An absorption theorem for minimal AF equivalence relations on Cantor sets

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Abstract

We prove that a 'small' extension of a minimal AF equivalence relation on a Cantor set is orbit equivalent to the AF relation. By a 'small' extension we mean an equivalence relation generated by the minimal AF equivalence relation and another AF equivalence relation which is defined on a closed thin subset. The result we obtain is a generalization of the main theorem in [GMPS2]. It is needed for the study of orbit equivalence of minimal $\mathbb{Z}^d$-systems for $d > 2$ [GMPS3], in a similar way as the result in [GMPS2] was needed (and sufficient) for the study of minimal $\mathbb{Z}^2$-systems [GMPS1].

1 Introduction

In the present paper we study equivalence relations on Cantor sets. By a Cantor set, we mean a compact, metrizable and totally disconnected space without isolated points. The topological orbit structure of countable group actions as homeomorphisms on Cantor sets has been studied by several authors [GPS1], [GMPS1]. More precisely, minimal $\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions on Cantor sets have been classified up to orbit equivalence. The strategy is to prove that the equivalence relation associated with the given minimal action is orbit equivalent to an AF relation (see Definition 1.1). To prove this, we need a delicate 'glueing' procedure, an essential part of which is done by the absorption theorem ([GPS2, Theorem 4.18], [GMPS2, Theorem 4.6]). Indeed, the result in [GMPS2] was sufficient for the study of orbit equivalence of minimal $\mathbb{Z}^2$-actions [GMPS1]. The aim of this paper is to prove a stronger version of the absorption theorem, which is needed for the study of minimal $\mathbb{Z}^d$-actions for $d > 2$ [GMPS3]. We refer to [GPS2] and [GMPS2] as both background and reference for specific results that we shall need in the sequel.

We will give a brief description of how a strengthening of the absorption theorem is needed in order to generalize the results for minimal $\mathbb{Z}^2$-actions to minimal $\mathbb{Z}^d$-actions. Let φ be a minimal free $\mathbb{Z}^d$-action on a Cantor set. For the associated equivalence relation R_φ, we will construct an increasing sequence of subrelations $R_0 \subset R_1 \subset \cdots \subset R_d = R_\varphi$ so that R_0 is a minimal AF equivalence relation with the relative topology from R_φ and each R_i is a 'small' extension of R_{i-1}. Then, we apply inductively the absorption theorem to

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$R_{i-1} \subset R_i$ and show that each R_i is orbit equivalent to an AF relation for $i = 1, 2, \dots, d$. In such a way, after d-times use of the absorption theorem, we can conclude that $R_d = R_\varphi$ is orbit equivalent to an AF relation and thus complete the classification up to orbit equivalence. One of the problems in this argument is to describe the difference between R_{i-1} and R_i. In the case of $d = 2$, we could find another compact relation K_i which is (locally) transverse to R_{i-1} so that R_i is generated by R_{i-1} and K_i (see [GMPS1]). For $d > 2$, however, we cannot find such a nice transverse relation, and so it is necessary to generalize the absorption theorem in [GMPS2]. The new absorption theorem (Theorem 3.2) in this paper does not need transverse relations and that is what is needed for the study of $\mathbb{Z}^d$-actions.

We collect notation and terminology relevant to this paper. Let X be a compact, metrizable and totally disconnected space and let $R \subset X \times X$ be an equivalence relation (we may call an equivalence relation just a relation). For a subset $A \subset X$, we set

$$R[A] = \{x \in X \mid \text{there exists } y \in A \text{ such that } (x, y) \in R\}.$$

The set $R[A]$ is called the R-saturation of A. For $x \in X$, we denote $R[\{x\}]$ by $R[x]$ and call it the R-orbit of x. We deal with only an equivalence relation with countable orbits (i.e. $R[x]$ is at most countable for each $x \in X$). When $R[x]$ is dense in X for each $x \in X$, we say that R is minimal. For a subset $A \subset X$, we denote $R \cap (A \times A)$ by $R|A$ and call it the restriction. When R and S are relations on X, we let $R \vee S$ denote the equivalence relation on X generated by R and S.

Suppose that R is equipped with a topology in which R is étale ([GPS2, Definition 2.1]). A closed subset $Y \subset X$ is called R-étale, if the restriction $R|Y = R \cap (Y \times Y)$ with the relative topology from R is étale. A subset $Y \subset X$ is called R-thin, if $\mu(Y)$ is zero for any R-invariant probability measure μ on X.

We collect several basic facts about étale equivalence relations. The reader should see [GPS2] and [GMPS2]. Let R be an étale relation on a Cantor set X. If $O \subset X$ is open, then its R-saturation $R[O]$ is also open. If R is compact, then the topology on R coincides with the topology from the product topology of $X \times X$. If R is compact and $O \subset X$ is clopen, then the R-saturation $R[O]$ is also clopen (and hence compact). One can easily show that a subrelation S of R is étale with respect to the relative topology from R if and only if S is an open subset of R. If $\mu(Y) = 0$ for a Borel subset Y of X and an R-invariant probability measure μ, then $\mu(R[Y])$ is also zero.

