Topological full groups of étale groupoids

Hiroki Matui
matui@math.s.chiba-u.ac.jp

Chiba University

October 30, 2012
C^*-Algebras, Dynamics, and Classification
Oberwolfach

Outline

(minimal) topological dynamical system on a Cantor set

(group action, equivalence relation, one-sided SFT...)

$\longrightarrow$ étale groupoid G whose $G^{(0)}$ is a Cantor set

$\longrightarrow$ topological full group $[[G]] \subset \mathrm{Homeo}(G^{(0)})$

Properties of $[[G]]$ (and its commutator subgroup $D([[G]])$):

- $[[G]]$ (and $D([[G]])$) 'remembers' G.
- $D([[G]])$ is (often) simple.
- What is $[[G]]/D([[G]])$?
- Is $[[G]]$ amenable?
- $[[G]]$ is sometimes finitely generated.
- $[[G]]$ is sometimes finitely presented.

Étale groupoids

A groupoid G is étale if the range map and the source map $r, s : G \to G^{(0)}$ are local homeomorphisms.
A groupoid G is essentially principal
if the interior of $\{g \in G \mid r(g) = s(g)\}$ is $G^{(0)}$.

Theorem (J. Renault 2008)

For essentially principal étale groupoids G_1 and G_2, the following are equivalent.

1. *G_1 is isomorphic to G_2.*
2. *There exists an isomorphism $\varphi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\varphi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.*

Topological full groups

From now on, we always assume that G is essentially principal and $G^{(0)}$ is a Cantor set.
A compact open set $U \subset G$ is a G-set if $r|U$ and $s|U$ are injective.
Then $\pi_U = (r|U) \circ (s|U)^{-1}$ is a partial homeomorphism on $G^{(0)}$.
The topological full group $[[G]]$ of G is defined by

$$[[G]] = \left\{ \pi_U \in \mathrm{Homeo}(G^{(0)}) \mid r(U) = s(U) = G^{(0)} \right\}.$$

Theorem (M 2012)

There exists a short exact sequence

$$1 \longrightarrow U(C(G^{(0)})) \longrightarrow N(C(G^{(0)}), C_r^*(G)) \longrightarrow [[G]] \longrightarrow 1.$$

Homology groups

$H_n(G)$ are the homology groups of the chain complex

$$0 \longleftarrow C_c(G^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(G^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(G^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \dots,$$

where $G^{(n)}$ is the space of composable strings of n elements.

For $\alpha = \pi_U \in [[G]]$, we have $\delta_1(1_U) = 0$, because $r(U) = s(U)$.
Thus 1_U is a 1-cycle.
We define the index map $I : [[G]] \to H_1(G)$ by $I(\alpha) = [1_U]$.
It is easy to see that I is a homomorphism.
Set $[[G]]_0 = \operatorname{Ker} I$.

We study the groups

$$D([[G]]) \subset [[G]]_0 \subset [[G]].$$

Examples of étale groupoids (1/3)

Let $\varphi : \Gamma \curvearrowright X$ be an essentially free action of a discrete group Γ on a Cantor set X.
$G_\varphi = \Gamma \times X$ is an étale groupoid with

$$(\gamma', \varphi_\gamma(x)) \cdot (\gamma, x) = (\gamma'\gamma, x), \quad (\gamma, x)^{-1} = (\gamma^{-1}, \varphi_\gamma(x)).$$

G_φ is called the transformation groupoid.
$C_r^*(G_\varphi)$ is canonically isomorphic to $C(X) \rtimes_{r,\varphi} \Gamma$.

$[[G_\varphi]]$ consists of $\alpha \in \mathrm{Homeo}(X)$ for which there exists a continuous map $c : X \to \Gamma$ such that $\alpha(x) = \varphi_{c(x)}(x)$ $\forall x \in X$.

$H_n(G_\varphi)$ are canonically isomorphic to the group homology $H_n(\Gamma, C(X, \mathbb{Z}))$.

