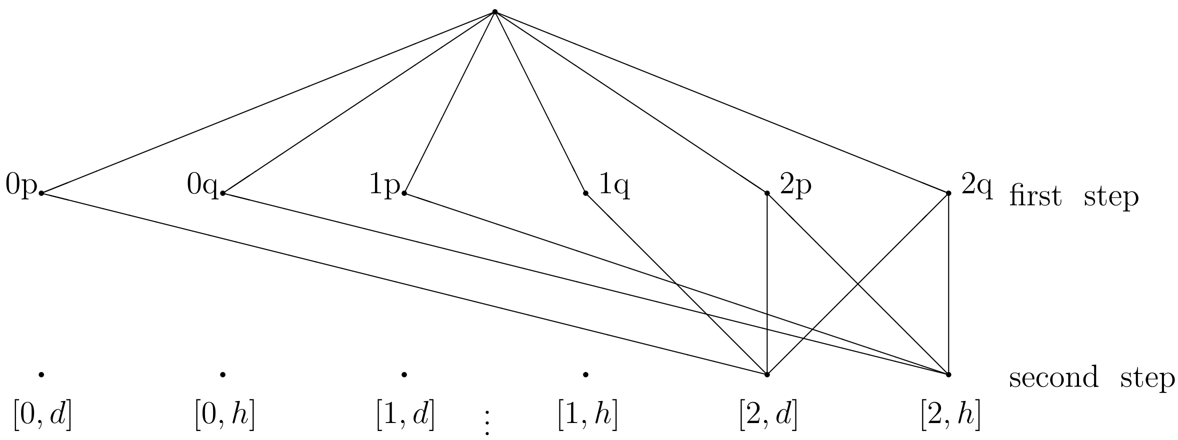

Fig. 3

By these observations, we get part of the Bratteli diagram of the AF-algebra $A_{\{y_0,y_1,y_2\}}$ in Fig.3. We won't write down the complete diagram, because the figure is too complicated. In the first step the symbol p means the clopen set

$$\{(a, *, *, \cdots)\} \cup \{(b, *, *, \cdots)\} \cup \{(e, *, *, \cdots)\} \cup \{(f, *, *, \cdots)\}$$

in X, and q the complement of p. The numbers $0,1,2$ represent each sheet, so that 0p means the clopen set

$$\{(0,a,*,*,\cdots)\}\cup\{(0,b,*,*,\cdots)\}\cup\{(0,e,*,*,\cdots)\}\cup\{(0,f,*,*,\cdots)\}$$

in Y, and the others are similar. Then the inclusion from the first step to the second step is represented by the matrix:

$$A=\begin{bmatrix}1&1&1&0&0&1\\1&1&0&1&1&0\\0&1&1&1&1&0\\1&0&1&1&0&1\\1&0&0&1&1&1\\0&1&1&0&1&1\end{bmatrix}.$$

In the third step, the clopen set $\{(2,h,h,*,*,\cdots)\}$ is transferred by ψ as follows:

$$[2,h]\leftarrow[1,d]\leftarrow[0,h]\leftarrow[2,d]\leftarrow(1,h,h,\cdots)\ or\ (1,h,d,\cdots),$$

and the clopen set $\{(2,d,d,*,*,\cdots)\}$ is transferred as follows:

$$[2,h]\leftarrow[1,h]\leftarrow[0,d]\leftarrow[2,d]\leftarrow(1,h,h,\cdots)\ or\ (1,h,d,\cdots).$$

So the inclusion from the second step to the third step is represented by A, too. The deeper steps are exactly the same. Hence the dimension group $K_0(A_{\{y_0,y_1,y_2\}})$ is obtained as the inductive limit of $\mathbf{Z}^6$ with the inclusion matrix A in every step. This dimension group has infinitesimal elements represented by ${}^t(1,1,-1,-1,0,0)$ and ${}^t(0,0,1,1,-1,-1)$ in the first step, which generates the subgroup $\mathbf{Z}^2$. Moreover, ${}^t(2,-2,-1,1,-1,1)$ is an eigenvector of A^4 with respect to the eigenvalue 9 and the elements of the dimension group represented by these vectors in each step are infinitesimal elements. Thanks to the exact sequence, we have that the dimension group $K^0(Y,\psi)$ has a non zero infinitesimal subgroup, which implies the system (Y,ψ) is not strong orbit equivalent to an odometer system. And the shift map acts on $K^0(Y,\psi)$ non trivially. The commutant group $C^+(\psi)$ is generated by ψ and the order three shift map.

