$\mathbb{Z}^N$-actions on UHF algebras of infinite type

Hiroki Matui
matui@math.s.chiba-u.ac.jp

Chiba University

June 26, 2010
Recent Developments in Operator Algebras
- on the occasion of the 77th birthday of Masamichi Takesaki -
University of Tokyo

Goal

Goal (too ambitious)

Classify outer actions of discrete amenable groups on simple classifiable C^*-algebras (Kirchberg algebras, UHF, AF, AT, AH algebras, the Jiang-Su alg. etc.) up to cocycle conjugacy.

Outer actions of finite groups on unital simple C^*-algebras do not always have the Rohlin property. In fact, if an action $\alpha : \Gamma \curvearrowright A$ of a finite group Γ on a unital simple C^*-algebra A has the Rohlin property, then $K_0(A)$ and $K_1(A)$ are completely cohomologically trivial as Γ-modules.

Goal (realistic)

Classify outer actions of $\mathbb{Z}^N$ (or "poly-infinite-cyclic" groups) on simple classifiable C^*-algebras up to cocycle conjugacy.

Cocycle conjugacy

Definition

Let $\alpha : \Gamma \curvearrowright A$ be an action of a countable discrete group Γ on a unital C^*-algebra A.
$(u_g)_{g\in\Gamma} \subset U(A)$ is called an α-cocycle if

$$u_g \alpha_g(u_h) = u_{gh} \quad \forall g, h \in \Gamma.$$

Two actions $\alpha, \beta : \Gamma \curvearrowright A$ are said to be cocycle conjugate, if

$$\exists \gamma \in \mathrm{Aut}(A), \quad \exists (u_g)_g \ \alpha\text{-cocycle}$$

$$\mathrm{Ad}\, u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1} \quad \forall g \in \Gamma.$$

Strong cocycle conjugacy further requires

$$\exists v_n \in U(A), \quad \|u_g - v_n \alpha_g(v_n^*)\| \to 0.$$

Strong outerness

An automorphism of the from $\mathrm{Ad}\, u$ is said to be inner.
An action $\alpha : \Gamma \curvearrowright A$ is said to be outer if α_g is not inner for every $g \in \Gamma \setminus \{e\}$.

Let $T(A)$ denote the set of tracial states and let π_τ be the GNS representation by $\tau \in T(A)$.
$\alpha \in \mathrm{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \mathrm{Ad}\, U$.
An action $\alpha : \Gamma \curvearrowright A$ is said to be strongly outer if α_g is not weakly inner for every $g \in \Gamma \setminus \{e\}$.

If $T(A) = \{\tau\}$, then

$$\alpha : \Gamma \curvearrowright A \text{ is strongly outer} \iff \bar{\alpha} : \Gamma \curvearrowright \pi_\tau(A)'' \text{ is outer.}$$

Main result

A UHF algebra A is said to be of infinite type if $A \otimes A \cong A$.

Theorem (M)

Let A be a UHF algebra of infinite type.
For $\alpha : \mathbb{Z}^N \curvearrowright A$, the following are equivalent.

1. *α is strongly outer.*
2. *α has the Rohlin property.*

Theorem (M)

Let A be a UHF algebra of infinite type. Any strongly outer actions of $\mathbb{Z}^N$ on A are mutually strongly cocycle conjugate to each other.

Group actions on AFD factors

- (A. Connes, V. F. R. Jones, A. Ocneanu 1980's)
  Actions of discrete amenable groups on the AFD II_1-factor.
- (C. E. Sutherland and M. Takesaki 1989)
  Actions of discrete amenable groups on AFD factors of type III_λ with $\lambda \neq 1$.
- (Y. Kawahigashi, C. E. Sutherland and M. Takesaki 1992)
  Actions of discrete abelian groups on the AFD factor of type III_1.
- (Y. Katayama, C. E. Sutherland and M. Takesaki 1998)
  Actions of discrete amenable groups on all AFD factors.
- (T. Masuda)
  A short proof, which is independent of types and is based on Evans-Kishimoto intertwining argument developed in C^*-algebras.

