Homology of étale groupoids on Cantor sets

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Outline

topological dynamical systems on Cantor sets
(group actions, equivalence relations, ...)

$\longrightarrow$ étale groupoids G

$$\longrightarrow \begin{cases} \text{homology group } H_n(G),\ n \geq 0 \text{ (Crainic-Moerdijk 2000)} \\ K\text{-group } K_i(C_r^*(G)),\ i = 0, 1 \\ \text{topological full group } [[G]] \end{cases}$$

Interaction between them?

- $\exists$ a homomorphism $\Phi_0 : H_0(G) \to K_0(C_r^*(G))$.
- $\exists$ a homomorphism $I : [[G]] \to H_1(G)$, called the index map.
- For certain G, $\exists$ a homomorphism $\Phi_1 : H_1(G) \to K_1(C_r^*(G))$.

Étale groupoids

A groupoid G is a 'group-like' algebraic object, in which the product may not be defined for all pairs in G.

- $g \in G$ is thought of as an arrow $\bullet \xleftarrow{g} \bullet$.
- $r : g \mapsto gg^{-1}$ is called the range map.
- $s : g \mapsto g^{-1}g$ is called the source map.
- $G^{(0)} = r(G) = s(G) \subset G$ is called the unit space.

G is an étale groupoid if G is equipped with a locally compact Hausdorff topology compatible with the groupoid structure and the range (or source) map is a local homeomorphism.

G is said to be principal if any g satisfying $r(g) = s(g)$ belongs to $G^{(0)}$. A principal groupoid G is identified with the equivalence relation $\{(r(g), s(g)) \in G^{(0)} \times G^{(0)} \mid g \in G\}$.

Examples of étale groupoids (1/3)

Let $\varphi : \Gamma \curvearrowright X$ be an action of a discrete group Γ on a locally compact Hausdorff space X by homeomorphisms.
$G_\varphi = \Gamma \times X$ is an étale groupoid with

$$(\gamma', x') \cdot (\gamma, x) = (\gamma'\gamma, x) \quad \text{if} \quad x' = \varphi^\gamma(x),$$

$$(\gamma, x)^{-1} = (\gamma^{-1}, \varphi^\gamma(x)).$$

G_φ is called the transformation groupoid.
The unit space $G_\varphi^{(0)} = \{e\} \times X$ is identified with X.

G_φ is principal if and only if φ is free, and in this case G_φ is identified with the étale equivalence relation on X induced by φ.

The groupoid C^*-algebra $C_r^*(G_\varphi)$ is canonically isomorphic to the crossed product C^*-algebra $C_0(X) \rtimes_r \Gamma$.

Examples of étale groupoids (2/3)

Let G be an étale groupoid whose unit space is a Cantor set.
G is called an elementary groupoid if G is principal and compact.
If G is elementary, then every G-orbit is finite and
$\exists$ clopen set $U \subset G^{(0)}$ which meets every G-orbit exactly once.

We say that G is an AF groupoid (or AF equivalence relation) if it can be written as an increasing union of elementary subgroupoids.

Let A be a finite set. The equivalence relation

$$\{(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \mid \exists n \in \mathbb{N} \quad \forall k \geq n \quad x_k = y_k\}$$

becomes an AF groupoid with a natural topology.

For an AF groupoid G, $C^*_r(G)$ is known to be an AF C^*-algebra, i.e. $C^*_r(G)$ is an inductive limit of finite dimensional C^*-algebras.

Examples of étale groupoids (3/3)

Let $\sigma : X \to X$ be a one-sided SFT.

$$G = \{(x, n, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \in \mathbb{N} \; n = k-l \; \sigma^k(x) = \sigma^l(y)\}$$

is an étale groupoid with

$$(x', n', y') \cdot (x, n, y) = (x', n'+n, y) \quad \text{if} \quad y' = x,$$

$$(x, n, y)^{-1} = (y, -n, x).$$

We call G an SFT groupoid. G is not principal because σ has periodic points.

The map $\rho : G \ni (x, n, y) \mapsto n \in \mathbb{Z}$ is a homomorphism and its kernel $K = \operatorname{Ker} \rho = \{(x, 0, y)\}$ is known to be an AF subgroupoid.

$C^*_r(G)$ is called the Cuntz-Krieger algebra.

