カントール極小$\mathbb{Z}^2$系の軌道同型による分類

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joint work with
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$X :=$ the Cantor set $\cong \{0,1\}^{\mathbb{N}}$

i.e. compact, metrizable

totally disconnected, perfect

$\varphi : \mathbb{Z}^d \to \mathrm{Homeo}(X)$ action

free $\overset{\text{def.}}{\Longleftrightarrow}$ $\forall n \in \mathbb{Z}^d \setminus \{0\}$, $\forall x \in X$

$\varphi^n(x) \neq x$

minimal $\overset{\text{def.}}{\Longleftrightarrow}$ $\forall x \in X$

$\{\varphi^n(x) \mid n \in \mathbb{Z}^d\}$ is dense

$$R_\varphi := \left\{ (x, \varphi^n(x)) \in X \times X \;\middle|\; \begin{array}{l} x \in X \\ n \in \mathbb{Z}^d \end{array} \right\}$$

equivalence relation induced by φ

Problem

$$\left.\begin{array}{l} \varphi : \mathbb{Z}^d \curvearrowright X \\ \psi : \mathbb{Z}^{d'} \curvearrowright X \end{array}\right\} \text{ free, minimal}$$

orbit equiv. $\overset{\text{def.}}{\Longleftrightarrow}$ $\exists\, h : X \to Y$ homeo.

s.t. $h \times h(R_\varphi) = R_\psi$

When are they orbit equiv. ?

$$M_\varphi := \{ \varphi\text{-invariant prob. measures} \}$$

Conjecture

T.F.A.E.

(1) φ is orbit equiv. to ψ

(2) $\exists\, h : X \to Y$ homeo.

s.t. $h_*(M_\varphi) = M_\psi$

GPS '95 $\qquad d = d' = 1$

GMPS '06 $\qquad 1 \leq d, d' \leq 2$

Example ①

$X := \{0, 1\}^{\mathbb{N}}$

$\varphi :=$ addition by $(1, 0, 0, 0, \cdots)$
with carry-over

$\rightsquigarrow (X, \varphi)$ Cantor minimal $\mathbb{Z}$-system
(adding machine)

$x_0 = (1, 1, 1, 1, \cdots)$

$\varphi(x_0) = (0, 0, 0, 0, \cdots)$

$\exists$ unique φ-inv. prob. measure μ

s.t. $\mu(\{\text{clopen subsets}\}) = [0,1] \cap \mathbb{Z}[\frac{1}{2}]$

$\theta :=$ Morse substitution

i.e. $\theta(0) = 01$, $\theta(1) = 10$

$\rightsquigarrow$ $Y \subset \{0,1\}^{\mathbb{Z}}$

$\psi :=$ shift

(Y, ψ) minimal subshift

$\exists$ unique ψ-inv. prob. measure ν

$\nu(\{\text{clopen subsets}\}) = [0,1] \cap \mathbb{Z}[\frac{1}{2}]$

$\therefore$ φ is orbit equiv. to ψ

Example ②

$T :=$ a Penrose tiling

$\Omega :=$ hull of T

i.e. closure of $\{T + p \mid p \in \mathbb{R}^2\}$

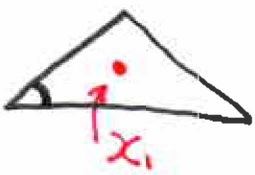
Ω : compact, metrizable

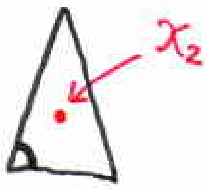
$\varphi : \mathbb{R}^2 \curvearrowright \Omega$ translation, free, minimal

prototiles = 

and their $\frac{\pi}{5}$-rotations

Pick up two prototiles t_1, t_2

$t_1 =$ 

$t_2 =$ 

Choose points x_1, x_2 in their interior

$$\Omega_{punc} := \left\{ T \in \Omega \;\middle|\; \begin{array}{l} \exists i \in \{1,2\},\ \exists P \in \mathbb{R}^2 \\ t_i + P \in T,\ x_i + P = 0 \end{array} \right\}$$

$\rightsquigarrow$ Cantor set

$$R_\varphi := \left\{ (T, \varphi^P(T)) \;\middle|\; \begin{array}{l} T, \varphi^P(T) \in \Omega_{punc} \\ P \in \mathbb{R}^2 \end{array} \right\}$$: equiv. relation on Ω_{punc}

$\exists$ unique invariant measure μ on Ω_{punc}

$$\mu(\{\text{clopen subsets}\}) = [0,1] \cap \left(\mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{5}}{2}\right)$$

θ := Fibonacci substitution

i.e. $\theta(0) = 01$, $\theta(1) = 0$

$\rightsquigarrow \quad Y \subset \{0,1\}^{\mathbb{Z}}$

τ ; shift

(Y, τ) minimal subshift

$\exists$ unique τ-inv. prob. measure on Y

$$\mu(\{\text{clopen subsets}\}) = [0,1] \cap \left(\mathbb{Z} + \mathbb{Z}\frac{1+\sqrt{5}}{2}\right)$$

