Group actions on simple stably finite C^*-algebras

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Notations

- A group Γ is always assumed to be countable, discrete and amenable.
- A C^*-algebra A is always assumed to be unital, simple and separable.
- $U(A)$ denotes the group of unitaries of A, and $\mathrm{Aut}(A)$ denotes the group of automorphisms of A.
- For $u \in U(A)$, $\mathrm{Ad}\, u \in \mathrm{Aut}(A)$ is given by $x \mapsto uxu^*$ and is called an inner automorphism.
- $\mathcal{Z}$ denotes the Jiang-Su algebra.

Cocycle actions

Definition

A pair (α, u) of a map $\alpha : \Gamma \to \mathrm{Aut}(A)$ and a map $u : \Gamma \times \Gamma \to U(A)$ is called a cocycle action of Γ on A if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$$

and

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk)$$

hold for any $g, h, k \in \Gamma$. We write $(\alpha, u) : \Gamma \curvearrowright A$.

We always assume $\alpha_1 = \mathrm{id}$, $u(g,1) = u(1,g) = 1$ for all $g \in \Gamma$.
When α_g is not inner for any $g \in \Gamma \setminus \{1\}$,
(α, u) is said to be outer.
When $u \equiv 1$, $\alpha : \Gamma \curvearrowright A$ is a genuine action.

Cocycle conjugacy

Definition

Two cocycle actions $(\alpha, u) : \Gamma \curvearrowright A$ and $(\beta, v) : \Gamma \curvearrowright B$ are said to be cocycle conjugate if there exist a family of unitaries $(w_g)_{g\in\Gamma}$ in B and an isomorphism $\theta : A \to B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \mathrm{Ad}\, w_g \circ \beta_g$$

and

$$\theta(u(g, h)) = w_g \beta_g(w_h) v(g, h) w_{gh}^*$$

hold for every $g, h \in \Gamma$.

Our eventual goal is

- to classify the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$,
- to classify (α, u) up to cocycle conjugacy and to determine when (α, u) is cocycle conjugate to a genuine action.

Twisted crossed product

Definition

For $(\alpha, u) : \Gamma \curvearrowright A$, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is the universal C^*-algebra generated by A and a family of unitaries $(\lambda_g^\alpha)_{g\in\Gamma}$ satisfying

$$\lambda_g^\alpha \lambda_h^\alpha = u(g,h)\lambda_{gh}^\alpha \quad \text{and} \quad \lambda_g^\alpha a \lambda_g^{\alpha *} = \alpha_g(a)$$

for all $g, h \in \Gamma$ and $a \in A$.

If (α, u) and (β, v) are cocycle conjugate via $\theta : A \to B$ and $(w_g)_g$, then $A \rtimes_{(\alpha,u)} \Gamma$ and $B \rtimes_{(\beta,v)} \Gamma$ are canonically isomorphic by

$$a \mapsto \theta(a) \quad \text{and} \quad \lambda_g^\alpha \mapsto w_g \lambda_g^\beta.$$

Group actions on injective factors

Theorem

Let M be an injective factor. Two cocycle actions (α, u) and (β, v) of Γ on M are strongly cocycle conjugate if and only if $\mathrm{Inv}(\alpha, u) = \mathrm{Inv}(\beta, v)$.

The invariant $\mathrm{Inv}(\alpha, u)$ consists of "centrally trivial part", "Connes-Takesaki module" and "characteristic invariant".

The theorem above has a long history: A. Connes (cyclic groups on II_1), V. F. R. Jones (finite groups on II_1), A. Ocneanu (on II_1), C. E. Sutherland and M. Takesaki (on III_λ with $\lambda \neq 1$), Y. Kawahigashi, C. E. Sutherland and M. Takesaki (abelian groups on III_1), Y. Katayama, C. E. Sutherland and M. Takesaki (all actions)...

and T. Masuda (all cocycle actions with a shorter proof).

$\mathcal{Z}$-stability of crossed product

Theorem (Y. Sato and M)

Let A be a stably finite C^*-algebra with finite nuclear dimension and with finitely many extremal tracial states. Let Γ be an elementary amenable group.
Let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action.
Then (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1) : \Gamma \curvearrowright A \otimes \mathcal{Z}$.
In particular, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is $\mathcal{Z}$-stable.

In order to prove this, it suffices to construct a unital embedding of $\mathcal{Z}$ into the fixed point algebra $(A^\infty \cap A')^\alpha$.

