On topological full groups of Cantor minimal systems

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Cantor minimal systems

A topological space X is called a Cantor set if X is compact, metrizable, totally disconnected (the closed and open sets form a base for the topology) and has no isolated points.
Any such X is homeomorphic to $\{0,1\}^{\mathbb{Z}}$.

A homeomorphism $\varphi \in \mathrm{Homeo}(X)$ is said to be minimal if for all $x \in X$ the set $\{\varphi^n(x) \mid n \in \mathbb{Z}\}$ is dense in X, or equivalently, there are no non-trivial closed φ-invariant subsets of X.

We call (X, φ) a Cantor minimal system.

T. Giordano, I. F. Putnam and C. F. Skau gave a complete classification of (X, φ) up to topological orbit equivalence.

Topological full groups

For a Cantor minimal system (X, φ), we set

$$[[\varphi]] = \{\psi \in \mathrm{Homeo}(X) \mid \exists c \in C(X, \mathbb{Z})\ \psi(x) = \varphi^{c(x)}(x)\}$$

and call it the topological full group. Clearly $[[\varphi]]$ is infinite ($\because [[\varphi]] \supset \langle \varphi \rangle \cong \mathbb{Z}$) and countable ($\because X$ has countably many clopen subsets).

There exists a homomorphism $I : [[\varphi]] \to \mathbb{Z}$ such that $I(\varphi) = 1$, called the index map. We write $[[\varphi]]_0 = \mathrm{Ker}\, I$.
Each $\psi \in [[\varphi]]$ gives rise to a unitary $u_\psi \in C(X) \rtimes_\varphi \mathbb{Z}$ satisfying

$$u_\psi f u_\psi^* = f \circ \psi^{-1} \quad \forall f \in C(X).$$

Then $I(\psi)$ is identified with the K_1-class of the unitary u_ψ.

The isomorphism theorem

Theorem (T. Giordano, I. F. Putnam and C. F. Skau 1999)

For Cantor minimal systems (X_1, φ_1) and (X_2, φ_2), the following conditions are equivalent.

1. φ_1 *is conjugate to* φ_2 *or* φ_2^{-1}.
2. $[[\varphi_1]]$ *is isomorphic to* $[[\varphi_2]]$.
3. $[[\varphi_1]]_0$ *is isomorphic to* $[[\varphi_2]]_0$.
4. $D([[\varphi_1]])$ *is isomorphic to* $D([[\varphi_2]])$.

This is a topological analogue of H. Dye's theorem for ergodic measure-preserving actions on a Lebesgue space.

There exist uncountably many Cantor minimal systems (e.g. topological entropy distinguishes), and so there exist uncountably many isomorphism classes of $[[\varphi]]$, $[[\varphi]]_0$ and $D([[\varphi]])$.

Homology groups

We let $H_*(\varphi)$ denote the homology groups $H_*(\mathbb{Z}, C(X,\mathbb{Z}))$ with coefficients in $C(X,\mathbb{Z})$ for $* = 0, 1$, i.e.

$$H_0(\varphi) = C(X,\mathbb{Z})/\{f - f\circ\varphi \mid f \in C(X,\mathbb{Z})\},$$

$$H_1(\varphi) \cong H^0(\mathbb{Z}, C(X,\mathbb{Z})) = C(X,\mathbb{Z})^{\mathbb{Z}} \cong \mathbb{Z}.$$

For any countable torsion-free abelian group $G \neq \mathbb{Z}$, there exists a Cantor minimal system (X,φ) such that $H_0(\varphi) = G$. We can think of the index map $I : [[\varphi]] \to \mathbb{Z}$ as a homomorphism onto $H_1(\varphi) \cong \mathbb{Z}$.

Odometers (1/2)

We identify $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ with $\{0, 1, \ldots, m-1\}$.
Let $(m_n)_{n=1}^{\infty}$ be a sequence of natural numbers such that m_n divides m_{n+1} and $m_n \to \infty$ as $n \to \infty$. Let $\rho_n : \mathbb{Z}_{m_{n+1}} \to \mathbb{Z}_{m_n}$ be the homomorphism such that $\rho_n(1) = 1$.

We let X be the inverse limit of $\mathbb{Z}_{m_n}$ under the map ρ_n, i.e.

$$X = \left\{ (x_n)_n \in \prod \mathbb{Z}_{m_n} \mid \rho_n(x_{n+1}) = x_n \right\}.$$

Define $\varphi \in \mathrm{Homeo}(X)$ by $\varphi((x_n)_n) = (x_n + 1)_n$.
(X, φ) is called the odometer of type $(m_n)_n$. It is easy to see

$$H_0(\varphi) \cong \{l/m_n \mid l \in \mathbb{Z},\ n \in \mathbb{N}\} \subset \mathbb{Q}.$$

The crossed product $C(X) \rtimes_\varphi \mathbb{Z}$ is the Bunce-Deddens algebra.

