tiling

$\mathcal{P} :=$ a finite set of polygons in $\mathbb{R}^2$ whose interior contain the origin.

ex. $\mathcal{P} = \{$  $\}$

$\mathcal{P} = \{$  and their $\frac{\pi}{5}$-rotations $\}$

$\mathcal{P} \ni p$, $\mathbb{R}^2 \ni t$, $p+t$ is called a tile, t is called the puncture of $p+t$

$T = \{ P \mid P \text{ tile} \}$ is a tiling

$$\underset{\text{def.}}{\Longleftrightarrow} \quad \bigcup_{P \in T} P = \mathbb{R}^2, \quad P, Q \in T,\ P \neq Q \Rightarrow P^\circ \cap Q^\circ = \phi$$

Two tilings T and T' are close

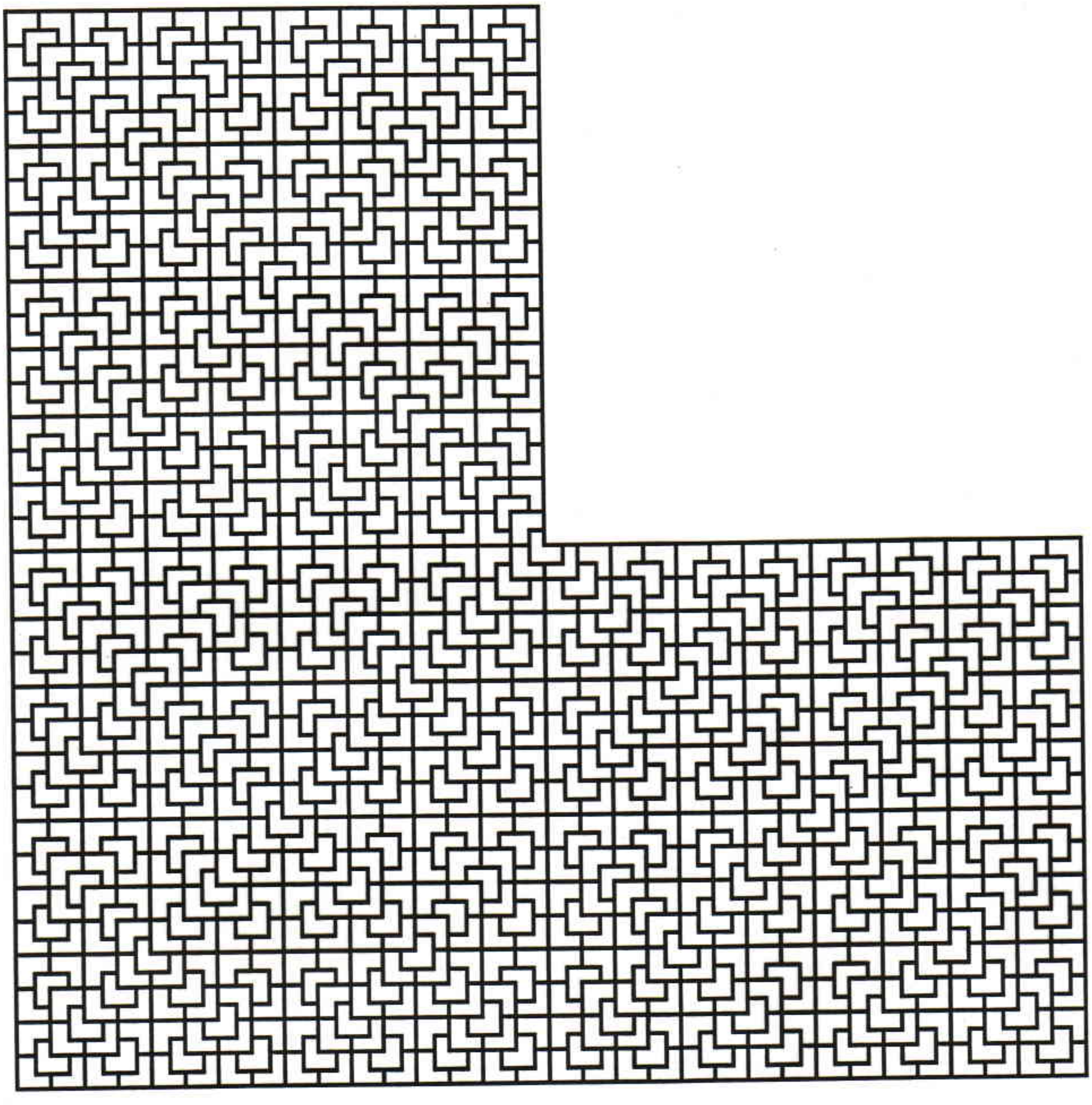
$\underset{\text{def.}}{\Longleftrightarrow}$ $\exists$ small $\varepsilon \in \mathbb{R}^2$, large $R > 0$ s.t. $T + \varepsilon$ and T' agree on $B(0, R)$

