カントール極小系の軌道同型による分類

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joint work with T. Giordano, I. F. Putnam and C. F. Skau

Preliminaries

$$\begin{aligned} X &= \text{Cantor set} \\ &= \text{compact, metrizable, totally disconnected, perfect} \\ &\cong \{a,b\}^{\mathbb{N}} \end{aligned}$$

For an equivalence relation $R \subset X \times X$,

$$R[x] = \{y \in X \mid (x,y) \in R\}$$

is called the R-orbit of x.
When $R[x]$ is dense in X for all $x \in X$, R is said to be minimal.
For an action $\varphi : G \curvearrowright X$ by homeomorphisms, we put

$$R_\varphi = \{(x, \varphi^g(x)) \mid x \in X,\ g \in G\}.$$

φ is said to be minimal if R_φ is minimal.
φ is said to be free if $\{g \in G \mid \varphi^g(x) = x\} = \{e\}$ for all $x \in X$.

Orbit equivalence

Definition

Let $R_i \subset X_i \times X_i$ $(i = 1, 2)$ be equivalence relations.
R_1 and R_2 are said to be orbit equivalent if there exists a homeomorphism $h : X_1 \to X_2$ such that $(h \times h)(R_1) = R_2$.

Let R be an equivalence relation on X.
A probability measure μ on X is said to be R-invariant if μ is γ-invariant for all $\gamma \in \mathrm{Homeo}(X)$ satisfying $(x, \gamma(x)) \in R$, $\forall x \in X$.

Put $M(R) = \{R\text{-invariant measures}\}$.

R is said to be uniquely ergodic if $M(R)$ is a singleton.

Classification

Theorem (Giordano-M-Putnam-Skau)

Let $\varphi_i : G_i \curvearrowright X_i$ $(i = 1, 2)$ be minimal actions of finitely generated abelian groups on Cantor sets. Then the following are equivalent.

1. R_{φ_1} *and* R_{φ_2} *are orbit equivalent.*
2. *There exists a homeomorphism* $h : X_1 \to X_2$ *such that* $h_*(M(R_{\varphi_1})) = M(R_{\varphi_2})$.

Corollary

When $M(R_{\varphi_i}) = \{\mu_i\}$ $(i = 1, 2)$, *the following are equivalent.*

1. R_{φ_1} *and* R_{φ_2} *are orbit equivalent.*
2. $\{\mu_1(U) \mid U \subset X_1 \text{ clopen subset}\}$ $= \{\mu_2(U) \mid U \subset X_2 \text{ clopen subset}\}$

Background

measurable dynamics (or measurable equivalence relations)
$\rightsquigarrow$ von Neumann algebras

Theorem (Krieger, Connes-Feldman-Weiss, etc)

Let $\varphi_i : G_i \curvearrowright (X_i, \mu_i)$ $(i = 1, 2)$ be essentially free ergodic actions of countable discrete amenable groups on standard probability spaces. Then the following are equivalent.

1. R_{φ_1} *and* R_{φ_2} *are measurably orbit equivalent.*
2. $\mathrm{vN}(R_{\varphi_1})$ *is isomorphic to* $\mathrm{vN}(R_{\varphi_2})$.

(minimal) topological dynamics (or equivalence relations)
$\rightsquigarrow$ (simple) C^*-algebras

- Classification of these C^*-algebras
- Relationship between dynamical systems and C^*-algebras

Cantor minimal $\mathbb{Z}$-systems

Theorem (Giordano-Putnam-Skau 1995)

Let $\varphi_i : \mathbb{Z} \curvearrowright X_i$ $(i = 1, 2)$ be minimal actions on Cantor sets. Then the following are equivalent.

1. (X_1, φ_1) *and* (X_2, φ_2) *are strongly orbit equivalent.*
2. $K^0(X_1, \varphi_1)$ *is isomorphic to* $K^0(X_2, \varphi_2)$*.*
3. $C^*(X_1, \varphi_1)$ *is isomorphic to* $C^*(X_2, \varphi_2)$*.*

Theorem (Giordano-Putnam-Skau 1995)

Let $\varphi_i : \mathbb{Z} \curvearrowright X_i$ $(i = 1, 2)$ be minimal actions on Cantor sets. Then the following are equivalent.

1. R_{φ_1} *and* R_{φ_2} *are orbit equivalent.*
2. *There exists a homeomorphism* $h : X_1 \to X_2$ *such that* $h_*(M(R_{\varphi_1})) = M(R_{\varphi_2})$*.*

Étale equivalence relations

Definition

An equivalence relation $R \subset X \times X$ with a topology is said to be étale if R is an r-discrete groupoid in this topology.
(In particular, $R \ni (x, y) \mapsto x \in X$ is a local homeomorphism.)