The following is the definition of AF equivalence relations.

Definition 1.1 ([GPS2, Definition 3.7, 4.1])**.** An étale equivalence relation R is called an AF relation, if there exists an increasing sequence $R_1 \subset R_2 \subset \dots$ of compact open subrelations of R such that $R = \bigcup_{n\in\mathbb{N}} R_n$. An equivalence relation R is said to be affable, if R is orbit equivalent to an AF relation.

We have to recall the notion of Bratteli diagrams. A Bratteli diagram (V, E) consists of a vertex set V and an edge set E, where V and E can be written as a countable disjoint union of non-empty finite sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \dots \text{ and } E = E_1 \cup E_2 \cup E_3 \cup \dots$$

with the following property: An edge e in E_n goes from a vertex in V_{n-1} to one in V_n, which we denote by $s(e)$ and $r(e)$, respectively. We require that there are no sinks, i.e. $s^{-1}(v) \neq \emptyset$ for all $v \in V$. If (V, E) has only one source $v_0 \in V$ —which necessarily entails $V_0 = \{v_0\}$— we will call (V, E) a standard Bratteli diagram.

For a standard Bratteli diagram (V, E),

$$X_{(V,E)} = \left\{ (e_1, e_2, \dots) \in \prod_{n \in \mathbb{N}} E_n \mid r(e_n) = s(e_{n+1}) \text{ for all } n \in \mathbb{N} \right\}$$

is called the infinite path space. Equipped with the relative topology from $\prod_{n\in\mathbb{N}} E_n$, $X_{(V,E)}$ is compact, metrizable and totally disconnected. For every $n \in \mathbb{N}$, let

$$R_n = \{(e, f) \in X_{(V,E)} \times X_{(V,E)} \mid e_k = f_k \text{ for all } k > n\},$$

where e_k and f_k denote the k-th edge of e and f, respectively. Give R_n the relative topology from $X_{(V,E)} \times X_{(V,E)}$. Then R_n is a compact étale equivalence relation. Let

$$AF(V, E) = \bigcup_n R_n$$

and give $AF(V, E)$ the inductive limit topology, so that $AF(V, E)$ is an AF equivalence relation.

It is known that $AF(V, E)$ is the prototype of an AF relation. More precisely, for any AF equivalence relation R on a compact, metrizable totally disconnected space X, there exists a standard Bratteli diagram (V, E) such that R is isomorphic to $AF(V, E)$ ([GPS2, Theorem 3.9]).

We need the following lemma in the next section. We have been unable to find a suitable reference in the literature, and so we include a proof for completeness.

Lemma 1.2. *Let X be a compact metrizable totally disconnected space. Suppose R and S are compact étale equivalence relations on X. If S is contained in R, then there exists a finite set K and a continuous map $\mu : X \to K$ such that*

$$S = \{(x, x') \in R \mid \mu(x) = \mu(x')\}.$$

Proof. First, we note that S is automatically open in R (see the comment following Definition 3.7 in [GPS2] for example).

Let Y be the quotient space of X by the relation S. From Proposition 3.2 of [GPS2] and its proof, we can see that Y is compact, metrizable and totally disconnected. Let us denote the quotient map by π.

For $f \in C(Y, \mathbb{Z})$, we define

$$R_f = \{(x, x') \in R \mid f(\pi(x)) = f(\pi(x'))\}.$$

It is easy to see that R_f is a closed subset of R and that S is contained in R_f. If (x, x') does not belong to S, then there exists $f \in C(Y, \mathbb{Z})$ such that $f(\pi(x)) \neq f(\pi(x'))$. Hence we have

$$S = \bigcap_{f \in C(Y,\mathbb{Z})} R_f.$$

Since S is open in R and R is compact, there exists a finite subset $A \subset C(Y, \mathbb{Z})$ such that

$$S = \bigcap_{f \in A} R_f.$$

Put $K = \{(f(y))_{f\in A} \in \mathbb{Z}^A \mid y \in Y\}$ and define $\mu : X \to K$ by $\mu(x) = (f(\pi(x)))_{f\in A}$. It is easy to see that K and μ have the desired properties. □

2 A splitting theorem

Let R be a minimal AF equivalence relation on a Cantor set X and let $Y \subset X$ be a closed, R-étale and R-thin subset. By Theorem 3.11 of [GPS2], $R|Y = R \cap (Y \times Y)$ with the relative topology is an AF equivalence relation on Y. Suppose that we are given an equivalence relation S on Y and that S is an open subset of $R|Y$. Note that S in the relative topology from R is also an AF equivalence relation on Y by [GPS2, Proposition 3.12 (ii)].

We would like to prove the following theorem in this section.