$T + t = \{ P + t \mid P \in T \}$

$\rightsquigarrow$ tiling space

$\mathbb{R}^2$- action by translation

The Chair Tiling



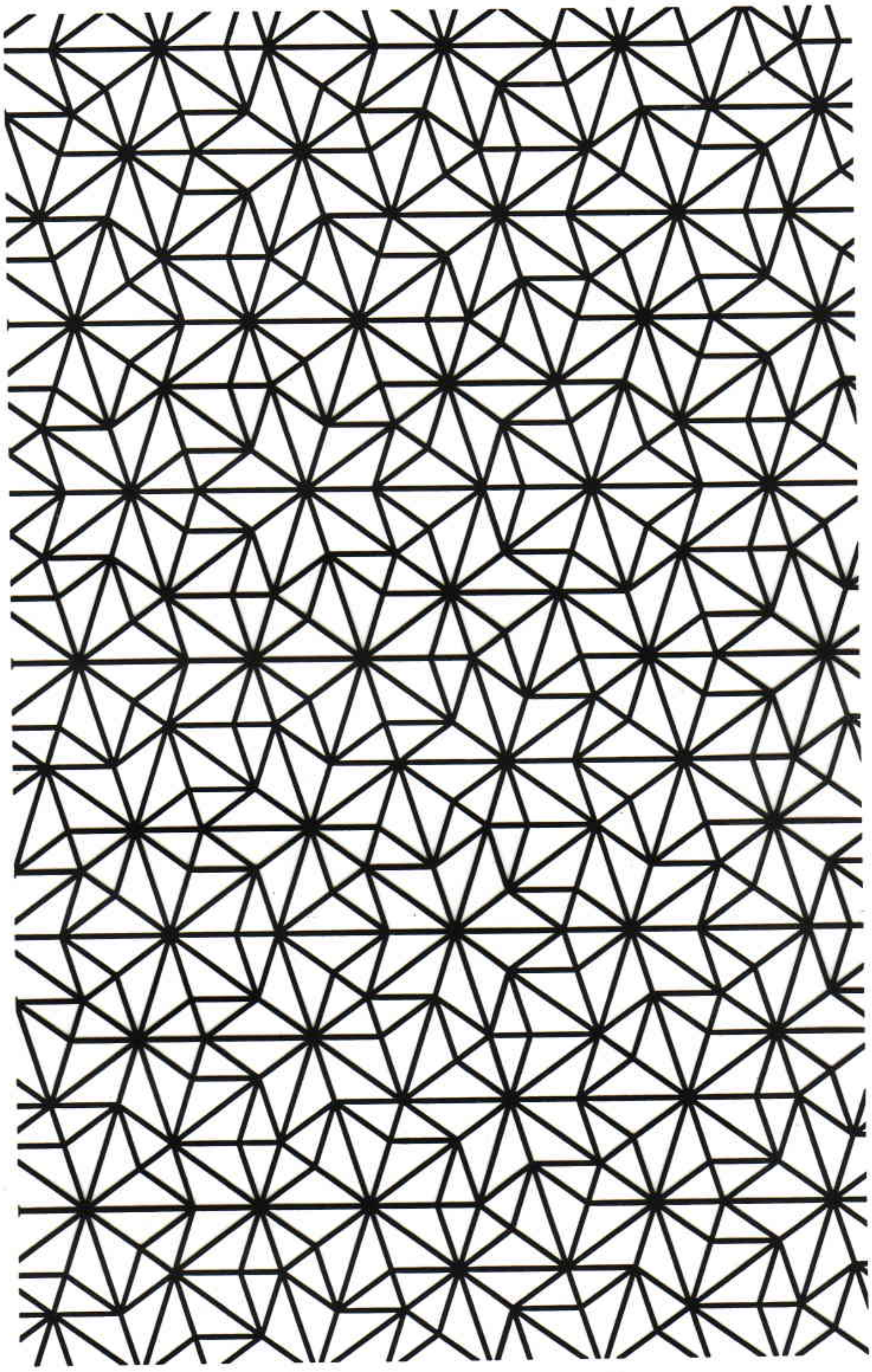


Figure 4. A Penrose tiling

substitution tiling

p tile $\quad \omega(p) = \{p_1, \cdots, p_n\}$ $\quad p_i$ tile

$$\lambda p = \bigcup p_i$$

$\omega^2(p) := \bigcup \omega(p_i)$, $\quad \omega^3(p), \omega^4(p), \cdots\cdots$