Odometers (2/2)

For $k \in \mathbb{N}$ and $l \in \mathbb{Z}_{m_k}$ we set $U(k,l) = \{(x_n)_n \in X \mid x_k = l\}$. Then $\{U(k,l) \mid l \in \mathbb{Z}_{m_k}\}$ is a clopen partition of X and $\varphi(U(k,l)) = U(k,l{+}1)$.

By definition, each $\psi \in [[\varphi]]$ is written as $\psi(x) = \varphi^{c(x)}(x)$.
Let $\Gamma_k \subset [[\varphi]]$ be the subgroup consisting of all ψ for which the function c is constant on each $U(k,l)$, $l \in \mathbb{Z}_{m_k}$.
Clearly $[[\varphi]]$ equals $\bigcup \Gamma_k$. Any $\psi \in \Gamma_k$ induces a permutation of $U(k,l)$, and so there exists a homomorphism from Γ_k to S_{m_k}.
Its kernel is isomorphic to $\mathbb{Z}^{m_k}$.

Proposition

When (X,φ) is an odometer, $[[\varphi]]$ can be written as an increasing union of subgroups of the form $\mathbb{Z}^m \rtimes S_m$.

Denjoy systems

Let $\alpha \in (0,1)$ be an irrational number and let X be the Cantor set obtained by cutting $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ at the points $n\alpha$, $n \in \mathbb{Z}$. In other words, X is the Gelfand spectrum of the abelian C^*-algebra generated by the characteristic functions of $[n\alpha, (n{+}1)\alpha) \subset \mathbb{T}$.

The α-rotation on $\mathbb{T}$ induces a minimal homeomorphism φ of X. We call (X, φ) a Denjoy system.

It is known that $H_0(\varphi)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Unlike the odometers, it is impossible to write $[[\varphi]]$ as an increasing union of some 'easy' groups.

Theorems

Let (X, φ) be a Cantor minimal system.

Theorem (M 2006)

1. *$D([[\varphi]])$ is simple.*
2. *$[[\varphi]]_0/D([[\varphi]])$ is isomorphic to $H_0(\varphi) \otimes \mathbb{Z}_2$.*
3. *$D([[\varphi]])$ is finitely generated iff φ is a subshift.*

Theorem (M 2011)

When $D([[\varphi]])$ is finitely generated, $D([[\varphi]])$ has exponential growth.

Elements of finite order (1/2)

For a clopen set $U \subset X$ such that U and $\varphi(U)$ are disjoint, we define $\sigma_U \in [[\varphi]]$ by

$$\sigma_U(x) = \begin{cases} \varphi(x) & x \in U \\ \varphi^{-1}(x) & x \in \varphi(U) \\ x & \text{otherwise.} \end{cases}$$

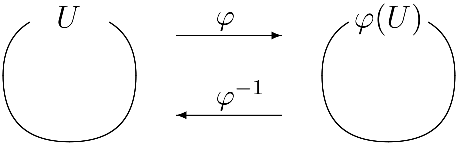
For a clopen set $U \subset X$ such that $\varphi^{-1}(U)$, U and $\varphi(U)$ are disjoint, we define $\gamma_U \in [[\varphi]]$ by

$$\gamma_U(x) = \begin{cases} \varphi(x) & x \in \varphi^{-1}(U) \cup U \\ \varphi^{-2}(x) & x \in \varphi(U) \\ x & \text{otherwise.} \end{cases}$$

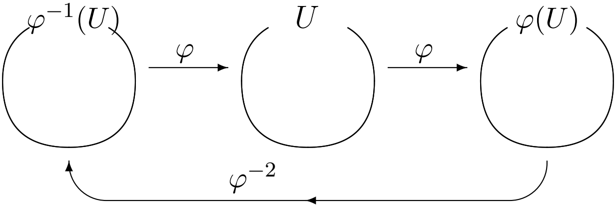
Since $\gamma_U = \sigma_U \circ \sigma_{\varphi^{-1}(U)} \circ \sigma_U \circ \sigma_{\varphi^{-1}(U)}$, γ_U belongs to $D([[\varphi]])$.

Elements of finite order (1/2)

σ_U is the transposition between U and $\varphi(U)$.



γ_U is the cyclic permutation on $\varphi^{-1}(U), U, \varphi(U)$.



Analogue of symmetric groups and alternating groups

Theorem

1. $D([[\varphi]])$ *is simple.*
2. $[[\varphi]]_0/D([[\varphi]])$ *is isomorphic to* $H_0(\varphi) \otimes \mathbb{Z}_2$.
3. $D([[\varphi]])$ *is finitely generated iff* φ *is a subshift.*

In the proof of this theorem, the following proposition plays an important role.