Theorem 2.1. *In the setting above, there exists an equivalence relation R' on X which satisfies the following.*

(1) *R' is an open subset of R.*

(2) *R' is minimal.*

(3) *$R'|Y$ is equal to S.*

(4) *$R'[Y]$ is equal to $R[Y]$.*

(5) *If $x \in X$ does not belong to $R[Y]$, then $R'[x] = R[x]$.*

(6) *Any R'-invariant probability measure on X is R-invariant.*

The property (3) of the above theorem means that, for every $y \in Y$, its R-orbit $R[y]$ splits into several R'-orbits and $R'[y] \cap Y$ equals $S[y]$. But, the property (5) means that if $R[x]$ does not meet Y, then $R[x]$ does not split. Note that (4) and (5) imply $R = R' \vee (R|Y)$.

At first, we need to represent the AF equivalence relation R on X by a Bratteli diagram. By Theorem 3.11 of [GPS2], there exists a standard Bratteli diagram (V, E), a subdiagram (W, F) (i.e. $W \subset V$, $F \subset E$) satisfying $r(F) \cup \{v_0\} = W$ and a homeomorphism $\pi : X \to X_{(V,E)}$ such that the following are satisfied.

- $\pi \times \pi$ induces an isomorphism from R to $AF(V, E)$.
- $\pi(Y)$ is equal to $\{(e_n)_n \in X_{(V,E)} \mid e_n \in F \text{ for all } n \in \mathbb{N}\}$.

Note that $\pi|Y \times \pi|Y$ induces an isomorphism between $R|Y$ and $AF(W, F)$. To simplify notation, we identify $X_{(V,E)}$ with X and omit π. We remark that (V, E) is a simple Bratteli diagram, because R is minimal. Moreover, (W, F) is a thin subdiagram of (V, E), because Y is R-thin in X.

Let R_0 be the trivial relation on X, that is, $R_0 = \{(x,x) \mid x \in X\}$. For $n \in \mathbb{N}$, we define

$$R_n = \{(x,x') \in X \times X \mid x_k = x'_k \text{ for all } k > n\},$$

where x_k and x'_k denote the k-th edge of infinite paths x and x', respectively. Notice that $R_0 \subset R_1 \subset R_2 \subset \dots$ and $R = \bigcup_n R_n$.

Since S is an AF relation, there exists an increasing sequence of compact open subrelations $S_1 \subset S_2 \subset S_3 \subset \dots$ in S such that $S = \bigcup_m S_m$. For any $m \in \mathbb{N}$, S_m is contained in $R|Y$ and $R|Y$ is a union of open subsets $R_n|Y$. It follows from the compactness of S_m that there exists an increasing sequence $n_1 < n_2 < \dots$ such that $S_m \subset R_{n_m}|Y$ for all $m \in \mathbb{N}$. By telescoping (V,E) to levels $0 < n_1 < n_2 < \dots$, we may assume that $S_n \subset R_n|Y = R_n \cap (Y \times Y)$ for all $n \in \mathbb{N}$.

Let v_0 be the unique vertex in V_0. For $v \in V_n$ and $w \in V_m$ with $0 \le n < m$, we denote the set of paths in (V,E) from v to w by $E(v,w)$. Let $F(v,w)$ be the set of paths $(e_1, e_2, \dots, e_{m-n})$ in $E(v,w)$ such that $e_i \in F$ for all $i = 1, 2, \dots, m-n$.

Lemma 2.2. *There exists an increasing sequence of non-negative integers* $\{n(k)\}_{k=0}^{\infty}$ *with* $n(0) = 0$ *such that*

$$|F(v_0,w)| \le \sum_{v \in V_{n(k-1)}} |E(v,w) \setminus F(v,w)|$$

for all $w \in W_{n(k)}$ *and* $k \in \mathbb{N}$.

Proof. By Lemma 4.12 of [GPS2], we can find $n(1) \ge 1$ such that $2|F(v_0,w)| \le |E(v_0,w)|$ for all $w \in W_{n(1)}$, which means

$$|F(v_0,w)| \le |E(v_0,w) \setminus F(v_0,w)|$$

for all $w \in W_{n(1)}$.

Put

$$L_n = \max_{v \in W_n} |F(v_0,v)|.$$

Let us find $n(2), n(3), n(4), \dots$ inductively. Suppose that $n(k-1)$ has been chosen. Since Y is R-thin, by Lemma 4.12 of [GPS2], there exists $n(k) > n(k-1)$ such that

$$(L_{n(k-1)} + 1)|F(v,w)| \le |E(v,w)|$$

for all $v \in W_{n(k-1)}$ and $w \in W_{n(k)}$. It follows that

$$\begin{aligned} |F(v_0,w)| &\le \sum_{v \in W_{n(k-1)}} L_{n(k-1)} |F(v,w)| \\ &\le \sum_{v \in W_{n(k-1)}} |E(v,w) \setminus F(v,w)| \\ &\le \sum_{v \in V_{n(k-1)}} |E(v,w) \setminus F(v,w)| \end{aligned}$$

for any $w \in W_{n(k)}$. □

From the lemma above, by telescoping (V, E) to levels $0 = n(0) < n(1) < n(2) < \dots$, we may assume that

$$|F(v_0, w)| \leq \sum_{v \in V_{n-1}} |E(v, w) \setminus F(v, w)| \qquad \text{for all } w \in W_n \text{ and } n \in \mathbb{N}.$$

Therefore, for $w \in W$, we can find a surjective map ρ_w from $\{e \in E \setminus F \mid r(e) = w\}$ to $F(v_0, w)$.