$\Omega :=$ the set of tilings T such that every finite subcollections of tiles is congruent to a subset of some $\omega^k(p)$

$\rightsquigarrow$ Ω is a compact metric space

$\varphi : \mathbb{R}^2 \curvearrowright \Omega$ translation $\qquad \omega(T) := \{\omega(p) \mid p \in T\}$

$\omega : \Omega \to \Omega$ the inflation map

Under suitable hypotheses,

ω is a homeomorphism

(Ω, φ) minimal dynamical system

$X := \{T \in \Omega \mid {}^\exists p \in T$ the puncture of p = the origin$\}$

$=$ "transversal" of $\Omega \cong$ Cantor set

$R := \{(T, T') \in X \times X \mid {}^\exists t \in \mathbb{R}^2 \ \ \varphi^t(T) = T'\}$

$=$ minimal étale equiv. relation on X

Thm R is affable

i.e. R is orbit equiv. to an AF-relation

i.e. $\exists$ Y Cantor set

$S \subset Y \times Y$ AF-relation

$h : X \to Y$ homeomorphism

s.t. $h \times h(R) = S$

Cor R is orbit equiv. to a Cantor minimal $\mathbb{Z}$-system.

AF-relation

$(V, E) :=$ Bratteli diagram of a unital simple AF-alg.

$X :=$ the infinite path space $\cong$ a Cantor set

$R_n := \{ (x, y) \in X \times X \mid x_k = y_k \text{ for } {}^\forall k \geq n \}$

with the relative topology from $X \times X$

$R_1 \subset R_2 \subset R_3 \subset \cdots\cdots$

$R := \bigcup R_n$ with the inductive limit topology

AF-relation

Absorption Theorem (Giordano - Putnam - Skau)

$R \subset X \times X$ minimal AF-relation

$B, B^* \subset X$ closed subsets

$\beta : B \to B^*$ homeomorphism

(1) $\forall \mu$ invariant measure $\mu(B) = \mu(B^*) = 0$

(2) B, B^* are R-étale

(3) $R \cap (B \times B^*) = \phi$

(4) $\beta \times \beta : R \cap (B \times B) \to R \cap (B^* \times B^*)$ homeo.

$\Longrightarrow$ $\widetilde{R} = R \vee \{(x, \beta(x)) \mid x \in B\}$

is orbit equiv. to R

In particular, $\widetilde{R}$ is affable

A "small" extension of an AF-relation is affable !!

Substitution tiling

Ω = tiling space $\qquad \varphi : \mathbb{R}^2 \curvearrowright \Omega$

$X \subset \Omega$ Cantor transversal

$R \subset X \times X$ $\qquad \omega : \Omega \to \Omega$ the inflation map

$T \in X$ $\quad [T]_R$ = R-equiv. class of T

$$= \{ \varphi^t(T) \in X \}$$

$$\xleftrightarrow{1:1} \{ \text{tiles in } T \}$$

$$\partial(T) := \bigcup_{p \in T} \text{boundary of } p \subset \mathbb{R}^2$$

$n \in \mathbb{N}$ $\quad \partial_n(T) := \lambda^n \, \partial(\omega^{-n}(T))$; boundary of all n-th "supertiles"

$$\mathbb{R}^2 \supset \partial(T) \supset \partial_1(T) \supset \partial_2(T) \supset \cdots\cdots$$

Define a subrelation $R_n \subset R$ by

$$[T]_{R_n} = \left\{ \varphi^t(T) \in X \;\middle|\; \begin{array}{l} \text{the origin and } t \in \mathbb{R}^2 \text{ belong to} \\ \text{the same connected component} \\ \text{of } \mathbb{R}^2 \setminus \partial_n(T) \end{array} \right\}$$

