AF embeddability of crossed products of AT algebras by the integers and its application

MATUI Hiroki

August 2001

Abstract

We will show that the crossed products of unital simple real rank zero AT algebras by the integers are AF embeddable. This is a generalization of N. Brown's AF embedding theorem. As an application, we will prove the AF embeddability of crossed product algebras arising from certain minimal dynamical systems induced by two commuting homeomorphisms.

1 Introduction

In [V2], Voiculescu raised the following question: for which $\mathbb{Z}^2$-actions, on a metrizable compact set X, is the crossed product $C(X) \rtimes_\alpha \mathbb{Z}^2$ embeddable into an AF algebra? In the $\mathbb{Z}$-action case, Pimsner solved this AF embeddability problem in [Pi]. He showed that the crossed product C^*-algebra arising from (X, T), where T is a homeomorphism on X, is AF embeddable if and only if T is pseudo-non-wandering, which means that there is no open subset $U \subset X$ such that $T(\overline{U}) \subset U$ and $U \setminus T(\overline{U}) \neq \emptyset$. For general discrete group actions including $\mathbb{Z}^2$, however, no general result has been known. In this paper we would like to give a partial answer for Voiculescu's question.

Our setting is as follows. Let X be a compact metric space and $T, S \in \mathrm{Homeo}(X)$ be two commuting homeomorphisms. We denote the crossed product C^*-algebra arising from (X, T, S) by $C^*(X, T, S)$. We say that (X, T, S) is a $\mathbb{Z}^2$-minimal system, if $\{T^n S^m(x) \ ; \ n, m \in \mathbb{Z}\}$ is dense in X for every $x \in X$. This is evidently equivalent to non-existence of a non-trivial closed subset which is T-invariant and S-invariant. (A subset E is said to be T-invariant if $T(E) = E$.) When (X, T, S) is a $\mathbb{Z}^2$-minimal system, there exists a probability measure supported on whole of X which is invariant under T and S. Hence, for every $(n, m) \in \mathbb{Z}^2$, $T^n S^m$ is pseudo-non-wandering. This fact seems to suggest the AF embeddability of the crossed product algebra $C^*(X, T, S)$. Actually we will prove that $C^*(X, T, S)$ is really AF embeddable if (X, T, S) is a $\mathbb{Z}^2$-minimal system and satisfies a certain condition which will be introduced in Section 4. This condition is necessary for reducing the problem to the special case that T itself is a minimal homeomorphism. If X is the Cantor set and (X, T) is a minimal system, then the crossed product C^*-algebra $C^*(X, T)$ is a simple unital AT algebra with real rank zero ([Pu]). Therefore the AF embeddability of $C^*(X, T, S)$ will be obtained as a corollary of the AF embeddability of crossed products of AT algebras by the integers, which is a generalization of [B, Corollary 4.10]. It should be also pointed out that quasidiagonality and stably finiteness follow immediately from AF embeddability.

Now we give an overview of each section below. In Section 2 we will generalize [KK, Theorem 3.1] to homomorphisms between simple unital AT algebras with real rank zero. It was proved in [E] that two homomorphisms between these algebras are approximately unitarily equivalent if and only if the induced homomorphisms on K-groups coincide. In the next section, however, we need a slight stronger equivalence relation, called asymptotically unitarily equivalence. By exactly the same way as in [KK], we will show that Ext and OrderExt groups can distinguish whether or not two homomorphisms are asymptotically unitarily equivalent. In Section 3 the generalization of [B, Corollary 4.10] will be proven. The strategy we will use is the same as in [B]. The only difference from the AF algebra case is that we have to take care of Bott elements in order to deduce the stability from the Rohlin property. The homotopy lemma of [BEEK] will play a very important role. As a corollary, we will also show that the crossed products of unital separable simple nuclear TAF algebras which satisfy UCT by the integers are AF embeddable. In Section 4 crossed product algebras arising from $\mathbb{Z}^2$-minimal systems are examined. As mentioned before, we will prove that these algebras are AF embeddable under a certain assumption. We will construct a Cantor minimal system by using ordered Bratteli diagram ([HPS]) and consider a skew product extension. In Section 5 examples of $\mathbb{Z}^2$-minimal systems will be given. It can be checked that our main theorem can be applied to many $\mathbb{Z}^2$-minimal systems. In fact the author does not know a $\mathbb{Z}^2$-minimal system for which our method does not work.

Acknowledgments.
The author would like to express his sincerest thanks to Professor IZUMI Masaki for his constant encouragement and helpful advice.

2 OrderExt and asymptotically unitarily equivalence

Let A and B be two unital C^*-algebras. We denote the set of all unital homomorphisms from A to B by $\mathrm{Hom}(A, B)$. The unitary groups of A and B are denoted by $U(A)$ and $U(B)$. Two unital homomorphisms $\alpha, \beta \in \mathrm{Hom}(A, B)$ are said to be approximately unitarily equivalent if there exists a sequence of unitaries $\{u_n\}_{n\in\mathbb{N}}$ in $U(B)$ such that $\mathrm{Ad}\, u_n\alpha(a)$ converges to $\beta(a)$ for all $a \in A$. If there exists a continuous map $u : [0, \infty) \to U(B)$ such that

$$\lim_{t\to\infty} \mathrm{Ad}\, u_t\alpha(a) = \beta(a)$$

for every a, we say that α and β are asymptotically unitarily equivalent. In this section we will show that α and β are asymptotically unitarily equivalent if and only if an invariant $\eta(\alpha, \beta)$ defined later is zero in the OrderExt group. The reader may refer to [KK, Section 2] for the definition of OrderExt groups.

Suppose A is simple and two homomorphisms $\alpha, \beta \in \mathrm{Hom}(A, B)$ are approximately unitarily equivalent. Define the mapping torus of α and β by

$$M_{\alpha,\beta} = \{x \in C([0, 1], B)\ ;\ x(0) = \alpha(a) \text{ and } x(1) = \beta(a) \text{ for some } a \in A\}.$$

From the exact sequence

$$0 \to SB \to M_{\alpha,\beta} \to A \to 0,$$

we get two exact sequences of abelian groups:

$$0 \to K_1(B) \to K_0(M_{\alpha,\beta}) \to K_0(A) \to 0$$

and

$$0 \to K_0(B) \to K_1(M_{\alpha,\beta}) \to K_1(A) \to 0.$$

We write the set of all tracial states of B by $T(B)$. For a unitary $u \in M_{\alpha,\beta}$ such that $t \mapsto u(t)$ is piecewise C^1,

$$T(B) \ni \tau \mapsto \int_0^1 \tau(\dot{u}(t)u(t)^*)\ dt$$

gives an affine map from $T(B)$ to $\mathbb{R}$. By exactly the same way as in [KK, Section 2], the map

$$R_{\alpha,\beta} : K_1(M_{\alpha,\beta}) \to \mathrm{Aff}(T(B)),$$

called the rotation map, is well defined. Therefore we obtain an element $\eta(\alpha,\beta)$ in

$$\mathrm{Ext}(K_0(A), K_1(B)) \oplus \mathrm{OrderExt}(K_1(A), K_0(B)).$$

Remark that $\eta(\alpha,\beta)$ is zero if and only if the following three conditions are satisfied:

(i) The extensions of $K_i(A)$ by $K_{1-i}(B)$ described above are trivial.