Homology of étale groupoids (1/2)

For any local homeomorphism $\pi : X \to Y$ between locally compact Hausdorff spaces, one can define $\pi_* : C_c(X, \mathbb{Z}) \to C_c(Y, \mathbb{Z})$ by

$$\pi_*(f)(y) = \sum_{x \in \pi^{-1}(y)} f(y).$$

For a (totally disconnected) étale groupoid G, let $G^{(n)}$ be the space of composable strings of n elements in G.
For each n and $i = 0, 1, 2, \ldots, n$, define $d_i : G^{(n)} \to G^{(n-1)}$ by

$$d_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, g_3, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & 1 \leq i \leq n-1 \\ (g_1, g_2, \ldots, g_{n-1}) & i = n. \end{cases}$$

It is easily checked that d_i is a local homeomorphism.

Homology of étale groupoids (2/2)

The homomorphisms

$$\delta_1 = s_* - r_*, \quad \delta_n = \sum_{i=0}^{n} (-1)^i d_{i*} \quad (n \geq 2)$$

yield the following chain complex:

$$0 \xleftarrow{\delta_0} C_c(G^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(G^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(G^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \dots .$$

We call $H_n(G) = \operatorname{Ker} \delta_n / \operatorname{Im} \delta_{n+1}$ the homology groups of G with constant coefficients $\mathbb{Z}$.
For a clopen subset $Y \subset G^{(0)}$, we can define the reduction $G|Y$ by

$$G|Y = \{g \in G \mid r(g) \in Y, \ s(g) \in Y\}.$$

Suppose that Y is G-full, i.e. $G^{(0)} = r(s^{-1}(Y))$. Then G and $G|Y$ are homologically similar, and hence have the isomorphic homology.

$H_0(G)$ and $K_0(C_r^*(G))$

- $C_c(G,\mathbb{Z})$ is generated by 1_U's for compact open subsets $U \subset G$ such that both $r|U$ and $s|U$ are one-to-one.
- For such U, $\delta_1(1_U) = 1_{s(U)} - 1_{r(U)}$, and so $H_0(G) = C(G^{(0)},\mathbb{Z})/\operatorname{Im}\delta_1 = C(G^{(0)},\mathbb{Z})/\langle 1_{s(U)} - 1_{r(U)}\rangle$.
- On the one hand, 1_U is regarded as an element of $C_r^*(G)$.
- One has $(1_U)^* = 1_{U^{-1}}$, $(1_U)^* \cdot 1_U = 1_{s(U)}$, $1_U \cdot (1_U)^* = 1_{r(U)}$.
- Thus, the two projections $1_{s(U)}$ and $1_{r(U)}$ in $C_r^*(G)$ are Murray-von Neumann equivalent via the partial isometry 1_U, and hence $[1_{s(U)}] = [1_{r(U)}]$ in $K_0(C_r^*(G))$.

This observation implies that there exists a natural homomorphism $\Phi_0 : H_0(G) \to K_0(C_r^*(G))$. It is unknown if Φ_0 is injective or not (even for free minimal actions of $\mathbb{Z}^N$).

Examples of homology groups (1/3)

Let $\varphi : \Gamma \curvearrowright X$ be an action of a discrete group Γ on a Cantor set X and let $G_\varphi = \Gamma \times X$ be the associated étale groupoid.
Then $H_n(G_\varphi)$ is canonically isomorphic to the usual group homology $H_n(\Gamma, C(X, \mathbb{Z}))$ of Γ with coefficients $C(X, \mathbb{Z})$.
Especially, $H_0(G_\varphi)$ is isomorphic to the coinvariants

$$C(X, \mathbb{Z}) / \langle f - f \circ \varphi_\gamma \mid f \in C(X, \mathbb{Z}) \rangle.$$

When $\Gamma = \mathbb{Z}^N$ and φ is free, the Chern character induces

$$\bigoplus H_{2n+i}(G_\varphi) \otimes \mathbb{Q} \;\cong\; K_i(C_r^*(G_\varphi)) \otimes \mathbb{Q} \quad i = 0, 1.$$

When $N = 1, 2$, the isomorphisms above hold without $\otimes \mathbb{Q}$.
For $\varphi : \mathbb{Z}^N \curvearrowright X$ arising from certain aperiodic tiling spaces (e.g. substitution tiling, projection method tiling), computation of $H_n(G_\varphi)$ is known (Anderson-Putnam, Gähler-Hunton-Kellendonk).

