Various examples of topological full groups

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Overview

dynamics on Cantor set X

$\downarrow$

$C_r^*(\mathcal{G})$
$K_i(C_r^*(\mathcal{G}))$

$\longleftarrow$

étale groupoid $\mathcal{G}$
with $\mathcal{G}^{(0)} = X$

$\longrightarrow$

topological
full group
$[[\mathcal{G}]] \subset \mathrm{Homeo}(X)$

$\downarrow$

$K_i(C_r^*(\mathcal{G}))$
vs. $H_n(\mathcal{G})$

homology group of $\mathcal{G}$
$H_n(\mathcal{G})$, $n = 0, 1, 2, \ldots$

index map
$I : [[\mathcal{G}]] \to H_1(\mathcal{G})$
$[[\mathcal{G}]]_{\mathrm{ab}}$ vs. $H_n(\mathcal{G})$

Étale groupoid

A groupoid $\mathcal{G}$ is a 'group-like' algebraic object, in which the product may not be defined for all pairs in $\mathcal{G}$.

- $g \in \mathcal{G}$ is thought of as an arrow $\bullet \xleftarrow{g} \bullet$.
- $r : g \mapsto gg^{-1}$ is called the range map.
- $s : g \mapsto g^{-1}g$ is called the source map.
- $\mathcal{G}^{(0)} = r(\mathcal{G}) = s(\mathcal{G}) \subset \mathcal{G}$ is called the unit space.

$\mathcal{G}$ is an étale groupoid if $\mathcal{G}$ is equipped with a locally compact Hausdorff topology compatible with the groupoid structure and the range (or source) map is a local homeomorphism.

In what follows, we assume that $\mathcal{G}$ is essentially principal ($\Longleftrightarrow$ topologically free) and that $\mathcal{G}^{(0)}$ is a Cantor set.

Topological full group

A compact open set $U \subset \mathcal{G}$ is called a $\mathcal{G}$-set if both $r|U$ and $s|U$ are injective.
The topological full group $[[\mathcal{G}]]$ is defined by

$$[[\mathcal{G}]] = \left\{ \gamma \in \mathrm{Homeo}(\mathcal{G}^{(0)}) \mid \exists \mathcal{G}\text{-set } U,\ \gamma = (r|U) \circ (s|U)^{-1} \right\}.$$

When $\varphi : \Gamma \curvearrowright X$ is a group action on a Cantor set X, $\mathcal{G}_\varphi = \Gamma \times X$ becomes an étale groupoid in a natural way.
In this situation,

$$\gamma \in [[\mathcal{G}_\varphi]] \iff \exists \text{conti. map } c : X \to \Gamma,\ \gamma(x) = \varphi_{c(x)}(x).$$

Isomorphism theorem

Theorem

For minimal groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$, the following are equivalent.

1. $\mathcal{G}_1$ *is isomorphic to* $\mathcal{G}_2$ *as an étale groupoid.*
2. $[[\mathcal{G}_1]]$ *is isomorphic to* $[[\mathcal{G}_2]]$ *as a group.*
3. $D([[\mathcal{G}_1]])$ *is isomorphic to* $D([[\mathcal{G}_2]])$ *as a group.*
4. *There exists an isomorphism* $\pi : C_r^*(\mathcal{G}_1) \to C_r^*(\mathcal{G}_2)$ *such that* $\pi(C(\mathcal{G}_1^{(0)})) = C(\mathcal{G}_2^{(0)})$.

Thus, $[[\mathcal{G}]]$ (or $D([[\mathcal{G}]])$) 'remembers' $\mathcal{G}$.

Homology group

$H_n(\mathcal{G})$ are the homology groups of the chain complex

$$0 \longleftarrow C(\mathcal{G}^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(\mathcal{G}^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \dots,$$

where $\mathcal{G}^{(n)}$ is the space of composable strings of n elements.

$$\delta_1(f)(x) = \sum_{s(g)=x} f(g) - \sum_{r(g)=x} f(g)$$

So,

$$H_0(\mathcal{G}) = C(\mathcal{G}^{(0)}, \mathbb{Z}) / \langle 1_{s(U)} - 1_{r(U)} \mid U \text{ is a } \mathcal{G}\text{-set} \rangle.$$

If U is a $\mathcal{G}$-set such that $s(U) = r(U) = G^{(0)}$, then $1_U \in \operatorname{Ker} \delta_1$.
Hence, one can define the index map $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$.