(ii) The range of the rotation map $R_{\alpha,\beta}$ coincides with the range of the canonical map $K_0(B) \to \mathrm{Aff}(T(B))$.

(iii) The exact sequence $0 \to \mathrm{Inf}(K_0(B)) \to \ker R_{\alpha,\beta} \to K_1(A) \to 0$ is trivial,

where $\mathrm{Inf}(K_0(B))$ denotes the infinitesimal subgroup

$$\{x \in K_0(B) \ ; \ \tau(x) = 0 \text{ for all } \tau \in T(B)\}.$$

It is easy to see that $\eta(\alpha,\beta) + \eta(\beta,\gamma)$ equals $\eta(\alpha,\gamma)$ for approximately unitarily equivalent homomorphisms α, β and γ in $\mathrm{Hom}(A, B)$.

We would like to prove the following theorem.

Theorem 1. *Let A and B be unital simple AT algebras with real rank zero. When $\alpha, \beta \in \mathrm{Hom}(A, B)$ are approximately unitarily equivalent homomorphisms, the following are equivalent.*

(i) *α and β are asymptotically unitarily equivalent.*

(ii) *$\eta(\alpha,\beta)$ defined above is zero in* $\mathrm{Ext}(K_0(A), K_1(B)) \oplus \mathrm{OrderExt}(K_1(A), K_0(B))$.

Because the proof is the same as that of [KK, Theorem 3.1], we give only a rough sketch of the proof. In the next section we actually need the above theorem only for the case that B is an AF algebra and $\mathrm{Inf}(K_0(B)) = 0$, and so the reader may consider that special case.

When u and v are almost commuting unitaries in a C^*-algebra A, we denote the Bott element associated with u and v by $B(u, v) \in K_0(A)$. We refer the reader to [BEEK] or [KK] for the Bott element.

At first we need to modify [KK, Lemma 3.4]. We say that the spectrum of a unitary $u \in U(M_n)$ is ε-dense with multiplicity m in $\mathbb{T}$, if for every $t \in \mathbb{T}$ the unitary u has m eigenvalues including multiplicity in the ε-neighborhood of t.

Lemma 1. *Let u be a unitary of $C(\mathbb{T}, M_n)$. Suppose $u(t)$ has n distinct eigenvalues and the spectrum is ε-dense with multiplicity m for all $t \in \mathbb{T}$. Then, for every k with $|k| \le m$, there exists a unitary $w \in C(\mathbb{T}, M_n)$ such that $||[u, w]|| < 2\varepsilon$ and the Bott element $B(u, w) \in K_0(C(\mathbb{T}, M_n)) \cong \mathbb{Z}$ is equal to k.*

Proof. We identify $C(\mathbb{T}, M_n)$ with a C^*-subalgebra of $C([0, 1], M_n)$. The unitary matrix $u(0)$ has n distinct eigenvalues in $\mathbb{T}$. We may assume $u(0)$ is a diagonal matrix. Let w be a permutation unitary of M_n such that each eigenvalues of $u(0)$ is permuted to the next eigenvalues in the counterclockwise order. Then, $||[u(0), w^k]|| < 2\varepsilon$ and $B(u(0), w^k) = k$ for $|k| < m$. We can find a unitary $v \in C([0, 1], M_n)$ such that $v(t)^*u(t)v(t)$ is a diagonal unitary, $v(0)$ is one and $v(1)$ equals w^l for some l. If we put $w'(t) = v(t)w^k v(t)^*$, the unitary w' is really in $C(\mathbb{T}, M_n)$. It is easy to see that $||[u, w']|| < 2\varepsilon$ and $B(u, w') = k$. □

Lemma 2. *Let A be a unital simple AT algebra with real rank zero and z be a unitary in A with full spectrum. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds: If $x \in K_0(A)$ satisfies $|\tau(x)| < \delta$ for every $\tau \in T(A)$, then there exists $w \in U(A)$ such that $||[z, w]|| < \varepsilon$ and $B(z, w) = x$.*

Proof. Let (V, E) be the Bratteli diagram associated with $K_0(A)$. We may assume that A is an inductive limit of $C_n \otimes C(\mathbb{T})$ with injective connecting homomorphisms and the finite dimensional C^*-algebra C_n is isomorphic to $\bigoplus_{v \in V_n} M_{h(v)}$, where $h(v)$ is the number of paths from V_0 to v.

By perturbing z a little bit, we may suppose that z is contained in $C_n \otimes C(\mathbb{T})$ for some n and the spectrum of z is ε-dense in $\mathbb{T}$ with multiplicity one. Because A is simple, we may further assume that the spectrum of $z(t)$ is ε-dense for every $t \in V_n \times \mathbb{T}$ ([DNNP, Proposition 2.1]). Let p_v be a minimal projection of $M_{h(v)} \subset C_n \otimes C(\mathbb{T})$ and put $p = \sum_{v \in V_n} p_v$. Define

$$\delta = \inf\{\tau(p)\ ;\ \tau \in T(B)\}.$$

For $m > n$ we denote the number of paths from V_n to $v \in V_m$ by e_v. The K_0 class of p is exactly $(e_v)_v \in \mathbb{Z}^{V_m}$.

Suppose $x \in K_0(A)$ satisfies $|\tau(x)| < \delta$ for all $\tau \in T(A)$. For sufficiently large $m > n$. there exists a representative $(x_v)_v \in \mathbb{Z}^{V_m}$ of x such that $|x_v|$ is less than e_v for every $v \in V_m$. Obviously the spectrum of $z(t)$ for $t \in V_m \times \mathbb{T}$ is ε-dense with multiplicity e_v. By perturbing z a little bit in $C_m \otimes C(\mathbb{T})$ again, we may assume that $z(t)$ has $h(v)$ distinct eigenvalues for every $t \in \{v\} \times \mathbb{T}$ and $v \in V_m$ (see [E, Theorem 4.1 (i)] for example). By virtue of Lemma 1, we get a unitary $w \in C_m \otimes C(\mathbb{T})$ such that

$$||[z, w]|| < 2\varepsilon, \quad B(z, w) = (x_v)_v \in K_0(C_m \otimes C(\mathbb{T})),$$

which completes the proof. □

Lemma 3. *Let A be a unital simple AT algebra with real rank zero. For every $z \in U(A)$, $x \in K_1(A)$ and $\varepsilon > 0$, there exists a unitary $w \in A$ such that $||[z, w]|| < \varepsilon$, $B(z, w) = 0$ and $[w] = x$.*

Proof. We use the same notation as in the proof of Lemma 2. We may assume that z is contained in a basic building block $C_n \otimes C(\mathbb{T})$ and all eigenvalues of $z(t)$ are distinct for every $t \in V_n \times \mathbb{T}$. Suppose $(x_v)_v \in \mathbb{Z}^{V_n}$ is a representative of x. For every $v \in V_n$ we can find continuous maps $\lambda : [0, 1] \to \mathbb{T}$ and $p : [0, 1] \to M_{h(v)}$ such that $p(t)$ is a rank one projection and $z(t)p(t) = p(t)z(t) = \lambda(t)p(t)$. Define $w_v(t) = e^{2\pi i x_v t} p(t) + (1 - p(t))$. Then, w_v is a unitary of $M_{h(v)} \otimes C(\mathbb{T})$ and commutes with z. It is clear that the K_1-class of the direct sum of w_v's equals x. □

We can prove Theorem 1 by using the above lemmas.