- R is rarely étale with the relative topology from $X \times X$, except in the special case that R itself is compact.
- There may exist a lot of choices of a topology by which R is étale.
- For a free action $\varphi : G \curvearrowright X$, R_φ is étale with the topology obtained by transferring the product topology on $X \times G$.

AF relations

Lemma

Let $R \subset X \times X$ be a compact étale equivalence relation on a compact metrizable totally disconnected space X. Then one has the following.

- *The topology on R is equal to the induced topology from $X \times X$.*
- $\sup\{\#R[x] \mid x \in X\}$ *is finite.*

Definition

An étale equivalence relation $R \subset X \times X$ on a compact metrizable totally disconnected space X is called an AF relation
if there exists an increasing sequence of compact open subrelations $R_1 \subset R_2 \subset \cdots \subset R$ such that $R = \bigcup_{n=1}^{\infty} R_n$.

Classification of AF relations

Theorem (Giordano-Putnam-Skau 1995)

Let $R_i \subset X_i \times X_i$ $(i = 1, 2)$ be minimal AF relations on Cantor sets. Then the following are equivalent.

1. R_1 *and* R_2 *are orbit equivalent.*
2. *There exists a homeomorphism* $h : X_1 \to X_2$ *such that* $h_*(M(R_1)) = M(R_2)$.
3. $(D(R_1), D(R_1)^+, [1]) \cong (D(R_2), D(R_2)^+, [1])$.

$$D(R) = C(X, \mathbb{Z}) / \{f \mid \mu(f) = 0 \ \forall \mu \in M(R)\}$$

$$D(R)^+ = \{[f] \in D(R) \mid f \geq 0\}$$

When R is a minimal AF relation, the triple $(D(R), D(R)^+, [1])$ is a unital simple dimension group with no infinitesimal elements.

Dimension groups

(D, D^+, u) satisfies the following.

- D is a countable torsion free abelian group and $D \ncong \mathbb{Z}$.
- D^+ is a positive cone,
  i.e. $D^+ + D^+ \subset D^+$, $D^+ \cap (-D^+) = \{0\}$ and
  $D^+ - D^+ = D$.
- $u \in D^+ \setminus \{0\}$.
- $\forall a_i \leq b_j$ $(i, j = 1, 2)$ in D, $\exists c \in D$ such that $a_i \leq c \leq b_j$.
- $\forall a, b \in D^+ \setminus \{0\}$ $\exists n \in \mathbb{N}$ s.t. $a \leq nb$.
- For any $a \in D$, there exists a homomorphism $\rho : D \to \mathbb{R}$ such that $\rho(D^+) \subset [0, \infty)$ and $\rho(a) \neq 0$.

Affability

Definition

We say that an equivalence relation R is affable if it can be given a topology making it an AF relation. This is the same as to say that R is orbit equivalent to an AF relation.

Theorem (Giordano-M-Putnam-Skau)

When $\varphi : \mathbb{Z}^N \curvearrowright X$ is a free minimal action on a Cantor set X, R_φ is affable.

It is known that

$$\begin{aligned} &\{\text{o.e. classes of minimal AF relations}\} \\ &= \{\text{o.e. classes of minimal } \mathbb{Z}\text{-actions}\} \\ &\supset \{\text{o.e. classes of minimal } \mathbb{Z}^N\text{-actions}\}. \end{aligned}$$

We do not know if they agree or not.

Example (1)

$$X = \{0,1\}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$$

$\varphi =$ addition by $(1,0,0,0,\dots)$ with carry-over

e.g. $\varphi(1,1,1,1,\dots) = (0,0,0,0,\dots)$.

$\rightsquigarrow (X,\varphi)$ Cantor minimal $\mathbb{Z}$-system (called the adding machine)

$$M(R_\varphi) = \{\mu\} \text{ i.e. uniquely ergodic}$$

$$\mu(\{\text{clopen sets}\}) = \mathbb{Z}[1/2] \cap [0,1]$$

$$(D, D^+, u) = (\mathbb{Z}[1/2], \mathbb{Z}[1/2]^+, 1)$$

Example (2)