Two conjectures

Let $\mathcal{G}$ be an essentially principal minimal étale groupoid with $\mathcal{G}^{(0)}$ a Cantor set.

HK Conjecture

There exist isomorphisms

$$\bigoplus_{i=0}^{\infty} H_{2i}(\mathcal{G}) \cong K_0(C_r^*(\mathcal{G})), \quad \bigoplus_{i=0}^{\infty} H_{2i+1}(\mathcal{G}) \cong K_1(C_r^*(\mathcal{G})).$$

AH Conjecture

There exists a short exact sequence

$$0 \longrightarrow H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \longrightarrow [[\mathcal{G}]]_{\mathrm{ab}} \xrightarrow{I} H_1(\mathcal{G}) \longrightarrow 0.$$

By 'weak AH', we mean $H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to [[\mathcal{G}]]_{\mathrm{ab}} \to H_1(\mathcal{G}) \to 0$ is exact.

Minimal $\mathbb{Z}$ actions

Let $\varphi : \mathbb{Z} \curvearrowright X$ be a minimal action and consider $\mathcal{G}_\varphi$.

$H_0(\mathcal{G}_\varphi) \cong K_0(C_r^*(\mathcal{G}_\varphi))$, $H_1(\mathcal{G}_\varphi) \cong K_1(C_r^*(\mathcal{G}_\varphi)) \cong \mathbb{Z}$ and $H_n(\mathcal{G}_\varphi) = 0$ for $n \geq 2$. So, HK conjecture holds.

AH conjecture holds, i.e. $[[\mathcal{G}_\varphi]]_{\mathrm{ab}} \cong \mathbb{Z} \oplus (H_0(\mathcal{G}_\varphi) \otimes \mathbb{Z}_2)$,
and $D([[\mathcal{G}_\varphi]])$ is simple.
$D([[\mathcal{G}_\varphi]])$ is finitely generated iff φ is expansive.

$\mathcal{G}_{\varphi_1} \cong \mathcal{G}_{\varphi_2}$ iff φ_1 is flip conjugate to φ_2 (Boyle-Tomiyama 1998).

$[[\mathcal{G}_\varphi]]$ is amenable (Juschenko-Monod 2013).

Minimal $\mathbb{Z}^N$ actions

Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be a free minimal action and consider $\mathcal{G}_\varphi$.

When $N = 1, 2$, HK conjecture holds. But, it's not known in general. The Chern character tells us that HK is true after tensoring the rationals.

We can show that weak AH holds for $\mathcal{G}_\varphi$, i.e.

$$H_0(\mathcal{G}_\varphi) \otimes \mathbb{Z}_2 \longrightarrow [[\mathcal{G}_\varphi]]_{\mathrm{ab}} \xrightarrow{\ I\ } H_1(\mathcal{G}_\varphi) \longrightarrow 0$$

is exact. But, we do not know if AH is true or not.
$D([[\mathcal{G}_\varphi]])$ is known to be simple.

$[[\mathcal{G}_\varphi]]$ is sometimes amenable and sometimes not (Elek-Monod 2013).

One-sided shifts of finite type (1/2)

Let $(\mathcal{V}, \mathcal{E})$ be an irreducible finite directed graph and
let A be the adjacency matrix.
Set

$$X = \{(x_k)_k \in \mathcal{E}^{\mathbb{N}} \mid t(x_k) = i(x_{k+1}) \quad \forall k \in \mathbb{N}\},$$

The one-sided shift on X is called a shift of finite type (SFT).
When $\#\mathcal{V}=1$ and $\#\mathcal{E}=N$, it's called the full shift over N symbols.

We can construct $\mathcal{G}_A$ from the SFT.
It is known $H_0(\mathcal{G}_A) \cong \mathrm{Coker}(\mathrm{id} - A^t)$, $H_1(\mathcal{G}_A) \cong \mathrm{Ker}(\mathrm{id} - A^t)$ and
$H_n(\mathcal{G}_A) = 0$ for $n \geq 2$.
The C^*-algebra $C_r^*(\mathcal{G}_A)$ is the Cuntz-Krieger algebra,
whose K_i-groups are isomorphic to H_i above.
So, HK conjecture holds for $\mathcal{G}_A$.