Proof of Theorem 1. The implication (i)⇒(ii) is obvious (see [KK]). Let us prove the other implication. As in Lemma 2, let (V, E) be the Bratteli diagram associated with $K_0(A)$ and suppose that A is the inductive limit of $C_n \otimes C(\mathbb{T})$ with injective connecting homomorphisms, where C_n is isomorphic to $\bigoplus_{v \in V_n} M_{h(v)}$. We denote the generating unitary of the center of $C_n \otimes C(\mathbb{T})$ by z_n. Let p_v be a minimal rank one projection of $M_{h(v)} \subset C_n$ for each $v \in V_n$ and define $z_v = z_n p_v + (1 - p_v)$. We may further assume that z_n equals $z_{n+1}a + z_{n+1}^* b + c$ for some partial isometries $a, b, c \in C_{n+1}$ for every n.

Suppose $\{\varepsilon_n\}_n$ is a decreasing sequence of positive real numbers. By applying Lemma 2 to $\beta(p_v)B\beta(p_v)$, $\beta(z_v p_v)$ for $v \in V_n$ and $\varepsilon_n/2$, we get a small number δ_v. Let δ_n be the minimum of $\varepsilon_n(4\pi)^{-1}$ and these δ_v's for $v \in V_n$. Because α is approximately unitarily equivalent to β, there exists $u_n \in U(B)$ such that $\mathrm{Ad}\, u_n \alpha | C_n = \beta | C_n$ and

$$||\,\mathrm{Ad}\, u_n \alpha(z_n) - \beta(z_n)|| < \sin 2\pi\delta.$$

For $v \in V_n$ we define

$$h_v = \frac{1}{2\pi i} \log \beta(z_v) u_n \alpha(z_v) u_n^*,$$

then h_v is in $\beta(p_v)B\beta(p_v)$ and $||h_v|| < \delta$. In the same way as in [KK], we can see that the affine function $T(B) \ni \tau \mapsto \tau(h_v)$ is in the range of the rotation map $R_{\alpha,\beta}$. From the assumption $\eta(\alpha,\beta) = 0$, there exists $x_v \in K_0(\beta(p_v)B\beta(p_v))$ such that $\tau(h_v) = \tau(x_v)$ for every $\tau \in T(\beta(p_v)B\beta(p_v))$. Therefore, by Lemma 2, we obtain a unitary $w_v \in \beta(p_v)B\beta(p_v)$ such that $||[\beta(z_v p_v), w_v)]|| < \varepsilon_n/2$ and $B(\beta(z_v p_v), w_v) = x_v$. By taking a direct sum of $h(v)$ copies of w_v, we get a partial unitary of B commuting with $\beta(M_{h(v)})$. Let w_n be the direct sum of these partial unitaries. Then, w_n is a unitary of $B \cap \beta(C_n)'$. When we replace u_n by $w_n u_n$, we can get the following:

$$\operatorname{Ad} u_n \alpha | C_n = \beta | C_n,$$
$$|| \operatorname{Ad} u_n \alpha(z_n) - \beta(z_n) || < \varepsilon_n$$

and $\tau(h_v) = 0$ for all $\tau \in T(B)$, where h_v is equal to

$$\frac{1}{2\pi i} \log \beta(z_v) u_n \alpha(z_v) u_n^*.$$

Thus we have shown [KK, Lemma 3.3] in our situation.

Lemma 3.5, 3.6 and 3.7 of [KK] are valid with no modification. If one uses Lemma 3 above, it is not so hard to see that [KK, Lemma 3.9] is also valid. Therefore we have chosen the above u_n's so that the Bott element $B(u_{n+1}^* u_n, \alpha(z_v))$ is zero for all $v \in V_n$ and $[u_{n+1}^* u_n \alpha(p_v)]$ is also zero in $K_1(\alpha(p_v)B\alpha(p_v))$. Note that the argument in this paragraph is not necessary, when B is an AF algebra and $\mathrm{Inf}(K_0(B)) = 0$.

Now we can apply the homotopy lemma [BEEK, Theorem 8.1] in a similar fashion to [KK] and conclude that α and β are asymptotically unitarily equivalent. □

3 Crossed products of AT algebras by the integers

The aim of this section is to generalize N. Brown's AF embedding theorem [B, Corollary 4.10]. Let us begin with the definition of the universal UHF algebra $\mathcal{U}$ and an automorphism $\sigma \in \mathrm{Aut}(\mathcal{U})$ with the Rohlin property ([B, Example 2.2]). The universal UHF algebra is the infinite tensor product C^*-algebra $\bigotimes_{n \in \mathbb{N}} M_n$. Let $u_n \in M_n$ be the unitary matrix satisfying $\operatorname{Ad} u_n(e_{i,j}) = e_{i+1,j+1}$, where $\{e_{i,j}\}_{i,j}$ is a system of matrix units in M_n and the addition is understood modulo n. The infinite product automorphism $\sigma = \otimes_n \operatorname{Ad} u_n$ has a special property, so called the Rohlin property.

In order to explain this property, we have to recall the idea of central sequence C^*-algebras. Let A be a C^*-algebra. Define a C^*-algebra and its ideal by

$$\ell^\infty(\mathbb{N}, A) = \left\{ (a_n)_{n \in \mathbb{N}} \ ; \ a_n \in A \text{ and } \sup_n ||a_n|| < \infty \right\}$$

and

$$c_0(\mathbb{N}, A) = \{ (a_n)_n \in \ell^\infty(\mathbb{N}, A) \ ; \ \lim a_n = 0 \}.$$

Let $A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ be the quotient C^*-algebra. The central sequence C^*-algebra A_∞ is defined by $A_\infty = A^\infty \cap A'$, where the original C^*-algebra A is embedded into A^∞ as constant sequences.

As explained in Section 2 of [B], the automorphism $\sigma \in \mathrm{Aut}(\mathcal{U})$ has the Rohlin property in the following sense: for every natural number N there exists a family of projections $E_0, E_1, \ldots, E_N \in \mathcal{U}_\infty$ such that $\sum_i E_i = 1$ and $\sigma(E_i) = E_{i+1}$ (with addition modulo N). This family of projections is sometimes called the Rohlin tower. Moreover, for any unital C^*-algebra B and automorphism $\beta \in \mathrm{Aut}(B)$, it is easy to see that the automorphism $\beta \otimes \sigma$ on $B \otimes \mathcal{U}$ also satisfies the same property. In general the Rohlin property does not require that the number of the Rohlin tower is one, and the height of the tower may not be exactly equal to the given natural number N (see [B, Definition 2.1]). In this paper, however, we will adopt the stronger version described above as definition of the Rohlin property for convenience.

At first we need a technical lemma.