Examples of homology groups (2/3)

Proposition

When G is an AF groupoid,

$$H_n(G) \cong \begin{cases} K_0(C_r^*(G)) & n = 0 \\ 0 & n \geq 1. \end{cases}$$

Proof.

$H_0(G) \cong K_0(C_r^*(G))$ is well-known (Elliott, Krieger).
Let $n \geq 1$. G is an increasing union of elementary subgroupoids G_k. Each G_k has a 'fundamental domain' $U_k \subset G_k^{(0)}$. Therefore $H_n(G_k) = H_n(G_k|U_k) = 0$, because $G_k|U_k$ is a trivial groupoid.
Hence we obtain $H_n(G) = 0$. □

In particular, one has $\bigoplus H_{2n+i}(G) \cong K_i(C_r^*(G))$.

Examples of homology groups (3/3)

Proposition

When G is the SFT groupoid associated with $A \in M_k(\mathbb{Z}_+)$,

$$H_n(G) \cong \begin{cases} \mathrm{Coker}(I_k - A^t) & n = 0 \\ \mathrm{Ker}(I_k - A^t) & n = 1 \\ 0 & n \geq 2. \end{cases}$$

Proof.

$\rho : G \ni (x, n, y) \mapsto n \in \mathbb{Z}$ is a homomorphism and $K = \mathrm{Ker}\,\rho$ is an AF subgroupoid. $H_0(K)$ is isomorphic to the dimension group of A^t. K is isomorphic to a reduction of the skew product $G \times_\rho \mathbb{Z}$. Then the Lindon-Hochschild-Serre spectral sequence applies. □

Again one has $\bigoplus H_{2n+i}(G) \cong K_i(C_r^*(G))$.

Summary

	H_0	H_1	H_2	H_3
AF	torsion free	0	0	0
minimal $\mathbb{Z}$-action	torsion free	$\mathbb{Z}$	0	0
Penrose	$\mathbb{Z}^8$	$\mathbb{Z}^5$	$\mathbb{Z}$	0
chair	$\mathbb{Z}[1/2]^3$	$\mathbb{Z}[1/2]^2$	$\mathbb{Z}$	0
Ammann-Kramer	$\mathbb{Z}^{181} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{72} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}$
SFT	$\mathrm{Coker}(I_k - A^t)$	$\mathrm{Ker}(I_k - A^t)$	0	0

Topological full group

If $U \subset G$ is a compact open subset such that both $r|U$ and $s|U$ are bijections between U and $G^{(0)}$, then $\alpha = (r|U) \circ (s|U)^{-1}$ is a homeomorphism on $G^{(0)}$.
We let $[[G]]$ be the set of all such homeomorphisms and call it the topological full group of G.

When $\varphi : \Gamma \curvearrowright X$ is a free action on a Cantor set X, any $\alpha \in [[G_\varphi]]$ is of the form

$$\alpha(x) = \varphi^{c(x)}(x) \quad \forall x \in X = G^{(0)},$$

where $c : X \to \Gamma$ is a continuous map.

For $\alpha = (r|U) \circ (s|U)^{-1} \in [[G]]$, one has

$$(1_U)^* \cdot 1_U = 1_U \cdot (1_U)^* = 1_{G^{(0)}},$$

$$1_U \cdot f \cdot (1_U)^* = f \circ \alpha^{-1} \quad \forall f \in C(G^{(0)}),$$

i.e. $1_U \in C_r^*(G)$ is a unitary normalizing $C(G^{(0)})$.

[[G]] for AF groupoids and $\mathbb{Z}$-actions

For an AF groupoid G, the following are known (Krieger).

- $[[G]]$ is written as an increasing union of finite direct sums of symmetric groups (in particular, locally finite).
- $[[G]]$ is a complete invariant for the isomorphism class of G.

For minimal actions $\varphi_i : \mathbb{Z} \curvearrowright X_i$ on Cantor sets, the following are equivalent (Giordano-Putnam-Skau, Boyle-Tomiyama).