Lemma 2.3. *There exist finite sets K_n, continuous maps $\lambda_n : X \to K_n$ and clopen subsets $U_n \subset X$ which satisfy the following.*

(1) *For every $n \in \mathbb{N}$, $S_n = \{(y, y') \in R_n \cap (Y \times Y) \mid \lambda_n(y) = \lambda_n(y')\}$.*

(2) *For every $n \in \mathbb{N}$, Y is contained in U_n.*

(3) *For every $n \in \mathbb{N}$, $\displaystyle\bigcap_{m \geq n} R_m[U_m] = R_n[Y]$.*

(4) *For every $n \in \mathbb{N} \setminus \{1\}$, if $(x, x') \in R_{n-1}$ and $\lambda_{n-1}(x) = \lambda_{n-1}(x')$, then $\lambda_n(x) = \lambda_n(x')$.*

(5) *For every $n \in \mathbb{N}$, if $x, x' \notin R_n[U_n]$, then $\lambda_n(x) = \lambda_n(x')$.*

(6) *For every $n \in \mathbb{N}$ and $y \in Y$, there exists $x \in R_n[y]$ such that if $(x, x') \in R_{n-1}$, then $\lambda_n(x') = \lambda_n(y)$.*

(7) *For every $n \in \mathbb{N} \setminus \{1\}$ and $y \in U_n$, we have*

$$\min_{v \in V_{n-1}} |E(v_0, v)| \times |\{x \in R_n[y] \cap U_n \mid \lambda_n(x) = \lambda_n(y)\}| \leq |\{x \in R_n[y] \mid \lambda_n(x) = \lambda_n(y)\}|.$$

(8) *For every $n \in \mathbb{N}$ and $x \in R_n[Y]$, there exists $y \in Y$ such that $(x, y) \in R_n$ and $\lambda_n(x) = \lambda_n(y)$.*

Proof. Since S_n is contained in $R_n|Y$, by applying Lemma 1.2, we get a finite set K_n and a continuous map $\mu_n : Y \to K_n$ such that

$$S_n = \{(y, y') \in R_n|Y \mid \mu_n(y) = \mu_n(y')\}. \tag{2.1}$$

For $k \in \mathbb{N}$, we define

$$Y_k = \{(x_n)_n \in X \mid x_n \in F \text{ for all } n = 1, 2, \dots, k\}.$$

The clopen sets Y_k's form a decreasing sequence and $\bigcap_k Y_k = Y$. For $w \in W$, let ρ_w be a surjective map from $\{e \in E \setminus F \mid r(e) = w\}$ to $F(v_0, w)$ as above.

First of all, let us find U_1 and $\lambda_1 : X \to K_1$. Put $U_1 = Y_2$. Then (2) for $n = 1$ is clear. Let $\tilde{\mu}_1 : U_1 \to K_1$ be an arbitrary continuous extension of $\mu_1 : Y \to K_1$. For $x \in U_1$, we define $\lambda_1(x) = \tilde{\mu}_1(x)$ This, together with (2.1), implies (1) for $n = 1$. On $X \setminus R_1[U_1]$, we fix an element of K_1 and let λ_1 be the constant map to this element, so that (5) is satisfied. Suppose that x is in $R_1[U_1] \setminus U_1$. Let x_k denote the k-th edge of the infinite path $x \in X$. It is easy to see $x_1 \notin F$ and $r(x_1) \in W$. Since $x_2 \in F$,

$$\tilde{x} = (\rho_{r(x_1)}(x_1), x_2, x_3, \dots) \in X$$

belongs to U_1. Hence we can define $\lambda_1(x) = \lambda_1(\tilde{x})$. One observes that $\lambda_1 : X \to K_1$ is continuous. To check (8), let $x \in R_1[Y] \setminus Y$. From $x_1 \notin F$, $r(x_1) \in W$ and $x_2 \in F$, we can see that x belongs to $R_1[U_1] \setminus U_1$. Obviously, $\tilde{x} = (\rho_{r(x_1)}(x_1), x_2, x_3, \dots)$ is in Y, and so (8) for $n = 1$ follows.