Lemma 4. *Let A be a unital simple AT algebra with real rank zero. For every finite subset F of A and $\varepsilon > 0$, there exist a finite subset G of the unit ball of A and $\delta > 0$ such that the following holds: If $\varphi : A \to B$ is a unital embedding to an AF algebra B and $u : [0,1] \to U(B)$ is a continuous path with $u(0) = 1$ satisfying*

$$||[u(t), \varphi(a)|| < \delta$$

for all $t \in [0,1]$ and $a \in G$, then there exists a continuous path $w : [0,1] \to U(B)$ with length less than 6π such that $w(0) = 1, w(1) = u(1)$ and

$$||[w(t), \varphi(a)]|| < \varepsilon$$

for all $t \in [0,1]$ and $a \in F$.

Proof. Use Lemma 7.1 or Theorem 8.1 of [BEEK]. See also [K, Lemma 4.4]. □

The following proposition is the key to Theorem 2.

Proposition 1. *Let A be a unital simple AT algebra with real rank zero and α be an automorphism of A. Let B be a unital simple AF algebra and $\varphi : A \to B$ be an embedding. Suppose there exists an automorphism $\beta \in \mathrm{Aut}(B)$ such that $\varphi\alpha$ and $\beta\varphi$ are asymptotically unitarily equivalent. Then, the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{Z}$ is AF embeddable.*

Proof. Since we can replace (B, β) with $(B \otimes \mathcal{U}, \beta \otimes \sigma)$, we may assume that β has the Rohlin property.

Let $F_1 \subset F_2 \subset F_3 \subset \ldots$ be an increasing sequence of finite subsets of the unit ball of A, and suppose the union $\bigcup F_n$ is dense in the unit ball of A. Let $k_1 < k_2 < k_3 < \ldots$ be a sequence of natural numbers which increase sufficiently rapidly and $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots$ be a sequence of positive real numbers which decrease sufficiently rapidly. By applying Lemma 4 to $\bigcup_{i=0}^{k_n-1} \alpha^{-i}(F_n)$ and ε_n, we get a finite subset G_n of the unit ball of A and $\delta'_n > 0$. We may assume $F_n \subset G_n$. Let $\delta_1 > \delta_2 > \delta_3 > \ldots$ be a decreasing sequence of positive real numbers such that $k_n^2 \delta_n < \delta'_n$ and $(k_n^2 + k_n)\delta_n < \varepsilon_n$.

Since $\varphi\alpha$ and $\beta\varphi$ are asymptotically unitarily equivalent, by perturbing β with an inner automorphism, we may assume the following: There exists a continuous map $u : [0, \infty) \to U(B)$ with $u(0) = 1$ such that

$$\varphi\alpha(a) = \lim_{t\to\infty} \mathrm{Ad}\, u(t)\beta\varphi(a)$$

holds for all $a \in A$ and

$$||\varphi\alpha(a) - \mathrm{Ad}\, u(t)\beta\varphi(a)|| < \frac{\delta_1}{2}$$

for $a \in \bigcup_{i=0}^{k_1-1} \alpha^{-i}(G_1)$ and $t \in [0, \infty)$.

For sufficiently large $s > 0$ we have

$$||\varphi\alpha(a) - \operatorname{Ad} u(t)\beta\varphi(a)|| < \frac{\delta_2}{2}$$

for all $t \in [s, \infty)$ and $a \in \bigcup_{i=0}^{k_2-1} \alpha^{-i}(G_2)$. For $t \in [0, s]$ we define

$$v_0(t) = 1,\ v_1(t) = u(t),\ v_2(t) = u(t)\beta(u(t)),$$
$$v_3(t) = u(t)\beta(u(t))\beta^2(u(t)), \ldots.$$

Then, for $a \in G_1$ we get

$$\varphi\alpha(a) \sim_{\delta_1/2} \beta\varphi(a) = \beta\varphi\alpha\alpha^{-1}(a) \sim_{\delta_1/2} \beta^2\varphi\alpha^{-1}(a) = \ldots,$$

and so

$$||\varphi\alpha(a) - \beta^k\varphi\alpha^{-k+1}(a)|| < \frac{k\delta_1}{2}$$

is obtained for all $a \in G_1$ and $k \leq k_1$. Hence we have

$$\begin{aligned} ||[v_k(t), \varphi\alpha(a)]|| &\leq \sum_{j=0}^{k-1} ||[\beta^j(u(t)), \varphi\alpha(a)]|| \\ &< \sum_{j=0}^{k-1} \left(||[\beta^j(u(t)), \beta^j\varphi\alpha^{-j+1}(a)]|| + j\delta_1\right) \\ &< k_1(\delta_1 + (k_1 - 1)\delta_1) = k_1^2\delta_1 < \delta_1', \end{aligned}$$

for all $t \in [0, s]$, $a \in G_1$ and $k \leq k_1$. Especially, we can apply Lemma 4 to the case $k = k_1$, and get a continuous map $w : [0, 1] \to U(B)$ satisfying $w(0) = v_{k_1}(s)$, $w(1) = 1$ and

$$||w(l) - w(l')|| < 6\pi|l - l'|$$
$$||[w(l), \varphi\alpha(a)]|| < \varepsilon_1$$

for $l \in [0, 1]$ and $a \in \bigcup_{i=0}^{k_1-1} \alpha^{-i}(F_1)$. Take a Rohlin tower $E_1, E_2, \ldots, E_{k_1} \in B_\infty$ and define

$$V = \sum_{j=1}^{k_1} v_{j-1}(s)\beta^{j-1}\left(w\left(\frac{j-1}{k_1 - 1}\right)\right) E_j \ \in\ B^\infty.$$

Obviously V is a unitary. Moreover we obtain

$$||V\beta(V)^* - u(s)|| < \frac{6\pi}{k_1 - 1}$$

and

$$||[V, \varphi\alpha(a)]|| < (k_1^2 + k_1)\delta_1 + \varepsilon_1 < 2\varepsilon_1$$

for all $a \in F_1$. Therefore there exists a unitary v_1 in B such that the above two inequalities hold for v_1 instead of V. We define $w_1 = v_1^* u(s)\beta(v_1)$, $\varphi_1 = \operatorname{Ad} v_1^*\varphi$, $\beta_1 = \operatorname{Ad} w_1\beta$ and $u_1(t) = v_1^* u(t + s)u(s)^* v_1$. Then we can check that

$$\varphi_1\alpha(a) = \lim_{t\to\infty} \operatorname{Ad} u_1(t)\beta_1\varphi_1(a)$$

holds for all $a \in A$ and

$$||\varphi_1\alpha(a) - \operatorname{Ad} u_1(t)\beta_1\varphi_1(a)|| < \frac{\delta_2}{2}$$

for all $t \in [0, \infty)$ and $a \in \bigcup_{i=0}^{k_2-1} \alpha^{-i}(G_2)$.