Strong outerness

Let $T(A)$ denote the set of tracial states and let π_τ be the GNS representation by $\tau \in T(A)$.

Definition

$\alpha \in \mathrm{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \mathrm{Ad}\, U$.
A cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ is said to be strongly outer if α_g is not weakly inner for every $g \in \Gamma \setminus \{1\}$.

If $T(A) = \{\tau\}$, then

$$(\alpha, u) : \Gamma \curvearrowright A \text{ is strongly outer}$$
$$\iff \quad (\bar{\alpha}, u) : \Gamma \curvearrowright \pi_\tau(A)'' \text{ is outer.}$$

Elementary amenable groups

Definition

The class of elementary amenable groups is defined as the smallest family of groups containing all finite groups and all abelian groups, and closed under the processes of taking subgroups, quotients, group extensions and increasing unions.

For instance, all solvable groups are elementary amenable.
There exist amenable groups which are not elementary (R. I. Grigorchuk).

Weak Rohlin property

Theorem

Let A be a nuclear stably finite C^*-algebra with finitely many extremal tracial states and let Γ be elementary.
Then any strongly outer cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ has the weak Rohlin property, i.e. for any $F \Subset \Gamma$ and $\varepsilon > 0$, there exist an (F, ε)-invariant $K \Subset \Gamma$ and a sequence $(e_n)_n$ of positive contractions in A such that

$$[e_n, a] \to 0, \quad \alpha_g(e_n)\alpha_h(e_n) \to 0, \quad \tau(e_n) \to |K|^{-1}$$

as $n \to \infty$ for all $a \in A$, $g, h \in K$ with $g \neq h$ and $\tau \in T(A)$.

A bounded sequence $(x_n)_n$ in A is called a central sequence if $[x_n, a] \to 0$ as $n \to \infty$ for all $a \in A$.

Property (SI)

Theorem

Let A be a stably finite C^*-algebra with finite nuclear dimension. Then A has the property (SI), i.e. for any central sequences $(x_n)_n$ and $(y_n)_n$ in A satisfying $0 \le x_n, y_n \le 1$,

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(x_n) = 0 \quad \text{and} \quad \inf_{m\in\mathbb{N}} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(y_n^m) > 0,$$

there exists a central sequence $(s_n)_n$ in A such that

$$\lim_{n\to\infty} \|s_n^* s_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n s_n - s_n\| = 0.$$

By using the weak Rohlin property and the property (SI), we can construct a unital embedding of $\mathcal{Z}$ into the fixed point algebra $(A^\infty \cap A')^\alpha$, which implies the $\mathcal{Z}$-absorption theorem.

$\mathbb{Z}$-actions on UHF algebras

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property, i.e. for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\sum_{i=0}^{m-1} \alpha^i(e_n) + \sum_{j=0}^{m} \alpha^j(f_n) \to 1.$$

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra. All strongly outer $\mathbb{Z}$-actions on A are cocycle conjugate to each other.

The proof uses the Evans-Kishimoto intertwining argument, which is an equivariant version of Elliott's intertwining argument.

Stability

Stability plays a central role in the Evans-Kishimoto intertwining argument. We would like to sketch it briefly.

We are given $\alpha, \beta : \Gamma \curvearrowright A$. Suppose that there exists a family $(u_g)_{g \in \Gamma}$ of unitaries in A^∞ such that

$$u_g \alpha_g(u_h) = u_{gh}, \; \beta_g(a) = (\operatorname{Ad} u_g \circ \alpha_g)(a) \quad \forall a \in A, \; g, h \in \Gamma.$$

Stability of α implies there exists a unitary $v \in A^\infty$ such that

$$u_g = v \alpha_g(v^*) \quad \forall g \in \Gamma.$$

Then we would have

$$\beta_g(a) = (\operatorname{Ad} v \circ \alpha_g \circ \operatorname{Ad} v^*)(a) \quad \forall a \in A, \; g \in \Gamma,$$

which may induce 'conjugacy' between α and β.

$\mathbb{Z}$-actions on AH algebras

We can generalize Kishimoto's results for UHF algebras to certain AH algebras.

Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth, real rank zero and finitely many extremal tracial states. Let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. If α_k is approximately inner for some $k \in \mathbb{N}$, then α has the Rohlin property.

Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth and real rank zero. If two actions $\alpha, \beta : \mathbb{Z} \curvearrowright A$ have the Rohlin property and $\alpha_1 \circ \beta_{-1}$ is asymptotically inner, then α and β are cocycle conjugate.