$\mathbb{Z}^N$-actions on simple C^*-algebras

- (A. Kishimoto 1995)
  Uniqueness of strongly outer $\mathbb{Z}$-actions on UHF algebras.
- (A. Kishimoto 1998, M 2010)
  Strongly outer $\mathbb{Z}$-actions on AT or AH algebras.
- (H. Nakamura 2000)
  Outer $\mathbb{Z}$-actions on Kirchberg algebras.
- (T. Katsura and M 2008)
  Strongly outer $\mathbb{Z}^2$-actions on UHF algebras.
- (M. Izumi and M 2010)
  Outer and locally KK-trivial $\mathbb{Z}^2$-actions on Kirchberg alg.
  Uniqueness of outer $\mathbb{Z}^N$-actions on $\mathcal{O}_2$, $\mathcal{O}_\infty$ etc.
- (Y. Sato, Y. Sato and M)
  Uniqueness of strongly outer $\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions on the Jiang-Su algebra $\mathcal{Z}$.

Rohlin property for $\mathbb{Z}^N$-actions (1)

Let $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{Z}^N$ be the canonical basis.
We let $A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ and $A_\infty = A^\infty \cap A'$.

Definition (Nakamura 1999)

$\alpha : \mathbb{Z}^N \curvearrowright A$ is said to have the Rohlin property, if for any $M \in \mathbb{N}$, there exist $R \in \mathbb{N}$, $m_r \in \mathbb{N}^N$ $(r = 1, 2, \ldots, R)$ and

$$\text{projections } e_g^{(r)} \in A_\infty \quad (r = 1, 2, \ldots, R,\ g \in \mathbb{Z}^N/m_r\mathbb{Z}^N)$$

such that

$$\sum_{r,g} e_g^{(r)} = 1, \quad \alpha_{\xi_i}(e_g^{(r)}) = e_{g+\xi_i}^{(r)} \quad (i = 1, 2, \ldots, N)$$

and each coordinate of m_r is not less than M.

Rohlin property for $\mathbb{Z}^N$-actions (2)

We can restate the definition of the Rohlin property as follows:

For any $i = 1, 2, \ldots, N$ and $M \in \mathbb{N}$, there exist $R \in \mathbb{N}$, $m_r \in \mathbb{N}$ $(r = 1, 2, \ldots, R)$ and

$$\text{projections } e_g^{(r)} \in A_\infty \quad (r = 1, 2, \ldots, R,\ g \in \mathbb{Z}/m_r\mathbb{Z})$$

such that

$$m_r \geq M, \quad \sum_{r,g} e_g^{(r)} = 1$$

and

$$\alpha_{\xi_i}(e_g^{(r)}) = e_{g+\xi_i}^{(r)}, \quad \alpha_{\xi_j}(e_g^{(r)}) = e_g^{(r)} \quad \forall j \neq i.$$

Thus,

$$\alpha : \mathbb{Z}^N \curvearrowright A \text{ has the Rohlin property}$$

$$\iff \exists \text{ Rohlin towers for } \alpha_{\xi_i} \text{ in } (A_\infty)^{\alpha_i'} \text{ for any } i = 1, 2, \ldots, N.$$

Asymptotically representable actions

An action $\alpha : \Gamma \curvearrowright A$ of a discrete group Γ is said to be approximately representable if there exists a family of unitaries $(v_g)_{g\in\Gamma}$ in A^∞ such that

$$v_g v_h = v_{gh}, \quad \alpha_g(v_h) = v_{ghg^{-1}}, \quad v_g a v_g^* = \alpha_g(a)$$

for any $g, h \in \Gamma$ and $a \in A$. Notice that

$$a \mapsto a, \quad \lambda_g^\alpha \mapsto v_g$$

induce a homomorphism from $A \rtimes_\alpha \Gamma$ to A^∞.

An action $\alpha : \Gamma \curvearrowright A$ of a discrete group Γ is said to be asymptotically representable if there exists a family of continuous paths of unitaries $(v_g(t))_{g\in\Gamma, t\in[0,\infty)}$ in A satisfying analogous properties.

Uniqueness of actions with the Rohlin property (1)

Proposition

Let A be a UHF algebra of infinite type and let $\alpha, \beta : \mathbb{Z}^N \curvearrowright A$ be two actions with the Rohlin property. Then α and β are cocycle conjugate.