By repeating this procedure, we get

$$\varphi_n = \operatorname{Ad} v_n v_{n-1} \dots v_2 v_1 \varphi$$

and

$$\beta_n = \operatorname{Ad} w_n w_{n-1} \dots w_2 w_1 \beta$$

for each n. Since we have

$$||[v_n, \varphi\alpha(a)]|| < 2\varepsilon_n$$

for $a \in F_n$, there exists $\varphi' \in \operatorname{Hom}(A, B)$ such that $\varphi'(a) = \lim_n \varphi_n(a)$ for every $a \in A$. Because $||w_n - 1||$ is less than $6\pi(k_n - 1)^{-1}$,

$$w = \lim_{n \to \infty} w_n w_{n-1} \dots w_2 w_1$$

exists. Moreover, for all $a \in F_n$, we have

$$||\varphi_n \alpha(a) - \beta_n \varphi_n(a)|| < \frac{\delta_n}{2},$$

which implies $\varphi' \alpha = \operatorname{Ad} w \beta \varphi'$. Since we have embedded (A, α) to $(B, \operatorname{Ad} w\beta)$ covariantly, the conclusion follows from [B, Corollary 4.10] (or from [V1, Theorem 3.6], when the original β is approximately inner.) $\square$

Theorem 2. *When A is a unital simple AT algebra with real rank zero and α is an automorphism of A, the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{Z}$ is AF embeddable.*

Proof. Let $\tau \in T(A)$ be an α-invariant tracial state. When we define $D = \tau(K_0(A))$ and $D^+ = D \cap \mathbb{R}^+$, the triple $(D, D^+, 1)$ is a simple dimension group. We can find a unital simple AF algebra B and $\varphi \in \operatorname{Hom}(A, B)$ such that K_0-group of B is isomorphic to $(D, D^+, 1)$ and φ_* is equal to τ on $K_0(A)$. From Elliott's theorem [E, Theorem 7.4] two homomorphisms φ and $\varphi\alpha$ are approximately unitarily equivalent.

Since $T(B)$ consists of one point, the range of the rotation map $R_{\varphi\alpha,\varphi}$ can be identified with a countable subgroup of $\mathbb{R}$ containing $D \cong K_0(B)$. Let D' be this countable group and D'^+ be $D' \cap \mathbb{R}^+$. There exists a unital simple AF algebra B' whose K_0-group is isomorphic to $(D', D'^+, 1)$. We can find $\psi \in \operatorname{Hom}(B, B')$ which induces the canonical inclusion from D to D'. Evidently the range of the rotation map $R_{\psi\varphi\alpha,\psi\varphi}$ is equal to D', which also implies that the extension of $K_1(A)$ by $K_0(B')$ is trivial because $\operatorname{Inf}(K_0(B)) = 0$. Hence we get $\eta(\psi\varphi\alpha, \psi\varphi) = 0$. Then, Theorem 1 tells us that two homomorphisms $\psi\varphi\alpha$ and $\psi\varphi$ are asymptotically unitarily equivalent. From Proposition 1, we get the desired AF embedding. $\square$

By using the recent classification result, we obtain the following corollary.

Corollary 1. *Let A be a unital separable simple nuclear TAF algebra which satisfies UCT. When α is an automorphism of A, the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{Z}$ is AF embeddable.*

Proof. Notice that A is a unital simple real rank zero AH algebra with slow dimension growth (see [L] and [EG]). Since $A \otimes \mathcal{U}$ is also such a kind of C^*-algebra and its K-groups are torsion free, the classification theorems in the above papers imply that $A \otimes \mathcal{U}$ is a unital simple AT algebra with real rank zero. Because we can apply Theorem 2 to $(A \otimes \mathcal{U}, \alpha \otimes id)$, the crossed product $A \rtimes_\alpha \mathbb{Z}$ is embedded into an AF algebra. $\square$

4 C^*-algebras arising from $\mathbb{Z}^2$-minimal systems

We will discuss the AF embeddability of the crossed product C^*-algebra $C^*(X,T,S)$ arising from a $\mathbb{Z}^2$-minimal system (X,T,S) in this section. The next proposition is our starting point.

Proposition 2. *Let T be a minimal homeomorphism on the Cantor set X and S be a homeomorphism which commutes with T. Then the crossed product C^*-algebra $C^*(X,T,S)$ is AF embeddable.*

Proof. The homeomorphism S induces the automorphism α of $C^*(X,T)$ and $C^*(X,T,S)$ is isomorphic to $C^*(X,T) \rtimes_\alpha \mathbb{Z}$. Thanks to [Pu, Theorem 2.1] the crossed product C^*-algebra $C^*(X,T)$ is a unital simple AT algebra with real rank zero (see also [HPS, Section 8]). We get the conclusion from Theorem 2. $\square$

Let T be a homeomorphism on a compact Hausdorff space X. We introduce the following condition called (#) for T:

(#) For every T-invariant non-empty open subset U there exists a T-invariant non-empty open subset V with $\overline{V} \subset U$.

The following lemma is an easy observation.

Lemma 5. (i) *If (X,T) is a minimal system and Y is a compact Hausdorff space, then $T \times id \in \mathrm{Homeo}(X \times Y)$ satisfies condition (#).*

(ii) *If (X,T) is a dynamical system and there exists a T-invariant metric on X, then T satisfies condition (#).*

(iii) *If $(T_s)_{s\in\mathbb{R}}$ is a minimal flow on X, then T_s satisfies condition (#) for every $s \in \mathbb{R}$.*

Proof. (i) Every $T \times id$-invariant open set takes the form of $X \times U$ for an open set U of Y. Since there exists a non-empty open set V with $\overline{V} \subset U$, we can check condition (#).

(ii) Let $d(\cdot,\cdot)$ be a T-invariant metric and U be a T-invariant open set. Take $x \in U$ and let W be an open ball with the center x and the radius $d(x,U^c)/2$. When we define $V = \bigcup_{n\in\mathbb{Z}} T^n(W \cap U)$, we can see that $d(U^c,V) = d(U^c, W \cap U)$ is not less than $d(x,U^c)/2$. Hence $\overline{V}$ is contained in U.

(iii) It suffices to show that T_1 satisfies condition (#). Let F be a minimal T_1-invariant closed subset. If $T_s(F) \cap F$ is not empty, from the minimality of F, we have $T_s(F) = F$. When we define $G = \{s \in \mathbb{R} \,;\, T_s(F) = F\}$, it is easily seen that G is a closed subgroup of $\mathbb{R}$ containing one. When G equals $\mathbb{R}$, one has $F = X$ and condition (#) is obviously satisfied. Suppose $G = n^{-1}\mathbb{Z}$ for $n \in \mathbb{N}$. Then, $\bigcup_{s\in[0,n^{-1})} T_s(F)$ gives a disjoint partition of X. We can prove that T_1 satisfies condition (#) in a similar fashion to (i). $\square$

Let (X,T,S) and (Y,T',S') be two $\mathbb{Z}^2$-minimal systems. A continuous map $\pi : Y \to X$ is called a factor map, when it satisfies $T\pi = \pi T'$ and $S\pi = \pi S'$. The factor map π induces a canonical embedding of $C^*(X,T,S)$ to $C^*(Y,T',S')$.