Examples of étale groupoids (2/3)

Let $(\mathcal{V}, \mathcal{E})$ be an irreducible finite directed graph and let M be the adjacency matrix. Set

$$X = \{(x_k)_k \in \mathcal{E}^{\mathbb{N}} \mid t(x_k) = i(x_{k+1}) \quad \forall k \in \mathbb{N}\},$$

$$G = \{((x_k)_k, (y_k)_k) \in X \times X \mid \exists n \ \forall k \geq n \ x_k = y_k\}.$$

G is a typical example of an AF groupoid.

$[[G]]$ is an increasing union of finite direct sum of symmetric groups, and

$$H_n(G) = \begin{cases} \lim(\mathbb{Z}^{\mathcal{V}} \xrightarrow{M^t} \mathbb{Z}^{\mathcal{V}} \xrightarrow{M^t} \dots) & n = 0 \\ 0 & n \geq 1. \end{cases}$$

Examples of étale groupoids (3/3)

Let $(\mathcal{V}, \mathcal{E})$, M and X be as before.
Let $\sigma : X \to X$ be the shift. (X, σ) is a one-sided SFT.
Set

$$G = \left\{ (x, n, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \in \mathbb{N},\ n = k - l,\ \sigma^k(x) = \sigma^l(y) \right\}.$$

G is an étale groupoid with

$$(x, n, y) \cdot (y, n', y') = (x, n+n', y'), \quad (x, n, y)^{-1} = (y, -n, x).$$

We call G an SFT groupoid. The subgroupoid $\{(x, 0, y) \in G\}$ is the AF groupoid mentioned in the previous slide.

We have

$$H_n(G) = \begin{cases} \operatorname{Coker}(\mathrm{id} - M^t) & n = 0 \\ \operatorname{Ker}(\mathrm{id} - M^t) & n = 1 \\ 0 & n \geq 2. \end{cases}$$

Isomorphism theorem

We always assume that G is essentially principal and $G^{(0)}$ is a Cantor set.

Theorem (M)

For minimal groupoids G_1 and G_2, the following are equivalent.

1. *G_1 is isomorphic to G_2 as an étale groupoid.*
2. *$[[G_1]]$ is isomorphic to $[[G_2]]$ as a group.*
3. *$[[G_1]]_0$ is isomorphic to $[[G_2]]_0$ as a group.*
4. *$D([[G_1]])$ is isomorphic to $D([[G_2]])$ as a group.*

This generalizes the result of T. Giordano, I. F. Putnam and C. F. Skau (for minimal $\mathbb{Z}$-actions) and the result of K. Matsumoto (for SFT groupoids).
The proof is based on an algebraic characterization of transpositions in $[[G]]$.

Almost finite groupoids

Definition (M 2012)

G is said to be almost finite if for any compact subset $C \subset G$ and $\varepsilon > 0$ there exists an elementary subgroupoid $K \subset G$ such that

$$\frac{|CKx \setminus Kx|}{|Kx|} < \varepsilon \quad \forall x \in G^{(0)}.$$

This may remind us of the Følner condition for amenable groups, but there is no direct relationship between them.
AF groupoids are clearly almost finite ($\because$ $\forall C \; \exists K \; C \subset K$).

Lemma (M 2012)

When $\varphi : \mathbb{Z}^N \curvearrowright X$ is free, G_φ is almost finite.

Purely infinite groupoids

Definition (M)

G is said to be purely infinite if for any clopen set $A \subset G^{(0)}$ there exist G-sets $U, V \subset G$ such that $s(U) = s(V) = A$, $r(U) \cup r(V) \subset A$ and $r(U) \cap r(V) = \emptyset$.

A purely infinite groupoid G admits no invariant probability measures on $G^{(0)}$.
If G is purely infinite, then $C_r^*(G)$ is purely infinite (M. Rørdam and A. Sierakowski 2012).

Lemma (M)

Any SFT groupoid G is purely infinite and minimal.