Let us consider the Cantor minimal system arising from the ordered Bratteli diagram Fig.2, where the order is induced by $a<b<d<e<c$ and $f<i<j<g<h$. The associated dimension group $K^0(X,\phi)$ is isomorphic to $\mathbf{Z}[1/5]$.

By the same argument we can see that the commutant subgroup of ϕ is trivial.

We can define the two to one covering map $\pi:(Y,\psi)\to(X,\phi)$ associated with $[1]\in\mathbf{Z}[1/5]/2\mathbf{Z}[1/5]$.

In the same way, we have the exact sequence:

$$0\to\mathbf{Z}\to K_0(A_{\{y_0,y_1\}})\to K^0(Y,\psi)\to 0,$$

where y_r for $r=0,1$ are the points $(r,c,c,\cdots)$ in Y. And the Bratteli diagram of the AF-algebra $A_{\{y_0,y_1\}}$ is represented by the matrix:

$$A=\begin{bmatrix}2&1&1&1\\2&1&1&1\\1&1&2&1\\1&1&2&1\end{bmatrix}.$$

Therefore we see that the infinitesimal subgroup of the ordered group $K_0(A_{\{y_0,y_1\}})$ is isomorphic to $\mathbf{Z}$, and the shift map γ acts on $K^0(Y,\psi)$ trivially. So, in this case, the dimension group $K^0(Y,\psi)$ is isomorphic to $\mathbf{Z}[1/5]$ with the unit 2. Thus the system (Y,ψ) is strong orbit equivalent to the odometer system of type $2\cdot 5^\infty$.

An easy computation shows that the Ext class $\eta_0\circ s(\gamma)$ is a non zero element of order two.

(4) Finally we give an example of a homeomorphism γ which satisfies $\eta_0(s(\gamma))=0$, but which is not in the topological full group.

Let (X,ϕ) be the Cantor minimal system represented by the diagram in Fig.4, where the order is $a<b<c<d$ and $e<g<f<h$. The dimension group $K^0(X,\phi)$ is isomorphic to

$$G=\{(n,\frac{l}{4^m})\ ;\ n,l\in\mathbf{Z}\ m\in\mathbf{N}\ n+l=0\mod 3\},$$

with the unit $(0,3)$ and has a unique trace given by

$$(n,\frac{l}{4^m})\mapsto\frac{1}{3}\frac{l}{4^m}.$$

We can construct a factor map from (X,ϕ) to the odometer system of type 2^∞ exactly as in Example(3). It is easily seen that the commutant subgroup of ϕ is trivial.

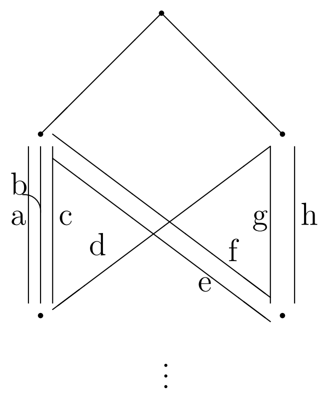

Fig. 4

We can construct the two to one covering map $\pi:(Y,\psi)\to(X,\phi)$ associated to the element $(-1,1)\in K^0(X,\phi)/2K^0(X,\phi)$. The corresponding $\mathbf{Z_2}$-valued cocycle is not zero on the clopen set