$\mathbb{Z}$-actions on $\mathcal{Z}$

Theorem (Y. Sato 2010)

All strongly outer $\mathbb{Z}$-actions on $\mathcal{Z}$ are cocycle conjugate to each other.

We sketch the proof. Let $\alpha, \beta : \mathbb{Z} \curvearrowright \mathcal{Z}$ be strongly outer.
(1) By the theorem mentioned before, we may replace α, β with $\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id} : \mathbb{Z} \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.
(2) $Z = \{f : [0,1] \to M_{2^\infty} \otimes M_{3^\infty} \mid f(0) \in M_{2^\infty},\ f(1) \in M_{3^\infty}\}$ is a unital subalgebra of $\mathcal{Z}$ (M. Rørdam and W. Winter 2010).
(3) By Kishimoto's result, $\alpha \otimes \mathrm{id}$ and $\beta \otimes \mathrm{id}$ are cocycle conjugate as actions on $\mathcal{Z} \otimes B$ with B being a UHF algebra.
(4) With some extra effort we get cocycle conjugacy on $\mathcal{Z} \otimes \mathcal{Z}$.

Cocycle actions of $\mathbb{Z}^2$ on AF algebras (1/2)

We write $\mathbb{Z}^2 = \langle a, b \mid bab^{-1} = a \rangle$.

Theorem (H. Nakamura 1999, M 2010, Y. Sato and M)

Let A be a unital simple AF algebra with finitely many extremal tracial states and let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ be a strongly outer cocycle action. Suppose that α_a^n and α_b^n are approximately inner for some $n \in \mathbb{N}$. Then (α, u) has the Rohlin property.

For a cocycle action $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$, we have

$$\begin{aligned}\alpha_b \circ \alpha_a &= \operatorname{Ad} u(b,a) \circ \alpha_{ba} \\ &= \operatorname{Ad} u(b,a) \circ \alpha_{ab} = \operatorname{Ad}(u(b,a)u(a,b)^*) \circ \alpha_a \circ \alpha_b\end{aligned}$$

Conversely, two single automorphisms commuting up to an inner automorphism give rise to a cocycle action of $\mathbb{Z}^2$.

Cocycle actions of $\mathbb{Z}^2$ on AF algebras (2/2)

For $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ satisfying $\alpha_g \in \overline{\mathrm{Inn}}(A)$ $\forall g \in \mathbb{Z}^2$, we introduce an invariant $c(\alpha, u) \in \mathrm{OrderExt}(K_0(A), K_0(A))$ as follows:
Consider the crossed product $B = A \rtimes_{\alpha_a} \mathbb{Z}$ by the first generator α_a. The second generator $\alpha_b \in \mathrm{Aut}(A)$ extends to $\tilde{\alpha}_b \in \mathrm{Aut}(B)$ by letting $\tilde{\alpha}_b(\lambda^{\alpha_a}) = \check{u}\lambda^{\alpha_a}$, where $\check{u} = u(b, a)u(a, b)^*$.
Let $\tilde{\eta}_0 : \overline{\mathrm{Inn}}(B) \to \mathrm{OrderExt}(K_1(B), K_0(B))$ be the homomorphism introduced by A. Kishimoto and A. Kumjian.
Define $c(\alpha, u) = \tilde{\eta}_0(\tilde{\alpha}_b) \in \mathrm{OrderExt}(K_1(B), K_0(B))$, which can be identified with $\mathrm{OrderExt}(K_0(A), K_0(A))$.

Theorem (M 2010, Y. Sato and M)

Let A be as before. Let (α, u) and (β, v) be strongly outer cocycle actions of $\mathbb{Z}^2$ such that $\alpha_g, \beta_g \in \overline{\mathrm{Inn}}(A)$.
If $c(\alpha, u) = c(\beta, v)$, then (α, u) and (β, v) are cocycle conjugate.

Cocycle actions of $\mathbb{Z}^2$ on UHF algebras

When A is a UHF algebra, we have

$$\mathrm{OrderExt}(K_0(A), K_0(A)) \cong \mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A)).$$

Corollary (T. Katsura and M 2008, Y. Sato and M)

Let A be a UHF algebra. There exists a natural bijective correspondence between the following two sets.

1. *Cocycle conjugacy classes of strongly outer cocycle actions of $\mathbb{Z}^2$ on A.*
2. $\mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A))$.