Proposition

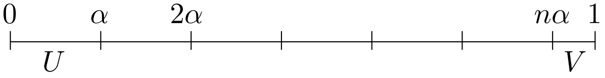
1. $[[\varphi]]_0$ *is generated by all the elements* σ_U.
2. $D([[\varphi]])$ *is generated by all the elements* γ_U.

Thus $[[\varphi]]_0$ and $D([[\varphi]])$ are regarded as analogue of symmetric groups and alternating groups acting on the Cantor set X.
We write $\mathrm{sgn} : [[\varphi]]_0 \to [[\varphi]]_0/D([[\varphi]]) \cong H_0(\varphi) \otimes \mathbb{Z}_2$.

Generators of Denjoy systems (1/2)

Let $\alpha \in (0, 1/6)$ be an irrational number and let (X, φ) be the Denjoy system induced by the α-rotation on $\mathbb{T}$.
We would like to see that $[[\varphi]]$ is generated by three elements.

Let $U \subset X$ be the clopen subset corresponding to $[0, \alpha)$ and let $V \subset X$ be the clopen subset corresponding to $[n\alpha, 1)$, where $n = \max\{k \in \mathbb{N} \mid k\alpha < 1\}$.



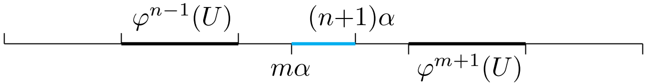
$H_0(\varphi) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by 1_U and 1_V, and so $\mathrm{sgn}(\sigma_U)$ and $\mathrm{sgn}(\sigma_V)$ generate $H_0(\varphi) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Hence $\langle \varphi, \sigma_U, \sigma_V, D([[\varphi]]) \rangle = [[\varphi]]$.

Generators of Denjoy systems (2/2)

We shall prove that $\langle \varphi, \sigma_U \rangle$ contains $D([[\varphi]])$.
Once this is done, we get $[[\varphi]] = \langle \varphi, \sigma_U, \sigma_V \rangle$.

First, $\varphi^n \circ \sigma_U \circ \varphi^{-n} = \sigma_{\varphi^n(U)}$ is in $\langle \varphi, \sigma_U \rangle$ for any $n \in \mathbb{Z}$.
Then, $\gamma_{\varphi^n(U)} = \sigma_{\varphi^n(U)} \circ \sigma_{\varphi^{n-1}(U)} \circ \sigma_{\varphi^n(U)} \circ \sigma_{\varphi^{n-1}(U)}$ is also in $\langle \varphi, \sigma_U \rangle$ for any $n \in \mathbb{Z}$.
Any clopen subset W of X is written as a disjoint union of clopen subsets O corresponding to intervals $[m\alpha, (n+1)\alpha)$.



Since $D([[\varphi]])$ is generated by the elements γ_O and

$$\gamma_O = \gamma_{\varphi^{m+1}(U)} \circ \gamma^{-1}_{\varphi^{n-1}(U)} \circ \gamma^{-1}_{\varphi^{m+1}(U)} \circ \gamma_{\varphi^{n-1}(U)}$$

belongs to $\langle \varphi, \sigma_U \rangle$, $D([[\varphi]])$ is contained in $\langle \varphi, \sigma_U \rangle$.

Growth of finitely generated groups

Let Γ be a finitely generated group and let S be a finite subset of generators of Γ.
For $k \in \mathbb{N}$, we denote by $\beta(\Gamma, S, k)$ the number of elements $\gamma \in \Gamma$ such that $\ell_S(\gamma) \leq k$.

The group Γ is said to have polynomial growth if there exist $c > 0$ and $d \geq 1$ such that $\beta(\Gamma, S, k) \leq ck^d$.
The group Γ is said to have exponential growth if there exist $c > 0$ and $v > 1$ such that $\beta(\Gamma, S, k) \geq cv^k$.

The property of being of exponential growth (resp. polynomial growth) does not depend on the choice of S.

If Γ contains a free semi-group on two generators, then Γ has exponential growth.

Lamplighter group

We call the wreath product

$$L = \mathbb{Z}_2 \wr \mathbb{Z} = \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \right) \rtimes \mathbb{Z}$$

the lamplighter group, where the semi-direct product is taken with respect to the shift action.

Let

$$a = (\ldots, 0, 0, 0, \overset{\overset{0}{\vee}}{1}, 0, 0, 0, \ldots) \in \bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \subset L$$

and let $t \in \mathbb{Z} \subset L$ be a generator of $\mathbb{Z}$. Clearly $L = \langle a, t \rangle$.
It is easy to see that t and ata generate a free semi-group.