The proof is by induction on N.
Let $\alpha' : \mathbb{Z}^{N-1} \curvearrowright A$ and $\beta' : \mathbb{Z}^{N-1} \curvearrowright A$ be the actions of $\mathbb{Z}^{N-1}$ generated by $\xi_1, \xi_2, \dots, \xi_{N-1}$.
By induction hypothesis, α' and β' are cocycle conjugate. In particular, the crossed products $A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$ and $A \rtimes_{\beta'} \mathbb{Z}^{N-1}$ are identified.
The automorphisms $\alpha_{\xi_N}, \beta_{\xi_N}$ of A extend to automorphisms $\tilde{\alpha}_{\xi_N}, \tilde{\beta}_{\xi_N}$ of the crossed products. We would like to compare $\tilde{\alpha}_{\xi_N}$ and $\tilde{\beta}_{\xi_N}$.

Uniqueness of actions with the Rohlin property (2)

(1) First, we observe that the crossed product $A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$ is a unital simple AT algebra with real rank zero.
(2) By using Kishimoto-Kumjian's theorem, we can show $\tilde{\alpha}_{\xi_N}$ is asymptotically inner, i.e
$\exists (u_t)_{t\in[0,\infty)}$ in $U(A \rtimes_{\alpha'} \mathbb{Z}^{N-1})$ such that $\tilde{\alpha}_{\xi_N} = \lim \operatorname{Ad} u_t$.
(3) By induction hypothesis, α' is asymptotically representable. Hence we may assume that $(u_t)_{t\in[0,\infty)}$ is in $U(A)$.
(4) The same is true for $\tilde{\beta}_{\xi_N}$, and so
$\exists (v_t)_{t\in[0,\infty)}$ in $U(A)$ such that $\tilde{\alpha}_{\xi_N} = \lim \operatorname{Ad} v_t \circ \tilde{\beta}_{\xi_N}$.
(5) α and β have the Rohlin property. Therefore $\tilde{\alpha}_{\xi_N}$ and $\tilde{\beta}_{\xi_N}$ have the Rohlin property (as single automorphisms) and the Rohlin projections can be taken from A.
(6) These items (together with a homotopy lemma) enables us to carry out the $\mathbb{Z}^{N-1}$-equivariant Evans-Kishimoto intertwining argument, which completes the proof of the proposition.

Rohlin type theorem (1)

Proposition

Let A be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property.

The proof is by induction on N. As before, let $\alpha' : \mathbb{Z}^{N-1} \curvearrowright A$ be the action of $\mathbb{Z}^{N-1}$ generated by $\xi_1, \xi_2, \dots, \xi_{N-1}$.
(1) Since $\bar{\alpha} : \mathbb{Z}^N \curvearrowright \pi_\tau(A)''$ is outer, for any $m \in \mathbb{N}$, one can find a Rohlin tower

$$e, \alpha_{\xi_N}(e), \dots, \alpha_{\xi_N}^{m-1}(e) \in (A_\infty)^{\alpha'}$$

such that the 'trace' of

$$1 - (e + \alpha_{\xi_N}(e) + \dots + \alpha_{\xi_N}^{m-1}(e))$$

is very small.

Rohlin type theorem (2)

(2) We would like to replace e to achieve $\alpha_{\xi_N}^m(e) = e$.
To this end it suffices to find a partial isometry $v \in (A_\infty)^{\alpha'}$ such that $v^*v = e$ and $vv^* = \alpha_{\xi_N}(e)$.
(3) By induction hypothesis, α' has the Rohlin property. Actions with the Rohlin property are already shown to be unique up to cocycle conjugacy. It follows that α' is approximately representable.
(4) In the same way as before, we can find a sequence $(u_n)_n$ of unitaries of A such that $\tilde{\alpha}_{\xi_N}(x) = \lim u_n x u_n^* \ \forall x \in A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$.
(5) Then we obtain v and achieve $\alpha_{\xi_N}^m(e) = e$.
(6) In the same way as the single automorphism case, we can construct two Rohlin towers in $(A_\infty)^{\alpha'}$ whose sum is equal to 1. The same is true for other generators, and so we can conclude that $\alpha : \mathbb{Z}^N \curvearrowright A$ has the Rohlin property.