$\therefore$ R_φ is orbit equiv. to R_τ

$\uparrow$ Penrose

$\uparrow$ Fibonacci

Observation

φ : adding machine on $X = \{0,1\}^{\mathbb{N}}$

$$X \cong \prod_{n=1}^{\infty} \mathbb{Z}_2 \qquad \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$$

$$\bigoplus_{n=1}^{\infty} \mathbb{Z}_2 \curvearrowright X \qquad \text{action}$$

$R :=$ equiv. relation induced by this action

- $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty} \in X$

  $(x,y) \in R \iff {}^{\exists} N \in \mathbb{N}$ s.t. $x_n = y_n \ {}^{\forall} n \geq N$

- $R \subset R_\varphi$

- $x_0 := (1,1,1,1,\cdots) \in X$

  if $x \notin R_\varphi[x_0]$, then $R_\varphi[x] = R[x]$

  $R_\varphi[x_0] = R[x_0] \cup R[\varphi(x_0)]$

i.e. $R_\varphi = R \vee \{(x_0, \varphi(x_0))\}$

$n = 1, 2, 3, \cdots$

$R_n :=$ equiv. relation induced by the action of $\bigoplus_{k=1}^{n} \mathbb{Z}_2$

$\rightsquigarrow \quad R_1 \subset R_2 \subset R_3 \subset \cdots, \quad R = \bigcup R_n$

R_n ; "finite" relation

$\rightsquigarrow$ R is Approximately Finite relation

Thm (GPS '95) "classification of AF relations"

$X_i :=$ Cantor sets

$R_i :=$ minimal AF relations on X_i $\quad i = 1, 2$

$M_i := \{ R_i\text{-invariant prob. measures} \}$

T.F.A.E.

(1) R_1 is orbit equiv. to R_2

(2) $\exists\, h : X_1 \to X_2$ homeo.

s.t. $\quad h_*(M_1) = M_2$

Problem

$\varphi : \mathbb{Z}^d \curvearrowright X$ free, minimal

$\overset{?}{\underset{?}{\Longrightarrow}}$ R_φ is orbit equivalent

to an AF relation ?????

Thm (GMPS '06) Yes for $d=2$!

Absorption Theorem (GMPS)

$R \subset X \times X$; minimal AF relation

$Y \subset X$: closed subset

$\mu(Y) = 0$ for ${}^\forall \mu$; R-inv. measure

$K \subset Y \times Y$: equiv. relation on Y

with several assumptions

$\Longrightarrow$ $R \vee K$ is orbit equiv. to R

Example

φ : adding machine on $X = \{0,1\}^{\mathbb{N}}$

free minimal $\mathbb{Z}$-action

$\rightsquigarrow$ $R_\varphi = R \vee \{(x_0, \varphi(x_0))\}$

$Y := \{x_0, \varphi(x_0)\}$ two pts set

$K := Y \times Y$

by the absorption thm, $R_\varphi \underset{\text{o.e.}}{\sim} R$

Strategy for $\mathbb{Z}^d$ actions

$\varphi : \mathbb{Z}^d \curvearrowright X$ free, minimal

Find a "large" AF subrelation $R \subset R_\varphi$

s.t. $R_\varphi = R \vee$ "something"

Apply the absorption thm. d times

to show R_φ is orbit equiv. to R

To find a "large" AF subrelation

$$R = \bigcup R_n, \quad R_1 \subset R_2 \subset R_3 \subset \cdots\cdots$$

in R_φ, we have to construct

a "large" finite subrelation R_n in R_φ

$x \in X$

$$R_\varphi[x] = \coprod_{i=1}^{\infty} R_n[x_i] \qquad \# R_n[x_i] < \infty$$

$$\begin{array}{c} R_\varphi[x] \\ \wr\!\parallel \\ \mathbb{Z}^d \end{array}$$

i.e. $\mathbb{Z}^d$ is partitioned into

(countably many) finite subsets

$\rightsquigarrow$ tiling of $\mathbb{R}^d$

We do NOT want to break

the topology of the Cantor set X

$x \in X$, $U \subset X$ clopen set

$$P := \{ p \in \mathbb{Z}^d \mid \varphi^p(x) \in U \}$$: hitting time

$\exists\, M_0, M_1 > 0$ s.t.