We would like to construct U_n and $\lambda_n : X \to K_n$ inductively. Let us assume that U_{n-1} and λ_{n-1} have been fixed. Let $\tilde{\mu}_n : Y_{n+1} \to K_n$ be an arbitrary continuous extension of $\mu_n : Y \to K_n$. We claim that there exists $k > n$ such that if $x, x' \in Y_k$ satisfies $(x, x') \in R_{n-1}$ and $\lambda_{n-1}(x) = \lambda_{n-1}(x')$, then $\tilde{\mu}_n(x) = \tilde{\mu}_n(x')$. Otherwise, for each $k > n$, we would have $x(k), x'(k) \in Y_k$ with $(x(k), x'(k)) \in R_{n-1}$, $\lambda_{n-1}(x(k)) = \lambda_{n-1}(x'(k))$ and $\tilde{\mu}_n(x(k)) \neq \tilde{\mu}_n(x'(k))$. We may assume that two sequences $x(k), x'(k)$ converge to $y, y' \in Y$, respectively, because X is compact and $\bigcap Y_k = Y$. By compactness of R_{n-1}, we also have $(y, y') \in R_{n-1}$. Combining this with $\lambda_{n-1}(y) = \lambda_{n-1}(y')$, by (1) for $n-1$, we get $(y, y') \in S_{n-1}$. On the other hand, by (2.1) and $\mu_n(y) \neq \mu_n(y')$, (y, y') does not belong to S_n, which contradicts $S_{n-1} \subset S_n$. Hence we can find $k > n$ which has the desired property. We put $U_n = Y_k$, so that

$$(x, x') \in R_{n-1}|U_n \text{ and } \lambda_{n-1}(x) = \lambda_{n-1}(x') \quad \Rightarrow \quad \tilde{\mu}_n(x) = \tilde{\mu}_n(x'). \tag{2.2}$$

Notice that Y is contained in U_n and U_n is contained in Y_{n+1}.

Next, we would like to define a continuous map $\lambda_n : X \to K_n$. Fix an element $\kappa_0 \in K_n$. Let $x \in R_{n-1}[U_n]$. If there exists $x' \in U_n$ such that $(x, x') \in R_{n-1}$ and $\lambda_{n-1}(x) = \lambda_{n-1}(x')$, then we define $\lambda_n(x) = \tilde{\mu}_n(x')$. This is well-defined because of (2.2). If there does not exist such $x' \in U_n$, then we define $\lambda_n(x) = \kappa_0$. Notice that this definition implies (1) and (4) for $x, x' \in R_{n-1}[U_n]$. For $x \notin R_n[U_n]$, we define $\lambda_n(x) = \kappa_0$, so that (5) is satisfied. Suppose that x is in $R_n[U_n] \setminus R_{n-1}[U_n]$. Let $x_k \in E$ denote the k-th edge of x. From $x \notin R_{n-1}[U_n]$, we can see that $x_n \in E \setminus F$. Since x is in $R_n[U_n]$, we also get $r(x_n) \in W$. By definition of $\rho_{r(x_n)}$, $\rho_{r(x_n)}(x_n)$ is in $F(v_0, r(x_n))$. It follows that

$$\tilde{x} = (\rho_{r(x_n)}(x_n), x_{n+1}, x_{n+2}, \dots)$$

belongs to U_n. Therefore we can define $\lambda_n(x) = \lambda_n(\tilde{x})$. We remark that, by definition, if $x, x' \in R_n[U_n] \setminus R_{n-1}[U_n]$ and $(x, x') \in R_{n-1}$, then $x_n = x'_n$, and hence $\rho_{r(x_n)}(x_n) = \rho_{r(x'_n)}(x'_n)$. Therefore $\lambda_n(x) = \lambda_n(x')$. Thus (4) for $x, x' \in R_n[U_n] \setminus R_{n-1}[U_n]$ is satisfied.

Let us check (6). Take $y \in Y$. By the surjectivity of $\rho_{r(y_n)}$, there exists $e \in E \setminus F$ such that $r(e) = r(y_n)$ and

$$\rho_{r(y_n)}(e) = (y_1, y_2, \dots, y_n).$$

Take an infinite path $x \in X$ such that $x_n = e$ and $x_k = y_k$ for all $k > n$. It is easy to see that x has the desired property.

We next verify (7). Take $y \in U_n$. Since U_n is contained in Y_{n+1}, by the same argument as above, we can choose $e \in E \setminus F$ such that $r(e) = r(y_n)$ and

$$\rho_{r(y_n)}(e) = (y_1, y_2, \dots, y_n).$$

Put

$$P_y = \{x \in X \mid x_n = e,\ x_k = y_k \text{ for all } k > n\}.$$

Notice that $|P_y|$ equals $|E(v_0, s(e))|$. It is clear that (x, y) belongs to R_n for every $x \in P_y$. From the definition of λ_n, we have $\lambda_n(x) = \lambda_n(y)$ for every $x \in P_y$. It is also clear that $x \notin U_n$ for any $x \in P_y$, because e is in $E \setminus F$. Finally, if $y, y' \in U_n$ are distinct, then P_y does not meet $P_{y'}$. This completes the proof of (7).