Moreover, genuine actions correspond to

$$\{r \in \mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A)) \mid r([1]) = 0\}.$$

Cocycle actions of $\mathbb{Z}^2$ on $\mathcal{Z}$

Let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be a cocycle action.
As before, put $\check{u} = u(b,a)u(a,b)^*$.
The following theorem says that the de la Harpe-Skandalis determinant $\Delta_\tau(\check{u}) \in \mathbb{R}/\mathbb{Z}$ is the complete invariant of (α, u).

Theorem (Y. Sato and M)

Let $(\alpha, u), (\beta, v) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be strongly outer cocycle actions. Then they are cocycle conjugate if and only if $\Delta_\tau(\check{u}) = \Delta_\tau(\check{v})$.

The proof uses the same idea as $\mathbb{Z}$-actions:

- (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes \mathcal{Z}$.
- We have already classified $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes B$ with B being a UHF algebra.
- Some extra effort gives the conclusion.

$\mathbb{Z}^N$-actions on UHF algebras of infinite type

A UHF algebra A is said to be of infinite type if $A \otimes A \cong A$.

Theorem (M)

Let A be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property.

Theorem (M)

Let A be a UHF algebra of infinite type. Then, all strongly outer actions of $\mathbb{Z}^N$ on A are mutually cocycle conjugate to each other.

Actions of the Klein bottle group (1/2)

We call $\Gamma = \langle a, b \mid bab^{-1} = a^{-1} \rangle$ the Klein bottle group.

Theorem (Y. Sato and M)

Let A be a UHF algebra and let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action. Then for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\alpha_a(e_n) - e_n \to 0, \quad \alpha_a(f_n) - f_n \to 0,$$
$$\sum_{i=0}^{m-1} \alpha_b^i(e_n) + \sum_{j=0}^{m} \alpha_b^j(f_n) \to 1.$$

Theorem (Y. Sato and M)

All strongly outer cocycle actions of Γ on a UHF algebra are mutually cocycle conjugate.

Actions of the Klein bottle group (2/2)

We sketch the proof.
Letting $\check{u} = u(b,a)u(a^{-1},b)^*$, we have $\alpha_b \circ \alpha_a = \mathrm{Ad}\,\check{u} \circ \alpha_a \circ \alpha_b$, and α_b extends to $\tilde{\alpha}_b \in \mathrm{Aut}(A \rtimes_{\alpha_a} \mathbb{Z})$ by $\tilde{\alpha}_b(\lambda^{\alpha_a}) = \check{u}\lambda^{\alpha_a *}$.

Suppose that we are given $(\alpha, u), (\beta, v) : \Gamma \curvearrowright A$.
By Kishimoto's theorem, $\exists \theta \in \mathrm{Aut}(A), w \in U(A)$ such that

$$\theta \circ \beta_a \circ \theta^{-1} = \mathrm{Ad}\,w \circ \alpha_a.$$

θ extends to the isomorphism $\tilde{\theta} : A \rtimes_{\beta_a} \mathbb{Z} \to A \rtimes_{\alpha_a} \mathbb{Z}$ by letting $\tilde{\theta}(\lambda^{\beta_a}) = w\lambda^{\alpha_a}$. By modifying θ in a 'suitable' way, we can make

$$\tilde{\theta} \circ \tilde{\beta}_b \circ \tilde{\theta}^{-1} \circ \tilde{\alpha}_b^{-1} \in \mathrm{Aut}(A \rtimes_{\alpha_a} \mathbb{Z})$$

asymptotically inner. Then the Evans-Kishimoto intertwining argument works and yields the conclusion.

Open problems

- Show the uniqueness of strongly outer cocycle actions of the Klein bottle group on $\mathcal{Z}$ (work in progress).
- Classify strongly outer actions of $\mathbb{Z}^N$ on a general UHF algebra when $N \geq 3$.
- Show the uniqueness of strongly outer actions of poly-$\mathbb{Z}$ groups on a UHF algebra of infinite type.
- Show the uniqueness of strongly outer $\mathbb{Z}^N$-actions on the Jiang-Su algebra $\mathcal{Z}$ when $N \geq 3$.
- Classify strongly outer $\mathbb{Z}^2$-actions on a unital simple AF algebra (as much as possible).
- Classify strongly outer $\mathbb{Z}$-actions on 'classifiable' C^*-algebras (as much as possible).

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