Lemma 6. *Let $\pi : Y \to X$ be a factor map between $\mathbb{Z}^2$-minimal systems (X,T,S) and (Y,T',S'). Suppose T satisfies condition (#). If either of the following holds, then T' also satisfies condition (#).*

(i) *The factor map π is almost one-to-one, that is, there exists $y \in Y$ with $\pi^{-1}(\pi(y)) = \{y\}$.*

(ii) *The factor map π is a local homeomorphism, that is, each point $y \in Y$ has a neighborhood U such that $\pi(U)$ is open and $\pi|U$ is a homeomorphism.*

Proof. (i) Suppose $U \subset Y$ is a T'-invariant non-empty open subset. Since (Y, T', S') is a $\mathbb{Z}^2$-minimal system, there exists $y \in U$ with $\pi^{-1}(\pi(y)) = \{y\}$. We can find a neighborhood W_0 of $\pi(y)$ such that $\pi^{-1}(W_0)$ is contained in U. Define $W = \bigcup_{n\in\mathbb{Z}} T^n(W_0)$. By the assumption, there exists a T-invariant non-empty open set V with $\overline{V} \subset W$. Then, we have $\pi^{-1}(\overline{V}) \subset \pi^{-1}(W) \subset U$, which implies condition (#).

(ii) Notice that $\pi^{-1}(x)$ is a finite set and its cardinality does not depend on x. Let U be a T'-invariant open set and define

$$k = \max\left\{\# \left(U \cap \pi^{-1}(x)\right) \ ; \ x \in X\right\}.$$

Then, $W = \{y \in U; \#(U \cap \pi^{-1}\pi(y)) = k\}$ is a T'-invariant open subset of U. There exists a T-invariant open set $V \subset X$ with $\overline{V} \subset \pi(W)$, because T satisfies condition (#). We would like to show that the T'-invariant open set $V' = W \cap \pi^{-1}(V)$ satisfies $\overline{V'} \subset W$. Suppose that a sequence $\{y_n\}_n \subset V'$ converges to $y \notin W$. Since y is contained in $\pi^{-1}(\overline{V})$, there exist k distinct preimages of $\pi(y)$ in W. Then, one can see that there exist $k+1$ preimages of $\pi(y_n)$ in W for sufficiently large n. This contradicts the assumption. $\square$

Let us prove the main theorem. In order to do that, we need a series of lemmas.

Lemma 7. *When (X, T) is a dynamical system, the following are equivalent.*

(i) *T satisfies condition (#).*

(ii) *T^n satisfies condition (#) for all $n \in \mathbb{N}$.*

(iii) *T^n satisfies condition (#) for some $n \in \mathbb{N}$.*

Proof. (i)⇒(ii). The proof goes by induction. Assume that the assertion has been shown for all natural numbers less than n. Let $U \subset X$ be a T^n-invariant open subset. Take the minimum natural number k such that $T^k(U) \cap U$ is not empty. If $k = n$, $W = U \cup T(U) \cup \cdots \cup T^n(U)$ is a T-invariant open set and we get a T-invariant open set V with $\overline{V} \subset W$. Then $U \cap V$ is a T^n-invariant open set and $\overline{U \cap V} \subset (W \backslash U)^c \cap \overline{V} \subset U$. Suppose $k < n$. When $U' = U \cap T^k(U) \cap \cdots \cap T^{(n-1)k}(U)$ is not empty, by the induction step, we can find a T^l-invariant open set V with $\overline{V} \subset U'$, where l is the greatest common divisor of k and n. Since V is also T^n-invariant, we have the conclusion. We may assume U' is empty. Let m be the natural number such that $U'' = U \cap T^k(U) \cap \cdots \cap T^{mk}(U)$ is not empty and $U \cap T^k(U) \cap \cdots \cap T^{(m+1)k}(U)$ is empty. Clearly U'' is a T^n-invariant open subset of U and $U'' \cap T^i(U'') = \emptyset$ for $i = 1, 2, \ldots, k-1, k$. By repeating this argument for U'', we will complete the proof.

(ii)⇒(iii) is obvious.

(iii)⇒(i). Suppose U is a T-invariant non-empty open subset. From the assumption there exists a T^n-invariant open subset V with $\overline{V} \subset U$. Then $V' = V \cup T(V) \cup \cdots \cup T^{n-1}(V)$ is a T-invariant open subset with $\overline{V'} \subset U$. $\square$

Lemma 8. *Let (X, T, S) be a $\mathbb{Z}^2$-minimal system and suppose T satisfies condition (#). Let a, b and c be three natural numbers and U be an open subset of X. If a and b are relatively prime, there exists a sequence of homeomorphisms $\phi_0, \phi_1, \ldots, \phi_l \in \mathrm{Homeo}(X)$ with $\phi_0 = id$ such that $\phi_i^{-1}\phi_{i+1}$ equals either of T^a, T^b, and T^cS for all $i = 0, 1, \ldots, l-1$ and $X = \bigcup_{i=0}^{l} \phi_i(U)$.*

Proof. Since $\bigcup_{k\in\mathbb{Z}} T^{ka}(U)$ is T^a-invariant and open, with the aid of Lemma 7 we obtain a T^a-invariant open subset V with $\overline{V} \subset \bigcup T^{ka}(U)$. Because $\overline{V}$ is compact, there exists n such that

$$U \cup T^a(U) \cup \cdots \cup T^{na}(U) \supset V.$$

Hence we have

$$\bigcup_{k=0}^{a-1} T^{kb+kna}(U \cup T^a(U) \cup \cdots \cup T^{na}(U)) \supset \bigcup_{k=0}^{a-1} T^{kb}(V).$$

Since $V' = \bigcup_{k=0}^{a-1} T^{kb}(V)$ is a T-invariant open subset, by the minimality, we can find a natural number m such that $V' \cup S(V') \cup \cdots \cup S^m(V') = X$. Therefore we get

$$\bigcup_{j=0}^{m} T^{cj} S^j \left(\bigcup_{k=0}^{a-1} T^{kb+kna}(U \cup T^a(U) \cup \cdots \cup T^{na}(U)) \right) = X,$$

which completes the proof. $\square$

The following lemma will be used to reduce the problem to Proposition 2. We refer the reader to [HPS] or [M, Section 2] for ordered Bratteli diagrams.

Lemma 9. *Let (X, T, S) be a $\mathbb{Z}^2$-minimal system and suppose T satisfies condition (#). Then there exist a Cantor minimal system (Y, ψ) and a continuous function $f : Y \to \mathbb{Z}$ such that $\gamma(x, y) = (TS^{f(y)}(x), \psi(y))$ is a minimal homeomorphism on $X \times Y$.*

Proof. We will construct a properly ordered simple Bratteli diagram $B = (V, E)$. We denote the Bratteli-Vershik system of B by (Y, ψ) and the unique minimal infinite path by $y_0 \in Y$. Define $V_0 = \{v_0\}$ and $V_n = \{a_n, b_n, c_n, d_n\}$ for every $n \in \mathbb{N}$. We connect the vertex v_0 to each vertex of V_1 by a single edge. These four edges in E_1 give a partition of Y, namely $Y = U_a \cup U_b \cup U_c \cup U_d$. We put a $\mathbb{Z}$-valued continuous function f by $f|U_a = f|U_b = 0$, $f|U_c = 1$ and $f|U_d = -1$. Inductively we will define the partially ordered edge set E_n for $n \geq 2$ so that the homeomorphism γ determined by f becomes minimal on $X \times Y$. We write the projections from $X \times Y$ to X and Y by π_X and π_Y. Let $\{U_n\}_{n \in \mathbb{N}}$ be an open basis of X. We write the number of paths from v to v' by $E(v, v')$. Suppose that the partially ordered edge sets $E_1, E_2, \ldots, E_n$ have been already defined so that the following properties are satisfied.

