Topological full groups of étale groupoids

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Overview

dynamics on Cantor set X

$\downarrow$

$C_r^*(\mathcal{G})$
$K_i(C_r^*(\mathcal{G}))$

$\longleftarrow$

étale groupoid $\mathcal{G}$
with $\mathcal{G}^{(0)} = X$

$\longrightarrow$

topological
full group
$[[\mathcal{G}]] \subset \mathrm{Homeo}(X)$

$\downarrow$

$K_i(C_r^*(\mathcal{G}))$
vs. $H_n(\mathcal{G})$

homology group of $\mathcal{G}$
$H_n(\mathcal{G})$, $n = 0, 1, 2, \ldots$

index map
$I : [[\mathcal{G}]] \to H_1(\mathcal{G})$
$[[\mathcal{G}]]_{\mathrm{ab}}$ vs. $H_n(\mathcal{G})$

Étale groupoid

A groupoid $\mathcal{G}$ is a 'group-like' algebraic object, in which the product may not be defined for all pairs in $\mathcal{G}$.

- $g \in \mathcal{G}$ is thought of as an arrow $\bullet \xleftarrow{g} \bullet$.
- $r : g \mapsto gg^{-1}$ is called the range map.
- $s : g \mapsto g^{-1}g$ is called the source map.
- $\mathcal{G}^{(0)} = r(\mathcal{G}) = s(\mathcal{G}) \subset \mathcal{G}$ is called the unit space.

$\mathcal{G}$ is an étale groupoid if $\mathcal{G}$ is equipped with a locally compact Hausdorff topology compatible with the groupoid structure and the range (or source) map is a local homeomorphism.
An arrow $\bullet \xleftarrow{g} \bullet$ is thought of as a germ at $s(g) = gg^{-1}$.

In what follows, we assume that $\mathcal{G}^{(0)}$ is a Cantor set.

Topological full group

A compact open set $U \subset \mathcal{G}$ is called a $\mathcal{G}$-set if both $r|U$ and $s|U$ are injective.
The topological full group $[[\mathcal{G}]]$ is defined by

$$[[\mathcal{G}]] = \left\{ \gamma \in \mathrm{Homeo}(\mathcal{G}^{(0)}) \mid \exists \mathcal{G}\text{-set } U,\ \gamma = (r|U) \circ (s|U)^{-1} \right\}.$$

Equivalently, $\gamma \in [[\mathcal{G}]]$ if and only if

$$\forall x \in \mathcal{G}^{(0)}\ \exists g \in s^{-1}(x),\ \gamma \text{ equals } g \text{ as a germ at } x.$$

When $\varphi : \Gamma \curvearrowright X$ is a group action on a Cantor set X, $\mathcal{G}_\varphi = \Gamma \times X$ becomes an étale groupoid in a natural way.
In this situation,

$$\gamma \in [[\mathcal{G}_\varphi]] \iff \exists \text{conti. map } c : X \to \Gamma,\ \gamma(x) = \varphi_{c(x)}(x).$$

Groupoid C^*-algebra

For an étale groupoid $\mathcal{G}$, the space $C_c(\mathcal{G}, \mathbb{C})$ of compactly supported continuous functions becomes a $*$-algebra by

$$(f_1 \cdot f_2)(g) = \sum_{h \in \mathcal{G}} f_1(gh) f_2(h^{-1}),$$

$$f^*(g) = \overline{f(g^{-1})}.$$

As a completion by a suitable norm,
we get a (reduced) groupoid C^*-algebra $C_r^*(\mathcal{G})$.

$C_r^*(\mathcal{G})$ contains the abelian subalgebra $C(\mathcal{G}^{(0)})$.
It is maximal, and its unitary normalizers generate $C_r^*(\mathcal{G})$.
Such a subalgebra $C(\mathcal{G}^{(0)})$ is called a Cartan subalgebra.

Isomorphism theorem

Theorem

For minimal groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$, the following are equivalent.

1. *$\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$ as an étale groupoid.*
2. *$[[\mathcal{G}_1]]$ is isomorphic to $[[\mathcal{G}_2]]$ as a group.*
3. *$D([[\mathcal{G}_1]])$ is isomorphic to $D([[\mathcal{G}_2]])$ as a group.*
4. *There exists an isomorphism $\pi : C_r^*(\mathcal{G}_1) \to C_r^*(\mathcal{G}_2)$ such that $\pi(C(\mathcal{G}_1^{(0)})) = C(\mathcal{G}_2^{(0)})$.*

Thus, $[[\mathcal{G}]]$ (or $D([[\mathcal{G}]])$) 'remembers' $\mathcal{G}$.