One-sided shifts of finite type (2/2)

The triple $(\mathrm{Coker}(\mathrm{id} - A^t), [u_A], \det(\mathrm{id} - A^t))$ is a complete invariant for the isomorphism class of $\mathcal{G}_A$ within SFT groupoids (Matsumoto-M 2014).

We have $[[\mathcal{G}_A]]_{\mathrm{ab}} \cong H_1(\mathcal{G}_A) \oplus (H_0(\mathcal{G}_A) \otimes \mathbb{Z}_2)$, i.e. AH holds.
It's also known $D([[\mathcal{G}_A]])$ is simple.

When X is the full shift over N symbols, $[[\mathcal{G}_A]]$ is canonically isomorphic to the Higman-Thompson group V_N.

$[[\mathcal{G}_A]]$ is of type F_∞ (in particular, finitely presented).
$[[\mathcal{G}_A]]$ has the Haagerup property.

Products of SFT (1/3)

Consider the product groupoid $\mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_m}$, where $\mathcal{G}_{A_i}$ is an SFT groupoid discussed in the previous slides.

Theorem

$\mathcal{G}_1 = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_m}$ and $\mathcal{G}_2 = \mathcal{G}_{B_1} \times \mathcal{G}_{B_2} \times \cdots \times \mathcal{G}_{B_n}$ are isomorphic if and only if

- $m = n$,
- *after permutation of indices,* $\mathrm{Coker}(\mathrm{id} - A_i^t) \cong \mathrm{Coker}(\mathrm{id} - B_i^t)$ *and* $\det(\mathrm{id} - A_i^t) = \det(\mathrm{id} - B_i^t)$ *for all* i,
- $(H_0(\mathcal{G}_1), [1_{\mathcal{G}_1^{(0)}}]) \cong (H_0(\mathcal{G}_2), [1_{\mathcal{G}_2^{(0)}}])$.

Products of SFT (2/3)

Theorem

For any étale groupoid $\mathcal{H}$ and any SFT groupoid $\mathcal{G}_A$,

$$H_p(\mathcal{H} \times \mathcal{G}_A) \cong (H_p(\mathcal{H}) \otimes H_0(\mathcal{G}_A)) \oplus (H_{p-1}(\mathcal{H}) \otimes H_1(\mathcal{G}_A)) \oplus \mathrm{Tor}(H_{p-1}(\mathcal{H}), H_0(\mathcal{G}_A)).$$

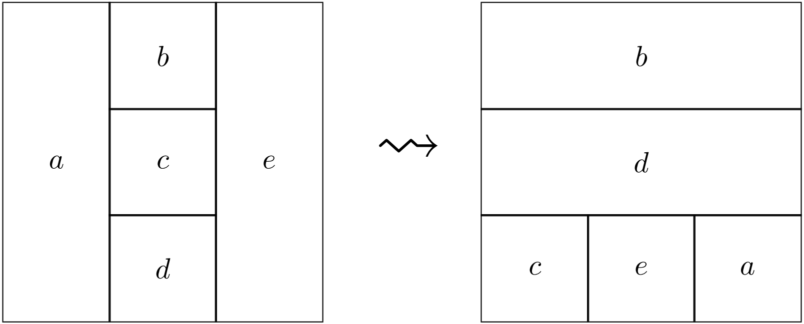
In particular, if HK holds for $\mathcal{H}$, then HK also holds for $\mathcal{H} \times \mathcal{G}_A$.

We can verify weak AH for $\mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_m}$,
but it's not clear if AH is true or not.
We know $D([[\mathcal{G}]])$ is simple.

When $\mathcal{G} = \mathcal{G}_N \times \mathcal{G}_N \times \cdots \times \mathcal{G}_N$ is the k-fold product of the full shifts over N symbols, $[[\mathcal{G}]]$ is known as a higher dimensional Thompson group kV_N and studied by Brin (2004) et al.