$$X = \{0,1\}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$$

$$R_n = \{(x,y) \in X \times X \mid x_k = y_k \ \forall k \geq n\}$$

$\rightsquigarrow R_1 \subset R_2 \subset R_3 \subset \dots$

$\rightsquigarrow R = \bigcup_n R_n$ is a minimal AF relation

$M(R) = \{\mu\}$ i.e. uniquely ergodic

$\mu(\{\text{clopen sets}\}) = \mathbb{Z}[1/2] \cap [0,1]$

$(D, D^+, u) = (\mathbb{Z}[1/2], \mathbb{Z}[1/2]^+, 1)$

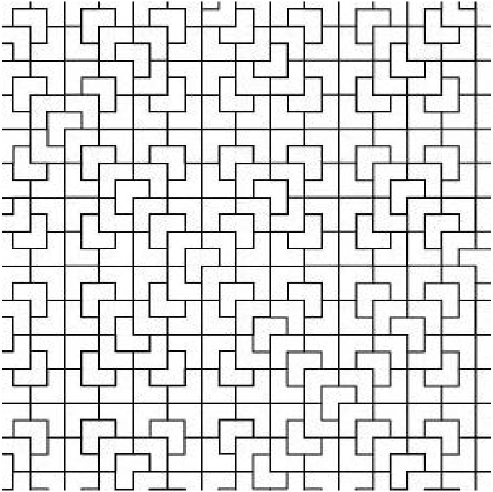
$R_\varphi[(0,0,0,\dots)] = R[(0,0,0,\dots)] \cup R[(1,1,1,\dots)]$

The identity map induces ‘almost’ orbit equivalence.

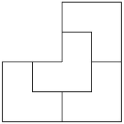
Tiling spaces

Let $\mathcal{C}$ be a finite collection of non-empty polyhedra in $\mathbb{R}^N$.
For $t \in \mathcal{C}$ and $p \in \mathbb{R}^N$, $t + p$ is called a tile. A collection T of tiles is called a tiling if the elements of T cover $\mathbb{R}^N$ with pairwise disjoint interiors.
We equip the set of tilings with a topology as follows:
Two tilings T and T' are close if there exist a small $\varepsilon \in \mathbb{R}^N$ and a large $R > 0$ such that $T + \varepsilon$ and T' agree on $B(0, R)$.
We obtain a topological space consisting of tilings and an action of $\mathbb{R}^N$ on it by translation.
Let T_0 be an aperiodic and repetitive tiling which satisfies the finite pattern condition.
Let Ω be the orbit closure of T_0, namely $\Omega = \overline{\{T_0 + p \mid p \in \mathbb{R}^N\}}$.
Then, it is known that Ω is compact and metrizable.
In addition, the natural $\mathbb{R}^N$ action φ on Ω is free and minimal.
Furthermore, we can find a Cantor transversal $X \subset \Omega$ and an induced equivalence relation R on X.

Example (3)



Chair tiling



$$(D, D^+, u) = (\mathbb{Z}[1/2], \mathbb{Z}[1/2]^+, 1)$$

Example (4)

Let θ be the Fibonacci substitution, i.e. $\theta(a) = ab$, $\theta(b) = a$.

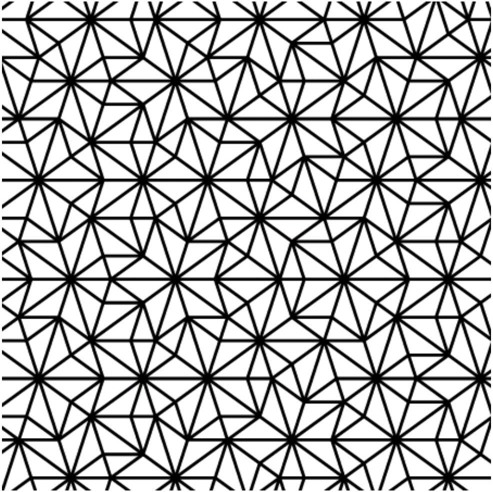
$$\begin{aligned} \theta(a) &= ab \\ \theta^2(a) &= aba \\ \theta^3(a) &= abaab \\ \theta^4(a) &= abaababa \\ \theta^5(a) &= abaababaabaab \\ &\vdots \end{aligned}$$

$\rightsquigarrow$ $X \subset \{a,b\}^{\mathbb{Z}}$ shift invariant closed subset, φ shift on X
$\rightsquigarrow$ (X,φ) Cantor minimal $\mathbb{Z}$-system

$$M(R_\varphi) = \{\nu\} \text{ i.e. uniquely ergodic}$$

$$\nu(\{\text{clopen sets}\}) = (\mathbb{Z} + \mathbb{Z}\lambda) \cap [0,1], \quad \lambda = \frac{1+\sqrt{5}}{2}$$

Example (5)



Penrose tiling

uniquely ergodic

$\nu(\{\text{clopen sets}\})$
$= (\mathbb{Z} + \mathbb{Z}\lambda) \cap [0,1]$,
where $\lambda = \frac{1+\sqrt{5}}{2}$

Absorption theorem (1)

Let R be an étale equivalence relation on a Cantor set X.