Conclusion (1)

Proposition

Let A be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. For any α-cocycle $(u_g)_{g\in\mathbb{Z}^N}$ in A and $\varepsilon > 0$, there exists a unitary $v \in A$ such that

$$\|u_{\xi_i} - v\alpha_{\xi_i}(v^*)\| < \varepsilon \quad \forall i = 1, 2, \dots, N.$$

(1) Consider two homomorphisms $\varphi, \psi : C^*(\mathbb{Z}^N) \to A \rtimes_\alpha \mathbb{Z}^N$ defined by

$$\varphi(\lambda_g) = \lambda^\alpha_g, \quad \psi(\lambda_g) = u_g\lambda^\alpha_g.$$

(2) Show $KK(\varphi) = KK(\psi)$ and $\tau \circ \varphi = \tau \circ \psi$.
(3) From these data, one can conclude that φ and ψ are approximately unitarily equivalent.

Conclusion (2)

(4) α is already known to be approximately representable, and so there exists $(v_n)_n$ in $U(A)$ such that $\psi = \lim \mathrm{Ad}\, v_n \circ \varphi$.
(5) This implies $\|u_g - v_n \alpha_g(v_n^*)\| \to 0$ for any $g \in \mathbb{Z}^N$.

Combining the three propositions above, we obtain the main result, i.e. any strongly outer actions of $\mathbb{Z}^N$ on a UHF algebra of infinite type are strongly cocycle conjugate to each other.

Remark

In general, strong cocycle conjugacy is strictly stronger than cocycle conjugacy.

Uniqueness of asymptotically representable actions

Let A be a unital simple AH algebra with real rank zero, slow dimension growth and finitely many extremal tracial states.

Theorem (M)

For an approximately representable action $\alpha : \mathbb{Z}^N \curvearrowright A$, the following are equivalent.

1. *α is strongly outer.*
2. *α has the Rohlin property.*

Theorem (M)

Let $\alpha, \beta : \mathbb{Z}^N \curvearrowright A$ be strongly outer, asymptotically representable actions. Then they are cocycle conjugate.

$\mathbb{Z}^2$-actions on UHF algebras

Let A be a UHF algebra. For each prime number p, we put

$$\zeta(p) = \sup\{k \mid [1] \text{ is divisible by } p^k \text{ in } K_0(A)\}.$$

For a $\mathbb{Z}^2$-action α on A, an invariant $[\alpha]$ is defined as an element in

$$\prod_{p \in P(A)} \mathbb{Z}/p^{\zeta(p)}\mathbb{Z} \quad \cong \quad \pi_1(\mathrm{Aut}(A)),$$

where $P(A) = \{p \mid 1 \leq \zeta(p) < \infty\}$.

Theorem (T. Katsura and M 2008)

Let A be a UHF algebra and let $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ be strongly outer actions. Then, α and β are strongly cocycle conjugate if and only if $[\alpha] = [\beta]$.

Strongly self-absorbing C^*-algebras

A C^*-algebra $A \neq \mathbb{C}$ is said to be strongly self-absorbing if there exists an isomorphism $\mu : A \to A \otimes A$ such that μ is approximately unitarily equivalent to $a \mapsto a \otimes 1$.

The known examples are UHF_∞, the Jiang-Su algebra $\mathcal{Z}$, $\mathcal{O}_2$, $\mathcal{O}_\infty$ and $\mathcal{O}_\infty \otimes \mathrm{UHF}_\infty$. Uniqueness of $\mathbb{Z}^N$-actions on these algebras has been obtained except for $\mathbb{Z}^N \curvearrowright \mathcal{Z}$ with $N \geq 3$.

algebras	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^N$
UHF_∞	Kishimoto '95	Katsura-M '08	M
$\mathcal{Z}$	Sato	Sato-M	?
Kirchberg	Nakamura '00	Izumi-M '10	

Why unique? — the key is $\pi_n(\mathrm{Aut}(A)) = 0$ for every $n \geq 0$.

Open problems

- Show that strongly outer actions of $\mathbb{Z}^N$ on a (general) UHF algebra have the Rohlin property when $N \geq 3$.
- Classify strongly outer actions of $\mathbb{Z}^N$ on a (general) UHF algebra when $N \geq 3$.
- Show the uniqueness of strongly outer $\mathbb{Z}^N$-actions on the Jiang-Su algebra $\mathcal{Z}$ when $N \geq 3$.
- Classify strongly outer $\mathbb{Z}^2$-actions on a unital simple AF algebra (as much as possible).
- Classify strongly outer $\mathbb{Z}$-actions on a unital simple $\mathcal{Z}$-stable C^*-algebra A (as much as possible).