Simplicity of commutator subgroups

Theorem (M)

Suppose that G is either almost finite or purely infinite.

1. *The index map $I : [[G]] \to H_1(G)$ is surjective.*
2. *Assume further that G is minimal. Then $D([[G]])$ is simple.*

It follows that the abelianization $[[G]]_{\mathrm{ab}} = [[G]]/D([[G]])$ has $H_1(G) \cong [[G]]/[[G]]_0$ as its quotient.

We may think of $[[G]]_0$ and $D([[G]])$ as 'symmetric group' and 'alternating group' acting on the Cantor set.
This is the reason why $D([[G]])$ is simple.

Minimal $\mathbb{Z}$-actions

Theorem (M 2006)

Let $\varphi : \mathbb{Z} \curvearrowright X$ be a minimal $\mathbb{Z}$-action on a Cantor set X.

1. *$[[G_\varphi]]_{\mathrm{ab}}$ is isomorphic to $(H_0(G_\varphi) \otimes \mathbb{Z}_2) \oplus \mathbb{Z}$.*
2. *$D([[G_\varphi]])$ is finitely generated if and only if φ is expansive.*
3. *$D([[G_\varphi]])$ is never finitely presented.*

Theorem (K. Juschenko and N. Monod 2012)

Let $\varphi : \mathbb{Z} \curvearrowright X$ be a minimal $\mathbb{Z}$-action on a Cantor set X.
Then $[[G_\varphi]]$ is amenable.

This provides the first examples of finitely generated simple amenable infinite groups.

Preliminaries

Let $(\mathcal{V}, \mathcal{E})$, M and (X, σ) be as before.
The SFT groupoid of (X, σ) (or of M) is

$$G = \left\{ (x, n, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \in \mathbb{N},\ n = k - l,\ \sigma^k(x) = \sigma^l(y) \right\}.$$

Any element $\alpha \in [[G]] \subset \mathrm{Homeo}(X)$ is locally equal to a partial homeomorphism of the form

$$(e_1, e_2, \dots, e_k, x_1, x_2, \dots) \mapsto (f_1, f_2, \dots, f_l, x_1, x_2, \dots),$$

where $(e_1, e_2, \dots, e_k)$ and $(f_1, f_2, \dots, f_l)$ are paths on the graph $(\mathcal{V}, \mathcal{E})$ such that $i(e_k) = i(f_l)$.

Higman-Thompson groups

In 1965 R. Thompson gave the first example of a finitely presented infinite simple group. G. Higman and K. S. Brown later generalized it to infinite families $F_{n,r} \subset T_{n,r} \subset V_{n,r}$ for $n \in \mathbb{N} \setminus \{1\}$ and $r \in \mathbb{N}$.

The group $V_{n,r}$ consists of PL right continuous bijections $f : [0, r) \to [0, r)$ with finitely many singularities, all in $\mathbb{Z}[1/n]$, slopes lying in powers of n, and mapping $\mathbb{Z}[1/n] \cap [0, r)$ to itself. $V_{n,r}$ is called the Higman-Thompson group.

It is known that $V_{n,r}$ is finitely presented, $D(V_{n,r})$ is simple, and $V_{n,r}/D(V_{n,r})$ is trivial when n is even and is $\mathbb{Z}_2$ when n is odd.

$F_{n,r}$ is a subgroup of $V_{n,r}$ consisting of continuous maps f.
$F_{n,r}$ is also finitely presented.
It is not yet known if $F_{n,r}$ is amenable or not.

Nekrashevych's observation

Theorem (V. V. Nekrashevych 2004)

When (X, σ) is the full shift over n symbols,
the topological full group $[[G]]$ is isomorphic to $V_{n,1}$.

The continuous map $\rho : \{0, 1, \dots, n-1\}^{\mathbb{N}} \to [0, 1]$ defined by

$$\rho((x_k)_k) = \sum_{k=1}^{\infty} \frac{x_k}{n^k}$$

induces the isomorphism $[[G]] \cong V_{n,1}$.