M_0 - separated i.e. $p \neq q \in P \Rightarrow d(p, q) \geq M_0$

M_1 - syndetic i.e. $\bigcup_{p \in P} B(p, M_1) = \mathbb{R}^d$

For $\forall p \in P$

$$T(p) := \{ x \in \mathbb{R}^d \mid d(x, p) \leq d(x, P) \}$$

Voronoi domain

$$\mathcal{T}_P := \{ T(p) \mid p \in P \}$$

Voronoi tessellation

$\rightsquigarrow$ "finite" subrelation of R_φ

$U_1, U_2, U_3, \cdots$

$\downarrow$ clopen sets in X getting smaller

for each $x \in X$

$P_1, P_2, P_3, \cdots$

separated, syndetic subsets of $\mathbb{R}^d$

getting thinner

$\downarrow$

for each $x \in X$

$\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \cdots$

Voronoi tessellations

each tile in $\mathcal{T}_\ell$ is getting larger

$\downarrow$

$R_1 \subset R_2 \subset R_3 \subset \cdots$

$R := \bigcup_n R_n$ AF subrelation of R_φ

Generically, for each $x \in X$,

we may expect $(d+1)$ possibilities

(1) $R_\varphi[x] = R[x]$ ← probability 1

(2) $\exists\, x_1 \in X$

s.t. $R_\varphi[x] = R[x] \amalg R[x_1]$

⋮

(d+1) $\exists\, x_1, \cdots, x_d \in X$

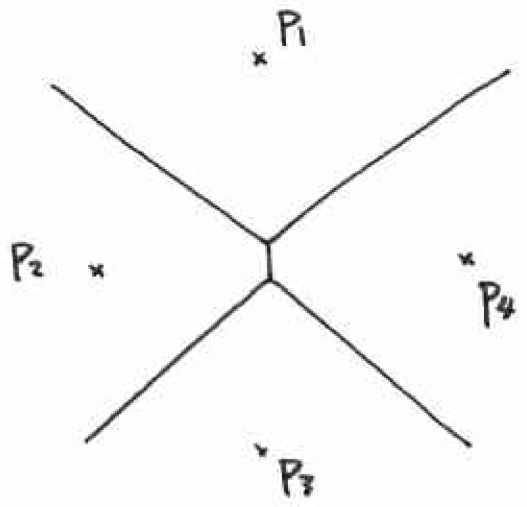
s.t. $R_\varphi[x] = R[x] \amalg R[x_1] \amalg \cdots \amalg R[x_d]$

d times application of the absorption thm,

⇝ R_φ is orbit equiv. to R

But we have a problem ......

$d=2$



Even if $d(P_i, P_j) \geq M$,
we may have
"a short edge"

Lem $\forall M \geq 1$ $\exists U \subset X$ clopen set
s.t. for each $x \in X$
P is M-separated, $2M$-syndetic

Let $\mathcal{T}_P$ be the Voronoi tessellation

Let $\mathcal{D}_P$ be its dual tessellation
(Delaunay tessellation)

i.e. vertices of $\mathcal{D}_P = P$

$\overline{pq}$ is an edge $\Leftrightarrow$ $T(p)$ and $T(q)$ share an edge

We can make $\mathcal{D}_P$ a triangulation

Lem $\forall\ \triangle P_1P_2P_3$ in $\mathcal{D}_P$

① $d(P_i, P_j) \geq M$

② radius of its circumsphere $\leq 2M$

i.e. the triangle is not so thin

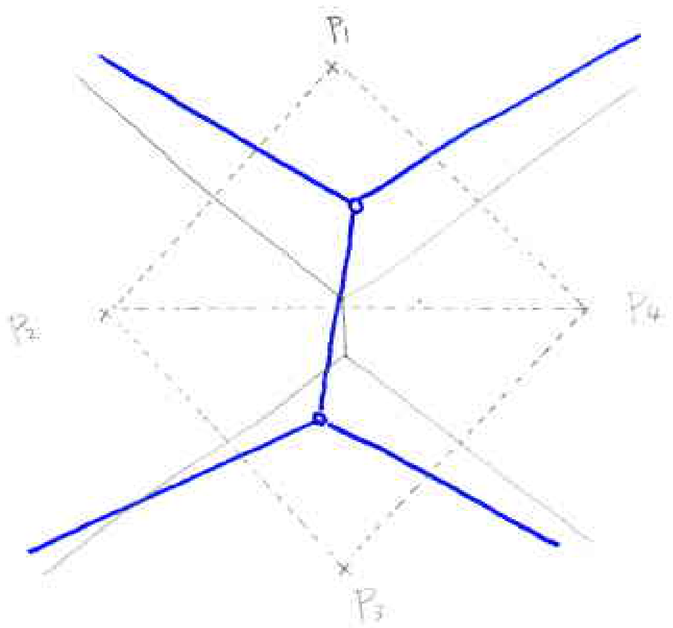
In particular, radius of its incircle is greater than cM (c : universal const.)

Rem vertex of a Voronoi domain

$=$ circumcentre of a triangle in $\mathcal{D}_P$

By moving the vertices of $\mathcal{T}_P$ to incentres, we get

modified Voronoi tessellation $\widetilde{\mathcal{T}}_P$



Then we have

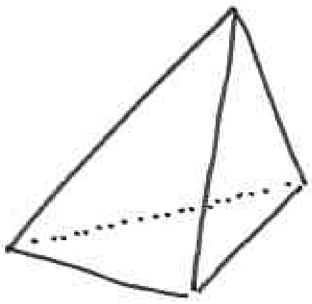
$$p, q \in P, \quad \tilde{T}(p) \cap \tilde{T}(q) = \phi$$

$$\Longrightarrow \quad d(\tilde{T}(p), \tilde{T}(q)) \geq cM$$

Problem in $d \geq 3$

$P \subseteq \mathbb{R}^3$ M-separated, 2M-syndetic

$\rightsquigarrow$ tetrahedron



length of each edge $\geq M$

radius of circumsphere $\leq 2M$

But it may be thin !!

" sliver "