Hence, if a finitely generated group Γ contains L,
then Γ has exponential growth.

Growth of topological full groups (1/2)

Theorem (M 2011)

For a Cantor minimal system (X, φ), the following are equivalent.

1. *(X, φ) is not an odometer.*
2. *$D([[\varphi]])$ contains the lamplighter group L.*
3. *$[[\varphi]]$ contains the lamplighter group L.*

In particular, if $D([[\varphi]])$ is finitely generated (or, equivalently, if (X, φ) is a minimal subshift), then $D([[\varphi]])$ has exponential growth.

As mentioned earlier, when (X, φ) is an odometer, the topological full group $[[\varphi]]$ is an increasing union of subgroups of the form $\mathbb{Z}^m \rtimes S_m$. Hence the third condition implies the first condition.

We sketch a proof of the implication from the first to the third.

Growth of topological full groups (2/2)

Suppose that (X,φ) is not an odometer. Choose a clopen subset U so that $U \cap \varphi(U) = \emptyset$. Let $\psi \in [[\varphi]]$ be the first return map on U. Then $\varphi \circ \psi \circ \varphi^{-1}$ is the first return map on $\varphi(U)$.
Set $t = \psi \circ (\varphi \circ \psi \circ \varphi^{-1}) \in [[\varphi]]$.

Since $(U, \psi|U)$ is not an odometer, there exists a clopen set $O \subset U$ such that for any finite subset $F \subset \mathbb{Z}$

$$\sum_{k \in F} 1_O \circ \psi^k \neq 0 \quad \text{in} \quad C(X, \mathbb{Z}_2).$$

Let $a = \sigma_O \in [[\varphi]]$.
Then

$$\langle t^k \circ a \circ t^{-k} \mid k \in \mathbb{Z} \rangle = \langle \sigma_{\psi^k(O)} \mid k \in \mathbb{Z} \rangle \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}_2,$$

and so $\langle a, t \rangle \cong L$.

Homology of $\mathbb{Z}^N$-actions

Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be a free action of $\mathbb{Z}^N$ on a Cantor set X by homeomorphisms. We write $H_*(\varphi) \cong H_*(\mathbb{Z}^N, C(X, \mathbb{Z}))$.
Examples of free minimal actions $\varphi : \mathbb{Z}^N \curvearrowright X$ are given by certain tilings on the Euclidean space.
For $\varphi : \mathbb{Z}^2 \curvearrowright X$ arising from the Penrose tiling on $\mathbb{R}^2$,

$$H_0(\varphi) \cong \mathbb{Z}^8, \quad H_1(\varphi) \cong \mathbb{Z}^5, \quad H_2(\varphi) \cong \mathbb{Z}.$$

For $\varphi : \mathbb{Z}^3 \curvearrowright X$ arising from the Ammann-Kramer tiling on $\mathbb{R}^3$,

$$H_0(\varphi) \cong \mathbb{Z}^{181} \oplus \mathbb{Z}_2, \quad H_1(\varphi) \cong \mathbb{Z}^{72} \oplus \mathbb{Z}_2,$$

$$H_2(\varphi) \cong \mathbb{Z}^{12}, \quad H_3(\varphi) \cong \mathbb{Z}.$$

In general, the Chern character induces

$$\bigoplus H_{2n+i}(\varphi) \otimes \mathbb{Q} \;\cong\; K_i(C(X) \rtimes_\varphi \mathbb{Z}^N) \otimes \mathbb{Q} \quad i = 0, 1.$$

Index map

For a free action $\varphi : \mathbb{Z}^N \curvearrowright X$, the topological full group $[[\varphi]]$ is defined in the same way:

$$[[\varphi]] = \{\psi \in \mathrm{Homeo}(X) \mid \exists c \in C(X, \mathbb{Z}^N) \ \psi(x) = \varphi^{c(x)}(x)\}$$

We can define a homomorphism I from the topological full group $[[\varphi]]$ to $H_1(\varphi) \cong H_1(\mathbb{Z}^N, C(X, \mathbb{Z}))$.

Theorem (M)

Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be a free action of $\mathbb{Z}^N$ on a Cantor set X by homeomorphisms.

1. $I : [[\varphi]] \to H_1(\varphi)$ *is surjective.*
2. $[[\varphi]]_0 = \mathrm{Ker}\, I$ *is generated by transpositions.*

Open problems

Problems

Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be a free minimal action on a Cantor set X.

- Is $D([[\varphi]])$ simple?
- What is $[[\varphi]]_0/D([[\varphi]])$?
- When is $D([[\varphi]])$ finitely generated?
- Is $[[\varphi]]$ amenable?