- For $Y \subset X$, we let $R|Y$ denote $R \cap (Y \times Y)$.
- We say that a closed subset Y is R-étale
  if $R|Y$ is étale with the induced topology from R.
- If R is AF and Y is R-étale, then $R|Y$ is again AF.
- We say that a closed subset Y is R-thin
  if $\mu(Y) = 0$ for all $\mu \in M(R)$.

Absorption theorem (2)

Theorem (M)

Let R be a minimal AF relation on a Cantor set X and let $Y \subset X$ be a closed subset which is R-étale and R-thin. Suppose that Q is an AF relation on Y such that $R|Y$ is an open subset of Q and $R|Y \hookrightarrow Q$ is continuous.
Then there exists a homeomorphism $h : X \to X$ such that

1. $(h \times h)(R \vee Q) = R$ *(thus, $R \vee Q$ is affable).*
2. $h(Y)$ *is R-étale and R-thin.*
3. $h|Y \times h|Y$ *is a homeomorphism from Q to $R|h(Y)$.*

Strategy

For a given $\varphi : \mathbb{Z}^N \curvearrowright X$, we find an AF subrelation $R \subset R_\varphi$, a closed subset $Y \subset X$ and another AF relation Q on Y so that $R_\varphi = R \vee Q$. Then apply the theorem above.

Construction of AF subrelations (1)

Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be a free minimal action.
We want to find a 'large' AF subrelation

$$R = \bigcup_n R_n, \quad R_1 \subset R_2 \subset \cdots \subset R_\varphi$$

in R_φ. Thus, we must find a 'large' compact subrelation R_n in R_φ.
For each $x \in X$, $R_\varphi[x]$ is partitioned into countably many R_n-orbits:

$$R_\varphi[x] = \bigcup_{i=1}^{\infty} R_n[x_i].$$

There exists a canonical bijection between $R_\varphi[x]$ and $\mathbb{Z}^N$.
So, we are thinking of a partition of $\mathbb{Z}^N$ into finite subsets.
$\rightsquigarrow$ tiling of $\mathbb{R}^N$

Construction of AF subrelations (2)

Take a (small) clopen subset U of X. Fix $x \in X$.
Consider

$$P = \{p \in \mathbb{Z}^N \mid \varphi^p(x) \in U\}.$$

Since φ is free, $\exists M_0 > 0$ such that P is M_0-separated, i.e.

$$p \neq q \in P \Rightarrow d(p,q) \geq M_0.$$

Since φ is minimal, $\exists M_1 > 0$ such that P is M_1-syndetic, i.e.

$$\bigcup_{p \in P} B(p, M_1) = \mathbb{R}^N.$$

Such a discrete subset of $\mathbb{R}^N$ is called a Delone set.

Construction of AF subrelations (3)

For each $p \in P$, we define

$$V(p) = \{q \in \mathbb{R}^N \mid d(q,p) = d(q,P)\}$$

and call it the Voronoi domain.
$T = \{V(p) \mid p \in P\}$ is called the Voronoi tessellation.
In such a way, we obtain a compact subrelation of R_φ.

$U_1, U_2, U_3, \dots$ clopen subsets of X getting smaller
$\rightsquigarrow P_1, P_2, P_3, \dots$ Delone sets of $\mathbb{R}^N$ getting thinner
$\rightsquigarrow T_1, T_2, T_3, \dots$ Voronoi tessellations
such that each tile is getting larger
$\rightsquigarrow R_1 \subset R_2 \subset R_3 \subset \cdots \subset R_\varphi$
$R = \bigcup_n R_n$ a 'large' AF subrelation of R_φ

Remarks

- We need to control the difference between R_φ and R.
- There are $N+1$ possibilities: $R_\varphi[x]$ may split into k distinct R-orbits ($k = 1, 2, \ldots, N+1$).
- So, we use the Absorption Theorem N times repeatedly.
- Furthermore, we must modify the Voronoi tessellations:
  For $p \in P$, we set

  $$V_w(p) = \{q \in \mathbb{R}^N \mid d(q,p)^2 - w(p) \leq d(q,p')^2 - w(p') \ \forall p' \in P\},$$

  where $w : P \to \mathbb{R}$ is called a weight function.