Homology group

$H_n(\mathcal{G})$ are the homology groups of the chain complex

$$0 \longleftarrow C(\mathcal{G}^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(\mathcal{G}^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \dots,$$

where $\mathcal{G}^{(n)}$ is the space of composable strings of n elements.

$$\delta_1(f)(x) = \sum_{s(g)=x} f(g) - \sum_{r(g)=x} f(g)$$

So,

$$H_0(\mathcal{G}) = C(\mathcal{G}^{(0)}, \mathbb{Z}) / \langle 1_{s(U)} - 1_{r(U)} \mid U \text{ is a } \mathcal{G}\text{-set} \rangle.$$

If U is a $\mathcal{G}$-set such that $s(U) = r(U) = G^{(0)}$, then $1_U \in \operatorname{Ker} \delta_1$.
Hence, one can define the index map $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$.

Minimal $\mathbb{Z}$ actions

Let $\varphi : \mathbb{Z} \curvearrowright X$ be a minimal action and consider $\mathcal{G}_\varphi = \mathbb{Z} \times X$.

$C_r^*(\mathcal{G}_\varphi)$ is 'classifiable by K-groups' (Putnam, Elliott).

$H_0(\mathcal{G}_\varphi) \cong K_0(C_r^*(\mathcal{G}_\varphi))$, $H_1(\mathcal{G}_\varphi) \cong K_1(C_r^*(\mathcal{G}_\varphi)) \cong \mathbb{Z}$ and $H_n(\mathcal{G}_\varphi) = 0$ for $n \geq 2$. So, $\bigoplus_n H_{2n+i} \cong K_i$ holds.

$D([[\mathcal{G}_\varphi]])$ is simple. $[[\mathcal{G}_\varphi]]_{\mathrm{ab}} \cong \mathbb{Z} \oplus (H_0(\mathcal{G}_\varphi) \otimes \mathbb{Z}_2)$,
i.e. Abelianization $\cong H_1 \oplus (H_0 \otimes \mathbb{Z}_2)$.
$D([[\mathcal{G}_\varphi]])$ is finitely generated iff φ is expansive.

$\mathcal{G}_{\varphi_1} \cong \mathcal{G}_{\varphi_2}$ iff φ_1 is flip conjugate to φ_2 (Boyle-Tomiyama 1998).

$[[\mathcal{G}_\varphi]]$ is amenable (Juschenko-Monod 2013).

One-sided shifts of finite type (1/3)

Let $(\mathcal{V}, \mathcal{E})$ be an irreducible finite directed graph and let A be the adjacency matrix.
Set

$$X = \{(x_n)_n \in \mathcal{E}^{\mathbb{N}} \mid t(x_n) = i(x_{n+1}) \quad \forall n \in \mathbb{N}\},$$

The one-sided shift on X is called a shift of finite type (SFT).
When $\#\mathcal{V}=1$ and $\#\mathcal{E}=k$, it's called the full shift over k symbols.

We can construct $\mathcal{G}_A$ from the SFT.

It is known $H_0(\mathcal{G}_A) \cong \mathrm{Coker}(\mathrm{id} - A^t)$, $H_1(\mathcal{G}_A) \cong \mathrm{Ker}(\mathrm{id} - A^t)$ and $H_n(\mathcal{G}_A) = 0$ for $n \geq 2$.

One-sided shifts of finite type (2/3)

The C^*-algebra $C^*_r(\mathcal{G}_A)$ is the Cuntz-Krieger algebra, which is generated by partial isometries $\{S_e \mid e \in \mathcal{E}\}$, subject to the following relations:

$$\sum_{e\in\mathcal{E}} S_e S_e^* = 1, \quad S_e^* S_e = \sum_{t(e)=i(f)} S_f S_f^*.$$

S_e corresponds to the operation of attaching $e \in \mathcal{E}$ to an infinite path $x \in X \subset \mathcal{E}^{\mathbb{N}}$, i.e. $x \mapsto ex$.

The Cuntz-Krieger algebra $C^*_r(\mathcal{G}_A)$ is 'classifiable by K-groups' (Rørdam, Kirchberg-Phillips).