$[[G]]$ for general SFT groupoids G may be thought of as a generalization of the Higman-Thompson group $V_{n,r}$.

The results

For G and $Y \subset G^{(0)}$, we let $G|Y = \{g \in G \mid r(g), s(g) \in Y\}$ be the reduction of G to Y.

Theorem (M)

Let G be an SFT groupoid and let $Y \subset X$ be a clopen set.

1. $[[G|Y]]$ *(and* $[[G|Y]]_0$ *and* $D([[G|Y]])$*) 'remembers'* $G|Y$.
2. $D([[G|Y]])$ *is simple.*
3. $[[G|Y]]$ *has the Haagerup property.*
4. $[[G|Y]]_{\mathrm{ab}}$ *is isomorphic to* $(H_0(G) \otimes \mathbb{Z}_2) \oplus H_1(G)$.
5. $[[G|Y]]$ *is of type* F_∞*, and hence is finitely presented.*
6. $[[G|Y]]_0$ *and* $D([[G|Y]])$ *are finitely generated.*

Brown's criterion

Theorem (K. S. Brown 1987)

Suppose that a group Γ admits a contractible Γ-complex Z such that the stabilizer of every cell is of type F_∞. Let $\{Z_q\}_{q\in\mathbb{N}}$ be a filtration of Z such that each Z_q is finite mod Γ. Suppose that the connectivity of the pair (Z_{q+1}, Z_q) tends to ∞ as q tends to ∞. Then Γ is of type F_∞.

Finite presentation (1/2)

Let $M = [2]$ (i.e. the full shift over 2 symbols).
$[[G]]$ is the Higman-Thompson group $V_{2,1}$ and
it is described by the following diagram (due to K. S. Brown).

$$\begin{tikzcd}
 & & \Sigma_7 & & \\
 & \Sigma_3 \times \Sigma_2 \arrow[ur] \arrow[dl] & & \Sigma_5 \arrow[ul] \arrow[dr] & \\
\Sigma_5 & & \Sigma_4 \arrow[ll] \arrow[rr] & & \Sigma_6
\end{tikzcd}$$

Finite presentation (2/2)

Let $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. We have $H_0(G) = H_1(G) = \mathbb{Z}$.

$[[G]]$ is described as follows.

$$\begin{tikzcd}
\Sigma_{6,2,4} \arrow[rr, "c"] \arrow[dd, "i"'] & & \Sigma_{8,8} & & \Sigma_{7,6} \arrow[ll, "c"'] \arrow[dd, "i"] \\
& \Sigma_{4,2,6} \arrow[ur] \arrow[dl] & & \Sigma_{6,7} \arrow[ul] \arrow[dr] & \\
\Sigma_{6,6} & & \Sigma_{5,6} \arrow[ll] \arrow[rr] & & \Sigma_{7,7} \\
& & \Sigma_{6,5} \arrow[ull, "i"] \arrow[urr, "c"'] & &
\end{tikzcd} \quad * \ \langle g \rangle$$

with relations " $g^{-1} i(\cdot) g = c(\cdot)$ "

References

- T. Giordano, I. F. Putnam and C. F. Skau, *Full groups of Cantor minimal systems*, Israel J. Math. 111 (1999), 285–320.
- K. Matsumoto, *Classification of Cuntz-Krieger algebras by orbit equivalence of topological Markov shifts*, to appear in Proc. Amer. Math. Soc.
- H. Matui, *Some remarks on topological full groups of Cantor minimal systems*, Internat. J. Math. 17 (2006), 231–251.
- H. Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, Proc. London Math. Soc. 104 (2012), 27–56.
- H. Matui, *Topological full groups of one-sided shifts of finite type*, preprint.
- J. Renault, *A Groupoid Approach to C^*-algebras*, Lecture Notes in Mathematics 793, Springer, Berlin, 1980.