Let us consider (8). Take $x \in R_n[Y]$. If x is in $R_{n-1}[Y]$, then by the induction hypothesis there exists $y \in Y$ such that $(x, y) \in R_{n-1}$ and $\lambda_{n-1}(x) = \lambda_{n-1}(y)$. It follows from (4) that $\lambda_n(x)$ is equal to $\lambda_n(y)$. Suppose $x \in R_n[Y] \setminus R_{n-1}[Y]$, which means $x_n \in E \setminus F$ and $x_k \in F$ for all $k > n$. Thus, $x \in R_n[U_n] \setminus R_{n-1}[U_n]$. As before, we put $\tilde{x} = (\rho_{r(x_n)}(x_n), x_{n+1}, x_{n+2}, \dots)$. Then, $\tilde{x}$ belongs to Y and $(x, \tilde{x}) \in R_n$, $\lambda_n(x) = \lambda_n(\tilde{x})$.

In this way, we can find U_n and $\lambda_n : X \to K_n$ for every $n \in \mathbb{N}$. Finally, let us check (3). Since U_m contains Y, $\bigcap_{m \geq n} R_m[U_m] \supset R_n[Y]$ is clear. By the construction of U_m, for every $m \in \mathbb{N}$,

$$R_m[U_m] \subset \{x \in X \mid x_{m+1} \in F\}.$$

As an immediate consequence, we have

$$\bigcap_{m \geq n} R_m[U_m] \subset \{x \in X \mid x_m \in F \text{ for all } m > n\} = R_n[Y].$$

□

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let K_n, $\lambda_n : X \to K_n$ and U_n be as in the lemma above. Define

$$R'_n = \{(x, x') \in R_n \mid \lambda_n(x) = \lambda_n(x')\}$$

for every $n \in \mathbb{N}$. It is clear that R'_n is an open subset of R_n. Moreover, by (4) of Lemma 2.3, R'_{n-1} is contained in R'_n. Put $R' = \bigcup R'_n$. Evidently R' is an equivalence relation and an open subset of R. By (1) of Lemma 2.3, we have $R'|Y = S$.

Let us show $R'[Y] = R[Y]$. Take $x \in R[Y]$. There exists $n \in \mathbb{N}$ such that $x \in R_n[Y]$. By (8) of Lemma 2.3, there exists $y \in Y$ such that $(x, y) \in R_n$ and $\lambda_n(x) = \lambda_n(y)$. Hence we get $(x, y) \in R'_n$, which means that x is in $R'[Y]$.

We would like to show condition (5) of Theorem 2.1. Suppose that x is not in $R[Y]$. In order to prove $R'[x] = R[x]$, take $x' \in R[x]$. We can find $n \in \mathbb{N}$ such that $(x, x') \in R_n$. By (3) of Lemma 2.3, there exists $m \geq n$ such that $R_m[U_m]$ does not contain x. Also, clearly $x' \notin R_m[U_m]$. It follows from (5) of Lemma 2.3 that $\lambda_m(x)$ is equal to $\lambda_m(x')$. By definition of R'_m, we get $(x, x') \in R'_m$. Therefore $R'[x] = R[x]$.

We now consider the minimality of R'. Take $x \in X$. We must show that $R'[x]$ is dense in X. If x is not in $R[Y]$, as shown in the last paragraph, $R'[x]$ is equal to $R[x]$. Since R is minimal, $R[x]$ is dense in X. Hence we may assume that x is in $R[Y]$. As shown above, $R[Y] = R'[Y]$. It follows that we can find $y \in Y$ such that $(x, y) \in R'$. Take a non-empty open subset $O \subset X$ arbitrarily. The minimality of R implies $R[O] = X$. Since X is compact and $R[O] = \bigcup R_n[O]$, we can find $n \in \mathbb{N}$ such that $R_n[O] = X$. By (6) of Lemma 2.3, there exists $z \in R_{n+1}[y]$ such that $R_n[z] \subset R'_{n+1}[y]$. From $(x, y) \in R'$, we have $R_n[z] \subset R'[x]$. Combining this with $R_n[O] = X$, we can conclude that $R'[x]$ meets O, which implies $R'[x]$ is dense in X.