Products of SFT (3/3)

$$[[\mathcal{G}_3 \times \mathcal{G}_3]] = 2V_3$$



Stein's group (1/2)

Let $P \subset \mathbb{R}_{>0}$ be the multiplicative subgroup freely generated by $n_1, n_2, \ldots, n_k \in \mathbb{N}$. Let $A = \mathbb{Z}[\frac{1}{n_1 n_2 \ldots n_k}] \subset \mathbb{R}$. Let $r \in \mathbb{N}$.

Stein's group $V(r, A, P)$ consists of right continuous bijections of $[0, r)$ which are piecewise linear, with finitely many discontinuities and singularities, all in A, slopes in P, and mapping $A \cap [0, r)$ to itself.

$V(r, \mathbb{Z}[1/n], \langle n \rangle)$ is the Higman-Thompson group $V_{n,r}$.

Stein (1992) proved that $V(r, A, P)$ is of type F_∞ and that $D(V(r, A, P))$ is simple. She also computed $V(r, A, P)_{\mathrm{ab}}$.

There exists an étale groupoid $\mathcal{G}$ such that $[[\mathcal{G}]] \cong V(r, A, P)$.

Stein's group (2/2)

Theorem

Let $\mathcal{G}$ be the étale groupoid corresponding to $P = \langle n_1, n_2, \ldots, n_k \rangle$. Then $H_0(\mathcal{G}) \cong \mathbb{Z}_d$ and $H_1(\mathcal{G}) \cong (\mathbb{Z}_d)^{k-1}$, where $d = \gcd(n_1 - 1, \ldots, n_k - 1)$. In particular, AH is true for $\mathcal{G}$.

HK conjecture is not yet verified in this case.

Theorem

Let $\mathcal{G}$ be the étale groupoid corresponding to $P = \langle n_1, \ldots, n_k \rangle$, and let $\mathcal{H}$ be the étale groupoid corresponding to $Q = \langle m_1, \ldots, m_l \rangle$. If $\mathcal{G} \cong \mathcal{H}$, then $k = l$ and $\mathbb{Z}[\frac{1}{n_1 n_2 \ldots n_k}] = \mathbb{Z}[\frac{1}{m_1 m_2 \ldots m_k}]$.

When $P = \langle 2, 3 \rangle$ and $Q = \langle 2, 5 \rangle$, their homology groups are trivial. But we can distinguish them by the theorem above.

Cleary's group (1/2)

Let $\beta > 0$ be an irrational number.
Let $P = \{\beta^n \mid n \in \mathbb{Z}\}$ and $A = \mathbb{Z}[\beta, \beta^{-1}]$.

Consider the group V_β consisting of right continuous bijections of $[0,1)$ which are piecewise linear, with finitely many discontinuities and singularities, all in A, slopes in P, and mapping $A \cap [0,1)$ to itself.

Cleary (1995, 2000) showed that V_β is of type F_∞
when $\beta > 0$ satisfies $\beta^2 + n\beta - 1 = 0$, $n \in \mathbb{N}$.

There exists an étale groupoid $\mathcal{G}_\beta$ such that $[[\mathcal{G}_\beta]] \cong V_\beta$.

K-groups of $C^*_r(\mathcal{G}_\beta)$ were computed for many values of β
by Carey-Phillips-Putnam-Rennie (2011).

Cleary's group (2/2)

Theorem

1. *When* $\beta > 0$ *satisfies* $\beta^2 + n\beta - 1 = 0$, $n \in \mathbb{N}$,

$$H_0(\mathcal{G}_\beta) = \mathbb{Z}_n, \quad H_1(\mathcal{G}_\beta) = \mathbb{Z}_2, \quad H_k(\mathcal{G}_\beta) = 0 \text{ for } k \geq 2.$$

2. *When* $\beta > 0$ *satisfies* $\beta^2 - n\beta + 1 = 0$, $n \in \mathbb{N} \setminus \{1, 2\}$,

$$H_0(\mathcal{G}_\beta) = \mathbb{Z}_{n-2}, \quad H_1(\mathcal{G}_\beta) = \mathbb{Z}, \quad H_2(\mathcal{G}_\beta) = \mathbb{Z},$$
$$H_k(\mathcal{G}_\beta) = 0 \text{ for } k \geq 3.$$

In both cases, HK conjecture and AH conjecture are true for $\mathcal{G}_\beta$.