It is known that

$$K_0(C^*_r(\mathcal{G}_A)) \cong \mathrm{Coker}(\mathrm{id} - A^t), \quad K_1(C^*_r(\mathcal{G}_A)) \cong \mathrm{Ker}(\mathrm{id} - A^t).$$

So, $\bigoplus_n H_{2n+i} \cong K_i$ holds for $\mathcal{G}_A$.

One-sided shifts of finite type (3/3)

The triple $(\mathrm{Coker}(\mathrm{id} - A^t), [u_A], \det(\mathrm{id} - A^t))$ is a complete invariant for the isomorphism class of $\mathcal{G}_A$ within SFT groupoids (Matsumoto-M 2014).

We have $[[\mathcal{G}_A]]_{\mathrm{ab}} \cong H_1(\mathcal{G}_A) \oplus (H_0(\mathcal{G}_A) \otimes \mathbb{Z}_2)$,
i.e. Abelianization $\cong H_1 \oplus (H_0 \otimes \mathbb{Z}_2)$ holds.
It's also known that $D([[\mathcal{G}_A]])$ is simple.

When X is the full shift over N symbols, $[[\mathcal{G}_A]]$ is canonically isomorphic to the Higman-Thompson group V_N.

$[[\mathcal{G}_A]]$ is of type F_∞ (in particular, finitely presented).
$[[\mathcal{G}_A]]$ has the Haagerup property.

Product of groupoid

For étale groupoids $\mathcal{G}$ and $\mathcal{H}$,
their product groupoid $\mathcal{G} \times \mathcal{H}$ is naturally defined.

Theorem

Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids.
For any $n \geq 0$, there exists a natural short exact sequence:

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(\mathcal{G}) \otimes H_j(\mathcal{H}) \longrightarrow H_n(\mathcal{G} \times \mathcal{H}) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(\mathcal{G}), H_j(\mathcal{H})) \longrightarrow 0.$$

It is easy to see $C_r^*(\mathcal{G} \times \mathcal{H}) = C_r^*(\mathcal{G}) \otimes C_r^*(\mathcal{H})$.
Künneth theorem for tensor products of C^*-algebras is also known.
So, (in many cases) $\bigoplus_n H_{2n+i} \cong K_i$ holds for products.

Product of SFT (1/4)

Consider the product groupoid $\mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_m}$, where $\mathcal{G}_{A_i}$ is an SFT groupoid discussed in the previous slides.

Theorem

$\mathcal{G}_1 = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_m}$ and $\mathcal{G}_2 = \mathcal{G}_{B_1} \times \mathcal{G}_{B_2} \times \cdots \times \mathcal{G}_{B_n}$ are isomorphic if and only if

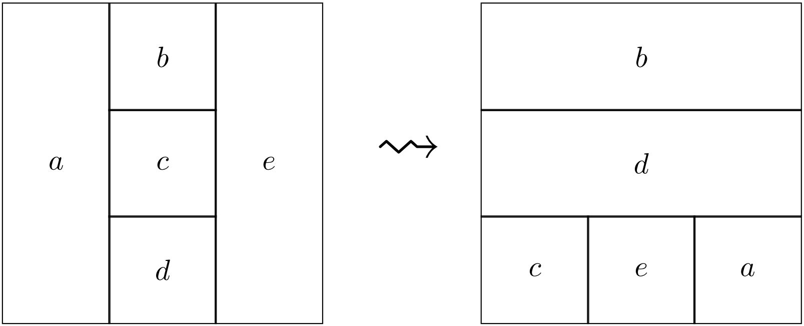
- $m = n$,
- *after permutation of indices, there exist isomorphisms $\varphi_i : \mathrm{Coker}(\mathrm{id} - A_i^t) \to \mathrm{Coker}(\mathrm{id} - B_i^t)$ such that*

$$(\varphi_1 \otimes \cdots \otimes \varphi_n)(u_{A_1} \otimes \cdots \otimes u_{A_n}) = u_{B_1} \otimes \cdots \otimes u_{B_n},$$

and $\det(\mathrm{id} - A_i^t) = \det(\mathrm{id} - B_i^t)$ for all i.

Product of SFT (2/4)

When $\mathcal{G} = \mathcal{G}_{[k]} \times \mathcal{G}_{[k]} \times \cdots \times \mathcal{G}_{[k]}$ is the n-fold product of the full shifts over k symbols, $[[\mathcal{G}]]$ is known as a higher dimensional Thompson group nV_k and studied by Brin (2004) et al.



$$[[\mathcal{G}_3 \times \mathcal{G}_3]] = 2V_3$$

Product of SFT (3/4)

Let $\mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n}$ be a product of SFT groupoids.