It remains for us to show the last condition. To do that, we would like to show that Y is R'-thin. From (7) of Lemma 2.3, for every $y \in U_n$, we have

$$\min_{v \in V_{n-1}} |E(v_0, v)| \times |R'_n[y] \cap U_n| \leq |R'_n[y]|.$$

Notice that R'_n is a compact relation. It follows that

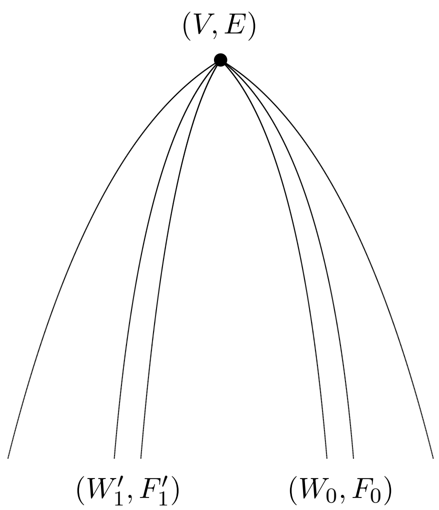
$$\mu(U_n) \leq \left(\min_{v \in V_{n-1}} |E(v_0, v)| \right)^{-1}$$

for every R'-invariant probability measure μ. The right-hand side converges to zero, because R is minimal. Since U_n contains Y, we get $\mu(Y) = 0$.

Let us show that any R'-invariant probability measure on X is R-invariant. Let μ be an R'-invariant probability measure and let $\gamma : O_1 \to O_2$ be a homeomorphism between clopen subsets $O_1, O_2 \subset X$ such that $(x, \gamma(x)) \in R$ for every $x \in O_1$, i.e. γ is a graph in R. It suffices to show $\mu(O_1) = \mu(O_2)$. Since Y is R'-thin and $R'[Y] = R[Y]$, we have $\mu(O_1) = \mu(O_1 \setminus R[Y])$ and $\mu(O_2) = \mu(O_2 \setminus R[Y])$. Clearly $\gamma(O_1 \setminus R[Y]) = O_2 \setminus R[Y]$ and $(x, \gamma(x)) \in R'$ for any $x \in O_1 \setminus R[Y]$. Hence we get $\mu(O_1 \setminus R[Y]) = \mu(O_2 \setminus R[Y])$, and so $\mu(O_1)$ is equal to $\mu(O_2)$. □

3 An absorption theorem

In this section, by using Theorem 2.1, we would like to prove the main theorem. We begin with a lemma.



Lemma 3.1. *Let $R \subset X \times X$ be a minimal AF equivalence relation on a Cantor set X and let $Y \subset X$ be a closed, R-étale and R-thin subset. Let Z be a compact metrizable totally disconnected space and let $Q \subset Z \times Z$ be an AF equivalence relation on Z. Then, there exists a continuous map $\pi : Z \to X$ such that the following are satisfied.*

(1) *π is a homeomorphism from Z to $\pi(Z)$.*

(2) *$\pi(Z)$ is a closed, R-étale and R-thin subset.*

(3) *$\pi(Z)$ does not meet $R[Y]$.*

(4) *$\pi \times \pi$ gives a homeomorphism from Q to $R \cap (\pi(Z) \times \pi(Z))$.*

Proof. As in the last section, we may assume that there exist a simple standard Bratteli diagram (V, E) and its thin subdiagram (W_0, F_0) such that the AF equivalence relation R on X is represented by (V, E) and $R|Y$ corresponds to (W_0, F_0). Similarly, by [GPS2, Theorem 3.9], we may assume that $Q \subset Z \times Z$ is represented by another standard Bratteli diagram (W_1, F_1).

We now transform the Bratteli diagram (V, E) by a succession of telescopings and microscopings so that the resulting diagram, which we again denote by (V, E), can be described as follows (see also the figure). There are two disjoint thin subdiagrams of (V, E). One is the subdiagram which is transformed from (W_0, F_0) above, and we retain the notation for it. The other thin subdiagram is a replica of (W_1, F_1), and we denote it by (W_1', F_1').

Let π denote the canonical homeomorphism from the infinite path space on (W_1, F_1), which is identified with Z, to the infinite path space on (W_1', F_1'), which is identified with a closed thin subset of X. Since (W_0, F_0) and (W_1', F_1') are disjoint, $\pi(Z)$ does not meet $R[Y]$. The other properties can be verified easily. □

We are now ready to give a proof of the main result. For étale equivalence relations Q and R, we say that Q is an étale extension of R, if Q contains R and the inclusion map from R to Q is continuous.

Theorem 3.2. *Let $R \subset X \times X$ be a minimal AF equivalence relation on a Cantor set X and let $Y \subset X$ be a closed, R-étale and R-thin subset. Suppose that an AF equivalence relation $Q \subset Y \times Y$ is an étale extension of $R|Y$. Then we can find a homeomorphism $h : X \to X$ such that the following are satisfied.*

(1) *$h \times h(R \vee Q) = R$, where $R \vee Q$ is the equivalence relation generated by R and Q.*

(2) *$h(Y)$ is a closed, R-étale and R-thin subset.*

(3) *$h|Y \times h|Y$ gives a homeomorphism from Q to $R|h(Y)$.*

In particular, $R \vee Q$ is affable.