$$\{(d,*,\cdots)\}\cup\{(g,*,\cdots)\}\cup\{(h,*,\cdots)\}.$$

We have the exact sequence:

$$0\to\mathbf{Z}\to K_0(A_{\{y_0,y_1\}})\to K^0(Y,\psi)\to 0,$$

where y_r for $r=0,1$ are the points $(r,h,h,\cdots)$ in $Y=\{0,1\}\times X$. The homeomorphism ψ transfers the clopen set $\{(1,h,*,\cdots)\}$ as follows:

$$(1,h,\cdots)\leftarrow(1,f,\cdots)\leftarrow(0,g,\cdots)\leftarrow(0,e,\cdots)\leftarrow(1,h,\cdots)\ or\ (1,d,\cdots),$$

and the clopen set $\{(1, d, *, \cdots)\}$ as follows:

$$(1, d, \cdots) \leftarrow (1, c, \cdots) \leftarrow (1, b, \cdots) \leftarrow (1, a, \cdots) \leftarrow (0, h, \cdots) \textit{ or } (0, d, \cdots).$$

The clopen sets $\{(0, h, *, \cdots)\}$ and $\{(0, d, *, \cdots)\}$ are also transferred in a similar way. When we denote these towers by $[1, h], [1, d], [0, h]$ and $[0, d]$, the clopen set $\{(1, h, h, *, \cdots\}$ goes through:

$$[1, h] \leftarrow [1, d] \leftarrow [0, h] \leftarrow [0, d] \leftarrow (1, h, h, *, \cdots) \textit{ or } (1, h, d, *, \cdots),$$

and the clopen set $\{(1, h, d, *, \cdots)\}$ goes through:

$$[1, h] \leftarrow [1, d] \leftarrow [0, d] \leftarrow [1, d] \leftarrow (0, h, h, *, \cdots) \textit{ or } (0, h, d, *, \cdots).$$

From this, we obtain the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

which represents the Bratteli diagram of the AF-algebra $A_{\{y_0,y_1\}}$.

Then we can compute the dimension group $K_0(A_{\{y_0,y_1\}})$. We have that the generator of the subgroup $\mathbf{Z}$ is represented by the vector ${}^t(1, 0, -1, 0)$, which is invariant under A, and the order two shift map γ acts trivially. The dimension group $K^0(Y, \psi)$ can be obtained as the quotient $K_0(A_{\{y_0,y_1\}})/\mathbf{Z}$, and it is order isomorphic to

$$\{(n, \frac{l}{4^m}) \ ; \ n, l \in \mathbf{Z} \ m \in \mathbf{N} \ n = l \mod 3\}$$

with the order unit $(0, 6)$ and a unique state given by

$$(n, \frac{l}{4^m}) \mapsto \frac{1}{6}\frac{l}{4^m}.$$

We would like to see that the Ext class $\eta_0(s(\gamma))$ is zero. The group $K^0(Y, \psi)$ is algebraically isomorphic to $\mathbf{Z} \oplus \mathbf{Z}[1/2]$, and so it suffices to consider the subgroup $\mathbf{Z}[1/2]$. We remark that the order unit is expressed by $0 \oplus 1 \in \mathbf{Z} \oplus \mathbf{Z}[1/2]$. Since the matrix A has the eigenvalue 4 with an eigenvector ${}^t(1, 1, 1, 1)$, we can choose the characteristic function f on the clopen set

$$\{(1, \overbrace{h, \cdots, h}^{m-1}, h, *, \cdots)\} \cup \{(1, \overbrace{h, \cdots, h}^{m-1}, d, *, \cdots)\}$$
$$\cup \{(0, h, \cdots, h, h, *, \cdots)\} \cup \{(0, h, \cdots, h, d, *, \cdots)\}$$

as the representative of the element $1/4^m \in \mathbf{Z}[1/2]$. These functions are invariant under γ. Hence $f - f \circ \gamma^{-1} = 0$, which implies the Ext class $\eta_0(s(\gamma))$ is zero (Theorem 5.5). The OrderExt class $\tilde{\eta}_1(s(\gamma))$ is also zero. Therefore the order two automorphism $s(\gamma)$ on $C^*(Y, \psi)$ is asymptotically inner by Theorem 4.4.