(i) Two natural numbers $E(v_0, a_n)$ and $E(v_0, b_n)$ are relatively prime.

(ii) $E(c_1, a_n) = E(d_1, a_n)$.

(iii) $E(c_1, b_n) = E(d_1, b_n)$.

(iv) $E(c_1, c_n) - E(d_1, c_n) = 1$.

(v) $E(c_1, d_n) - E(d_1, d_n) = -1$.

(vi) Every maximal edges and minimal edges in E_n goes through a_{n-1} and every pair of vertices in V_{n-1} and V_n is connected.

(vii) For $x \in X$ there exists a non-negative integer k less than $E(v_0, a_n)$ such that $\pi_X \gamma^k(x, y_0)$ is contained in U_{n-1} and the initial $n-2$ edges of $\pi_Y \gamma^k(x, y_0) = \psi^k(y_0)$ coincide with y_0.

We would like to define the partially ordered edge set E_{n+1}. By applying Lemma 8 to (X, T^{-1}, S^{-1}), U_n and three natural numbers $E(v_0, a_n)$, $E(v_0, b_n)$ and $E(v_0, c_n)$, we obtain a sequence of homeomorphisms $\phi_0, \phi_1, \ldots, \phi_l$ with $\phi_0 = id$. At first, let us consider $r^{-1}(a_{n+1})$. For $i = 1, 2, \ldots, l$ we determine the source vertex of i-th edge e_i in $r^{-1}(a_{n+1})$ by the homeomorphism $\phi_{i-1}\phi_i^{-1}$, that is, we define

$$s(e_i) = \begin{cases} a_n & \phi_{i-1}\phi_i^{-1} = T^{E(v_0, a_n)} \\ b_n & \phi_{i-1}\phi_i^{-1} = T^{E(v_0, b_n)} \\ c_n & \phi_{i-1}\phi_i^{-1} = ST^{E(v_0, c_n)}. \end{cases}$$

This definition for $n = 1$ has ambiguity because $E(v_0.a_1) = E(v_0, b_1) = 1$. But we may choose a_1 or b_1 freely when $n = 1$. Lemma 8 and the definition above tell us that the property (vii) in the $n+1$-st step is satisfied. By adding more edges greater than e_l to $r^{-1}(a_{n+1})$, we can achieve the property (ii) and (vi) in the $n + 1$-st step. For the other edge sets $r^{-1}(b_{n+1})$, $r^{-1}(c_{n+1})$ and $r^{-1}(c_{n+1})$ we have much flexibility. It is easy to achieve the other properties and make the Bratteli diagram simple. By repeating this procedure, we get a properly ordered simple Bratteli diagram $B = (V, E)$.

From the construction we can show that the closure of the γ-orbit $\{\gamma^n(x, y_0); n \in \mathbb{Z}\}$ contains $X \times \{y_0\}$ for every $x \in X$. Let $x, x' \in X$ be two points. When U is an open neighborhood of x', there exists a sufficiently large number n such that $x' \in U_n \subset U$. From the property (vii), we can find a natural number k so that $\pi_X \gamma^k(x, y_0)$ is contained in U_n and $\psi^k(y_0)$ is very close to y_0. Hence the closure of $\{\gamma^n(x, y_0); n \in \mathbb{Z}\}$ contains (x', y_0).

Let us check that the minimality of γ. Take $(x, y) \in X \times Y$ arbitrarily. We would like to show that the closure of the γ-orbit of (x, y) is $X \times Y$. Since (Y, ψ) is minimal and X is compact, we can find a sequence of integers $\{m_n\}_n$ such that $\gamma^{m_n}(x, y)$ converges to a point in $X \times \{y_0\}$. Hence the closure of the γ-orbit of (x, y) contains the whole of $X \times \{y_0\}$. By using the minimality of (Y, ψ) again, we get the conclusion. □

Although the following lemma may be well-known, the author would like to present the proof for the reader's convenience.

Lemma 10. *Let (X, T, S) be a $\mathbb{Z}^2$-minimal system and suppose X is not a finite set. Then there exist a $\mathbb{Z}^2$-minimal system (Y, T', S') and an almost one-to-one factor map $\pi : Y \to X$ such that Y is the Cantor set.*

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be an open basis of X. Define $U_{0,n} = \overline{U_n}$ and $U_{1,n} = U_n^c$. It is clear that

$$X' = \left\{ y \in \{0, 1\}^{\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}} \ ; \ \bigcap_{n,k,l} T^k S^l (U_{y(n,k,l),n}) \text{ is not empty} \right\}$$

is a closed subset of $\{0, 1\}^{\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}}$. Obviously there exists a continuous surjection $\pi : X' \to X$. The map π is one-to-one at least on the points of

$$\bigcap_{n,k,l} T^k S^l (U_n \cup \overline{U_n}^c),$$

which is not empty by the Baire category theorem. We denote the subshifts on the second and third coordinates of $\{0, 1\}^{\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}}$ by T' and S'. By Zorn's lemma we can find a minimal T'-invariant and S'-invariant non-empty closed subset Y in X'. Then (Y, T', S') is a $\mathbb{Z}^2$-minimal system and $\pi : Y \to X$ gives an almost one-to-one factor map. Since Y is totally disconnected and X is not a finite set, we can see that Y is the Cantor set. □

Now we are ready to prove the main theorem.

Theorem 3. *Let (X, T, S) be a $\mathbb{Z}^2$-minimal system. If $T^n S^m$ satisfies condition (#) for some $(n, m) \in \mathbb{Z}^2 \setminus 0$, then the crossed product C^*-algebra $C^*(X, T, S)$ is AF embeddable.*

Proof. We may assume that n and m are relatively prime because of Lemma 7. By replacing generators of the $\mathbb{Z}^2$-action, we may further assume that T satisfies condition (#). From Lemma 6 (i) and 10, it suffices to consider the case that X is the Cantor set.

By Lemma 9 we can find a Cantor minimal system (Y, ψ), a continuous function $f : Y \to \mathbb{Z}$ and a minimal homeomorphism γ on $X \times Y$ determined by f. Since we can apply Proposition

2 to the Cantor minimal system $(X \times Y, \gamma)$ and the homeomorphism $\tau = S \times id$, the crossed product C^*-algebra $C^*(X \times Y, \gamma, \tau)$ is AF embeddable. We denote the implementing unitaries corresponding to γ and τ by u and v. Then, for any $g \in C(X \times Y)$ we have $ugu^* = g\gamma^{-1}$ and $vgv^* = g\tau^{-1}$. Define a unitary $w \in C(X \times Y, \gamma, \tau)$ by

$$w = \sum_{n \in \mathbb{Z}} uv^{-n} 1_{X \times f^{-1}(n)},$$

where 1_E denotes the characteristic function on E and the sum is actually a finite sum. It is easily seen that

$$w(g \otimes 1_Y)w^* = gT^{-1} \otimes 1_Y, \quad v(g \otimes 1_Y)v^* = gS^{-1} \otimes 1_Y$$

for every $g \in C(X)$, and so a unital homomorphism $\pi : C^*(X, T, S) \to C^*(X \times Y, \gamma, \tau)$ is obtained. Because there are faithful conditional expectations $E : C^*(X, T, S) \to C(X)$ and $F : C^*(X \times Y, \gamma, \tau) \to C(X \times Y)$ such that $\pi E = F\pi$, we can show that π is an embedding. $\square$

5 Examples

At the final section we would like to give examples of $\mathbb{Z}^2$-minimal systems to which Theorem 3 can be applied.