Proof. The proof idea is the same as in the proof of the absorption theorem [GMPS2, Theorem 4.6], namely constructing countable disjoint replicas of $R|Y$, respectively Q, inside a "big" equivalence relation, and use the extension result [GPS2, Lemma 4.15].

Let $Z = (Y \times \mathbb{N}) \cup \{\infty\}$ be the one-point compactification of $Y \times \mathbb{N}$. Set

$$\widetilde{Q} = \{((y, n), (y', n)) \in Z \times Z \mid (y, y') \in Q, n \in \mathbb{N}\} \cup \{(\infty, \infty)\}.$$

Since Q is an AF relation, there exists an increasing sequence of compact open subrelations $Q_n \subset Q$ such that $Q = \bigcup_{n \in \mathbb{N}} Q_n$. For every $n \in \mathbb{N}$, we put

$$\widetilde{Q}_n = \{((y, k), (y', k)) \in Z \times Z \mid (y, y') \in Q_n, k = 1, 2, \ldots, n\} \cup \{(z, z) \mid z \in Z\}.$$

It is not so hard to see that $\widetilde{Q}_n$ is a compact étale relation on Z with the relative topology from $Z \times Z$. In addition, we have $\widetilde{Q}_n \subset \widetilde{Q}_{n+1}$ and $\widetilde{Q} = \bigcup_n \widetilde{Q}_n$. It follows that $\widetilde{Q}$ is an AF equivalence relation with the inductive limit topology. By Lemma 3.1, there exists a continuous map $\pi : Z \to X$ such that the following properties are satisfied.

- π is a homeomorphism from Z to $\pi(Z)$.
- $\pi(Z)$ is a closed, R-étale and R-thin subset.
- $\pi(Z)$ does not meet $R[Y]$.
- $\pi \times \pi$ is a homeomorphism from $\widetilde{Q}$ to $R|\pi(Z) = R \cap (\pi(Z) \times \pi(Z))$.

From the second and third conditions, it follows that $Y \cup \pi(Z)$ is also R-étale and R-thin.

We define an equivalence relation S on Z by

$$S = \{((y,n),(y',n)) \in \widetilde{Q} \mid (y,y') \in R\} \cup \{(\infty,\infty)\}.$$

It is a routine matter to verify that S is an open subrelation of $\widetilde{Q}$. Therefore $\pi \times \pi(S)$ is an open subrelation of $\pi \times \pi(\widetilde{Q}) = R|\pi(Z)$. By Theorem 2.1, there exists a minimal open subrelation $R' \subset R$ such that the following properties are satisfied.

- $R'|\pi(Z) = \pi \times \pi(S)$.
- $R'[\pi(Z)] = R[\pi(Z)]$.
- If x is not in $R[\pi(Z)]$, then $R'[x] = R[x]$. In particular, $R'|Y = R|Y$.
- Any R'-invariant probability measure on X is R-invariant.

Evidently $Y \cup \pi(Z)$ is R'-étale and R'-thin, and we have

$$R = R' \vee (R|\pi(Z)) = R' \vee (\pi \times \pi(\widetilde{Q}))$$

and

$$R \vee Q = R' \vee Q \vee (\pi \times \pi(\widetilde{Q})).$$

It is also easy to see

$$\begin{aligned} R'|(Y \cup \pi(Z)) &= (R'|Y) \cup (R'|\pi(Z)) \\ &= (R|Y) \cup (R'|\pi(Z)) \\ &\cong (R|Y) \cup S \\ &\cong S, \end{aligned}$$

where the last homeomorphism is obtained by an obvious shift map sending n to $n+1$, cf. definition of S. We define a homeomorphism $h : Y \cup \pi(Z) \to \pi(Z)$ by $h(y) = \pi(y,1)$ for $y \in Y$, $h(\pi(y,n)) = \pi(y,n+1)$ for $(y,n) \in Z$ and $h(\pi(\infty)) = \pi(\infty)$. Then

$$h \times h : R'|(Y \cup \pi(Z)) \to R'|\pi(Z)$$

is a homeomorphism. Note also that $h \times h$ implements an isomorphism between $Q \vee (\pi \times \pi(\widetilde{Q}))$ (which is a relation on $Y \cup \pi(Z)$) and $\pi \times \pi(\widetilde{Q})$ (which is a relation on $\pi(Z)$). This is an immediate consequence of the definition of $\widetilde{Q}$ and π. By [GPS2, Lemma 4.15], h extends to a homeomorphism $\tilde{h} : X \to X$ such that $\tilde{h} \times \tilde{h}(R') = R'$. It is clear that $\tilde{h} \times \tilde{h}(R \vee Q)$ equals R. Besides, $h(Y) = \pi(Y \times \{1\})$ is R-étale and R-thin. We can also check that $\tilde{h} \times \tilde{h}$ induces a homeomorphism from $Q \subset Y \times Y$ to $R|\tilde{h}(Y)$, which completes the proof. $\square$

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