8 Problems

In our final section, we would like to present some open problems.

(1) We defined a new invariant for the group $T(\phi)$ in Section 5. However, in our examples, the range of the invariant is trivial except for the case of odometer systems or systems that are strong orbit equivalent to odometer systems. Does the invariant take non-trivial values in the case of a Cantor minimal system which is not strong orbit equivalent to an odometer system?

(2) In Section 7, we constructed an example where $\eta_0(s(\gamma))$ becomes zero for a non-trivial element γ of $T(\phi)$. But we don't know why this phenomenon occurs. Can we find a characterization of the kernel of $\eta_0 \circ s$ in terms of the dynamical systems?

References

[1] K. R. Davidson, *C^*-algebras by examples*, Fields Institute Monographs 6, Amer. Math. Soc., Providence, RI,1996.

[2] E. G. Effros, *Dimensions and C^*-algebras*, Conf. Board Math. Sci.46, Amer. Math. Soc., Providence, R.I., 1981.

[3] G. A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. reine angew. Math., 443 (1993), 179-219.

[4] G. A. Elliott and M. Rørdam, *The automorphism group of the irrational rotation C^*-algebra*, Commun. Math. Phys., 155 (1993), 3-26.

[5] T. Giordano, I. F. Putnam and C. F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. reine angew. Math., 469 (1995), 51-111.

[6] T. Giordano, I. F. Putnam and C. F. Skau, *Full groups of Cantor minimal systems*, Israel J. Math., 111 (1999), 285-320.

[7] T. Giordano, I. F. Putnam and C. F. Skau, *K-theory and asymptotic index for certain almost one-to-one factors*, Math. Scand., 89 (2001), 297-319.

[8] E. Glasner, B. Weiss, *Weak orbit equivalence of Cantor minimal systems*, Intern. J. Math., 6 (1995), 559-579.

[9] R. H. Herman, I. F. Putnam and C. F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Intern. J. Math., 3 (1992), 827-864.

[10] Kishimoto A. and A. Kumjian, *The Ext class of an approximately inner automorphism*, Trans. Amer. Math. Soc., 350 (1998), no. 10, 4127-4148.

[11] Kishimoto A. and A. Kumjian, *The Ext class of an approximately inner automorphism, II*, J. Operator Theory, 46 (2001), 99-122.

[12] Kishimoto A., *Automorphisms of AT algebras with the Rohlin property*, J. Operator Theory, 40 (1998), 277-294.

[13] Kishimoto A., *Unbounded derivations in AT algebras*, J. Funct. Anal., 160 (1998), 270-311.

[14] G. W. Mackey, *Ergodic theory and virtual groups*, Math. Ann., 166 (1966), 187-207.

[15] W. Parry, *Abelian group extensions of discrete dynamical systems*, Z. Wahrsch. Verw. Geb., 13 (1969), 95-113.

[16] G. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, New York 1979.

[17] I. F. Putnam, *The C^*-algebras associated with minimal homeomorphisms of the Cantor set*, Pacific J. Math., 136 (1989), 329-353.

[18] I. F. Putnam, K. Schmidt and C. F. Skau, *C^*-algebras associated with Denjoy homeomorphisms of the circle*, J. Operator Theory, 16 (1986), 99-126.

[19] Tomiyama J., *Invitation to C^*-algebras and Topological Dynamics*, World Sc. Adv. Ser. Dyn. Sys. 3, World Scientific, Singapore 1987.

[20] Tomiyama J., *Topological full groups and structure of normalizers in transformation group C^*-algebras*, Pacific J. Math., 173 (1996), 571-583.

[21] R. Zimmer, *Extensions of ergodic group actions*, Illinois J. Math., 20 (1976), 373-409.

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