(1) Many examples of minimal $\mathbb{R}$-flows on compact connected manifolds are known ([FH]). Let $\alpha : \mathbb{R} \times X \to X$ be such a minimal flow. If s and t are rationally independent real numbers, then $T = \alpha_t$ and $S = \alpha_s$ induce a $\mathbb{Z}^2$-minimal system on X. From Lemma 5 (iii) we can see that $C^*(X, T, S)$ is AF embeddable.

(2) We would like to consider Toeplitz sequences over $\mathbb{Z}^2$. The reader may refer to [W] for the usual $\mathbb{Z}$ case. Define

$$RF(\mathbb{Z}^2) = \{H \subset \mathbb{Z}^2 \ ; \ H \text{ is a subgroup of } \mathbb{Z}^2 \text{ with } \mathbb{Z}^2/H \text{ finite } \}.$$

Let A be a finite set. For $\xi : \mathbb{Z}^2 \to A$ and $H \in RF(\mathbb{Z}^2)$ we denote by $\mathrm{Per}(\xi, H)$ the set of all $p \in \mathbb{Z}^2$ such that $\xi(p) = \xi(p+q)$ for all $q \in H$. We call ξ a Toeplitz sequence over $\mathbb{Z}^2$, when

$$\mathbb{Z}^2 = \bigcup_{H \in RF(\mathbb{Z}^2)} \mathrm{Per}(\xi, H).$$

Note that this definition of Toeplitz sequences is valid not only for $\mathbb{Z}^2$ or $\mathbb{Z}$ but also for all residually finite groups.

Let T and S be the subshift over $\mathbb{Z}^2$, that is, for $\eta \in A^{\mathbb{Z}^2}$ we define $T(\eta)(n, m) = \eta(n+1, m)$ and $S(\eta)(n, m) = \eta(n, m+1)$. Let X be the closure of $\{T^n S^m(\xi); n, m \in \mathbb{Z}\}$ for a Toeplitz sequence ξ. Then, by the same way as the $\mathbb{Z}$ case, one can prove that (X, T, S) is a $\mathbb{Z}^2$-minimal system. We call $H \in RF(\mathbb{Z}^2)$ a essential period of ξ, if $\mathrm{Per}(\xi, H)$ is not empty and $\mathrm{Per}(\xi, H') \subsetneq \mathrm{Per}(\xi, H)$ holds for all $H' \in RF(\mathbb{Z}^2)$ with $H \subsetneq H'$. For $H \in RF(\mathbb{Z}^2)$, let T_H and S_H be the translations on $\mathbb{Z}^2/H$ by $(1, 0)$ and $(0, 1)$ of $\mathbb{Z}^2$. Let H be an essential period of ξ. For $p \in \mathbb{Z}^2$, we can see that the closure of

$$\{T^n S^m(\xi) \in X \ ; \ (n, m) \in H + p\}$$

is contained in the closed set

$$U_{H,p} = \{\eta \in X \ ; \ \eta(q - p) = \xi(q) \text{ for all } q \in \mathrm{Per}(\xi, H)\}.$$

When p runs over representatives of all cosets in $\mathbb{Z}^2/H$, it can be checked that $U_{H,p}$'s give a disjoint partition of X. Therefore $U_{H,p}$ is a clopen subset of X and there exists a factor map $\pi_H : (X,T,S) \to (\mathbb{Z}^2/H, T_H, S_H)$. By considering all essential periods, we obtain a factor map π from (X,T,S) to a $\mathbb{Z}^2$-minimal system induced by two rotations over a compact abelian group and π is one-to-one at ξ. From Lemma 5 (ii) and Lemma 6 (i), it follows that $T \in \mathrm{Homeo}(X)$ satisfies condition (#). Hence $C^*(X,T,S)$ is AF embeddable.

References

[BEEK] Bratteli, O.; Elliott, G. A.; Evans, D. E.; Kishimoto, A.; *Homotopy of a pair of approximately commuting unitaries in a simple C^*-algebra*, J. Funct. Anal. 160 (1998), 466–523.

[B] Brown, N. P.; *AF embeddability of crossed products of AF algebras by the integers*, J. Funct. Anal. 160 (1998), 150–175.

[DNNP] Dadarlat, M.; Nagy, G.; Nemethi, A.; Pasnicu, C.; *Reduction of topological stable rank in inductive limits of C^*-algebras*, Pacific J. Math. 153 (1992), 267–276.

[E] Elliott, G. A.; *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. 443 (1993), 179–219.

[EG] Elliott, G. A.; Gong, G.; *On the classification of C^*-algebras of real rank zero, II*, Ann. of Math. 144 (1996), 497–610.

[FH] Fathi, A.; Herman, M; *Existence de difféomorphismes minimaux*, Asterisque 49 (1977), 37–59.

[HPS] Herman, R. H.; Putnam, I. F.; Skau, C. F.; *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. 3 (1992), 827–864.

[K] Kishimoto A.; *Unbounded derivations in AT algebras*, J. Funct. Anal. 160 (1998), 270–311.

[KK] Kishimoto A.; Kumjian, A.; *The Ext class of an approximately inner automorphism, II*, J. Operator Theory 46 (2001), 99–122.

[L] Lin H.; *Classification of simple C^*-algebras of tracial topological rank zero*, to appear in Duke Math. J.

[M] Matui H.; *Topological orbit equivalence of locally compact Cantor minimal systems*, Ergodic Theory Dynam. Systems 22 (2002), 1871–1903.

[Pi] Pimsner, M. V.; *Embedding some transformation group C^*-algebras into AF algebras*, Ergodic Theory Dynam. Systems 3 (1983), 613–626.

[Pu] Putnam, I. F.; *On the topological stable rank of certain transformation group C^*-algebras*, Ergodic Theory Dynam. Systems 10 (1990), 197–207.

[V1] Voiculescu, D.; *Almost inductive limit automorphisms and embedding into AF-algebras*, Ergodic Theory Dynam. Systems 6 (1986), 475–484.

[V2] Voiculescu, D.; *Around quasidiagonal operators*, Integral Equations Operator Theory 17 (1993), 137–149.

[W] Williams, S.; *Toeplitz minimal flows which are not uniquely ergodic*, Z. Wahrsch. Verw. Gebiete 67 (1984), 95–107.

e-mail; matui@kusm.kyoto-u.ac.jp
Department of Mathematics,
Kyoto University,
Kitasirakawa-Oiwaketyô, Sakyôku,
Kyôto, 606-8502,
Japan.