1. $[[G_{\varphi_1}]]$ is isomorphic to $[[G_{\varphi_2}]]$.
2. G_{φ_1} is isomorphic to G_{φ_2}.
3. φ_1 is conjugate to φ_2 or φ_2^{-1}.

For a minimal action $\varphi : \mathbb{Z} \curvearrowright X$ on a Cantor set,

- $D([[G_\varphi]])$ is simple and $[[G_\varphi]]/D([[G_\varphi]]) \cong \mathbb{Z} \oplus (H_0(G_\varphi) \otimes \mathbb{Z}_2)$.
- $D([[G_\varphi]])$ is finitely generated if and only if φ is expansive.

Almost finite groupoid

Definition

An étale groupoid G on a Cantor set is said to be almost finite if for any compact subset $C \subset G$ and $\varepsilon > 0$ there exists an elementary subgroupoid $K \subset G$ such that

$$\frac{|CKx \setminus Kx|}{|Kx|} < \varepsilon \quad \forall x \in G^{(0)}.$$

This may remind us of the Følner condition for amenable groups, but there is no direct relationship between them.
AF groupoids are clearly almost finite ($\because \ \forall C \ \exists K \ C \subset K$).

Lemma (M)

When $\varphi : \mathbb{Z}^N \curvearrowright X$ is free, G_φ is almost finite.

An almost finite groupoid admits an invariant probability measure on $G^{(0)}$, and so SFT groupoids are not almost finite.

$H_0(G)$ and $[[G]]$

Theorem (M)

Let G be an étale almost finite groupoid. For two G-full clopen subsets $U, V \subset G^{(0)}$, the following are equivalent.

1. $[1_U]$ *equals* $[1_V]$ *in* $H_0(G)$.
2. *There exists* $\alpha \in [[G]]$ *such that* $\alpha(U) = V$.
3. *There exists a unitary* $w \in C_r^*(G)$ *normalizing* $C(G^{(0)})$ *such that* $w1_U w^* = 1_V$.

Theorem (M)

Suppose that G is almost finite and minimal. For two clopen subsets $U, V \subset G^{(0)}$, the following are equivalent.

1. $\mu(U)$ *equals* $\mu(V)$ *for any invariant measure* $\mu \in M(G)$.
2. *There exists* $\alpha \in [G]$ *such that* $\alpha(U) = V$.

Index map

Let G be an étale essentially principal groupoid on a Cantor set. For $\alpha = (r|U) \circ (s|U)^{-1} \in [[G]]$,

$$\delta_1(1_U) = s_*(1_U) - r_*(1_U) = 1_{s(U)} - 1_{r(U)} = 0, \text{ i.e. } 1_U \in \operatorname{Ker} \delta_1.$$

We define the index map $I : [[G]] \to H_1(G)$ by $I(\alpha) = [1_U]$. It is easy to see that I is a homomorphism.

When $\varphi : \mathbb{Z} \curvearrowright X$ is minimal, the index map takes its values in $H_1(G_\varphi) \cong \mathbb{Z}$. For $\alpha \in [[G_\varphi]]$, we have

$$I(\alpha) = \int_{G^{(0)}} c(x)\, d\mu(x) \in \mathbb{Z},$$

where $\alpha(x) = \varphi^{c(x)}(x)$ and $\mu \in M(G)$ (Giordano-Putnam-Skau). In this case, $I(\alpha)$ is also understood as the Fredholm index of certain Fredholm operators.

$H_1(G)$ and $K_1(C_r^*(G))$

As mentioned before, for $\alpha = (r|U) \circ (s|U)^{-1} \in [[G]]$, 1_U can be thought of as a unitary of $C_r^*(G)$. Hence one can define a homomorphism $J : [[G]] \to K_1(C_r^*(G))$ by $J(\alpha) = [1_U]$.

Theorem (M)

If G is almost finite, then we have the following.

- *$I : [[G]] \to H_1(G)$ is surjective.*
- *Any $\alpha \in \operatorname{Ker} I$ is written as a product of four elements in $[[G]]$ of finite order. In particular, $\operatorname{Ker} I$ is contained in $\operatorname{Ker} J$.*

Corollary (M)

If G is almost finite, then there exists a homomorphism $\Phi_1 : H_1(G) \to K_1(C_r^*(G))$ such that $\Phi_1 \circ I = J$.

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