Again, $D([[\mathcal{G}]])$ is simple.

As for the abelianization $[[\mathcal{G}]]_{\mathrm{ab}}$ of the topological full group, we have the following.

Theorem

1. *There exists an exact sequence:*

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{\ j\ } [[\mathcal{G}]]_{\mathrm{ab}} \xrightarrow{\ I\ } H_1(\mathcal{G}) \longrightarrow 0.$$

2. *The homomorphism j is not always injective.*
3. *The index map I does not always have a right inverse.*

Product of SFT (4/4)

Recall that the higher dimensional Thompson group nV_k is the topological full group of the n-fold product $\mathcal{G} = \mathcal{G}_{[k]} \times \cdots \times \mathcal{G}_{[k]}$.

Theorem

1. *When* $n = 1$, $(1V_k)_{\mathrm{ab}} \cong \mathbb{Z}_{k-1} \otimes \mathbb{Z}_2$.
2. *When* $n = 2$,
$$(2V_k)_{\mathrm{ab}} \cong \begin{cases} \mathbb{Z}_{k-1} & k \in 2\mathbb{Z} \\ \mathbb{Z}_{k-1} \oplus \mathbb{Z}_2 & k \in 4\mathbb{Z}+1 \\ \mathbb{Z}_{2k-2} & k \in 4\mathbb{Z}+3. \end{cases}$$
3. *When* $n \geq 3$,
$$(nV_k)_{\mathrm{ab}} \cong \begin{cases} (\mathbb{Z}_{k-1})^{n-1} & k \in 2\mathbb{Z} \text{ or } k \in 4\mathbb{Z}+3 \\ (\mathbb{Z}_{k-1})^{n-1} \oplus \mathbb{Z}_2 & k \in 4\mathbb{Z}+1. \end{cases}$$

Cleary's group (1/2)

Let $\beta > 0$ be an irrational number.
Let $P = \{\beta^n \mid n \in \mathbb{Z}\}$ and $A = \mathbb{Z}[\beta, \beta^{-1}]$.

Consider the group V_β consisting of right continuous bijections of $[0, 1)$ which are piecewise linear, with finitely many discontinuities and singularities, all in A, slopes in P, and mapping $A \cap [0, 1)$ to itself.

Cleary (1995, 2000) showed that V_β is of type F_∞
when $\beta > 0$ satisfies $\beta^2 + n\beta - 1 = 0$, $n \in \mathbb{N}$.

There exists an étale groupoid $\mathcal{G}_\beta$ such that $[[\mathcal{G}_\beta]] \cong V_\beta$.

K-groups of $C_r^*(\mathcal{G}_\beta)$ were computed for many values of β
by Carey-Phillips-Putnam-Rennie (2011).
Also, $C_r^*(\mathcal{G}_\beta)$ is 'classifiable by K-groups'.

Cleary's group (2/2)

Theorem

1. *When* $\beta > 0$ *satisfies* $\beta^2 + n\beta - 1 = 0$, $n \in \mathbb{N}$,

$$H_0(\mathcal{G}_\beta) = \mathbb{Z}_n, \quad H_1(\mathcal{G}_\beta) = \mathbb{Z}_2, \quad H_k(\mathcal{G}_\beta) = 0 \text{ for } k \geq 2.$$

2. *When* $\beta > 0$ *satisfies* $\beta^2 - n\beta + 1 = 0$, $n \in \mathbb{N} \setminus \{1, 2\}$,

$$H_0(\mathcal{G}_\beta) = \mathbb{Z}_{n-2}, \quad H_1(\mathcal{G}_\beta) = \mathbb{Z}, \quad H_2(\mathcal{G}_\beta) = \mathbb{Z}, \\ H_k(\mathcal{G}_\beta) = 0 \text{ for } k \geq 3.$$

In both cases, we have

$$\bigoplus_n H_{2n+i}(\mathcal{G}_\beta) \cong K_i(C_r^*(\mathcal{G}_\beta)) \quad i = 0, 1$$

and

$$[[\mathcal{G}_\beta]]_{\mathrm{ab}} \cong (H_0(\mathcal{G}_\beta) \otimes \mathbb{Z}_2) \oplus H_1(\mathcal{G}_\beta).$$