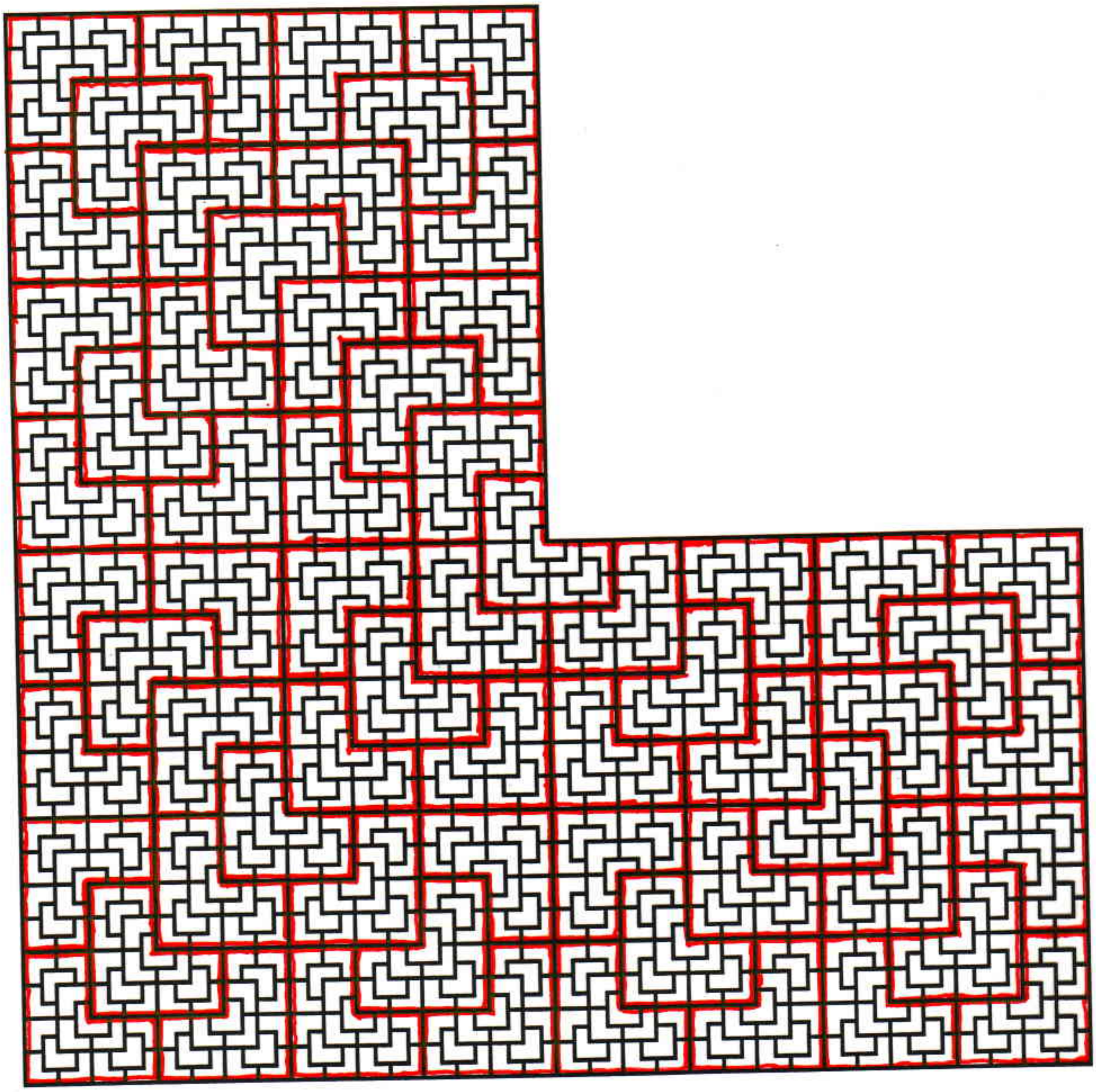
Then we have $R_1 \subset R_2 \subset R_3 \subset \cdots$

$R_{AF} := \bigcup R_n$ $\quad$; AF-subrelation of R

$$[T]_{R_{AF}} = \left\{ \varphi^t(T) \in X \;\middle|\; \begin{array}{l} \text{the origin and } t \in \mathbb{R}^2 \text{ belong to} \\ \text{the same connected component} \\ \text{of } \mathbb{R}^2 \setminus \partial_\infty(T) \end{array} \right\}$$

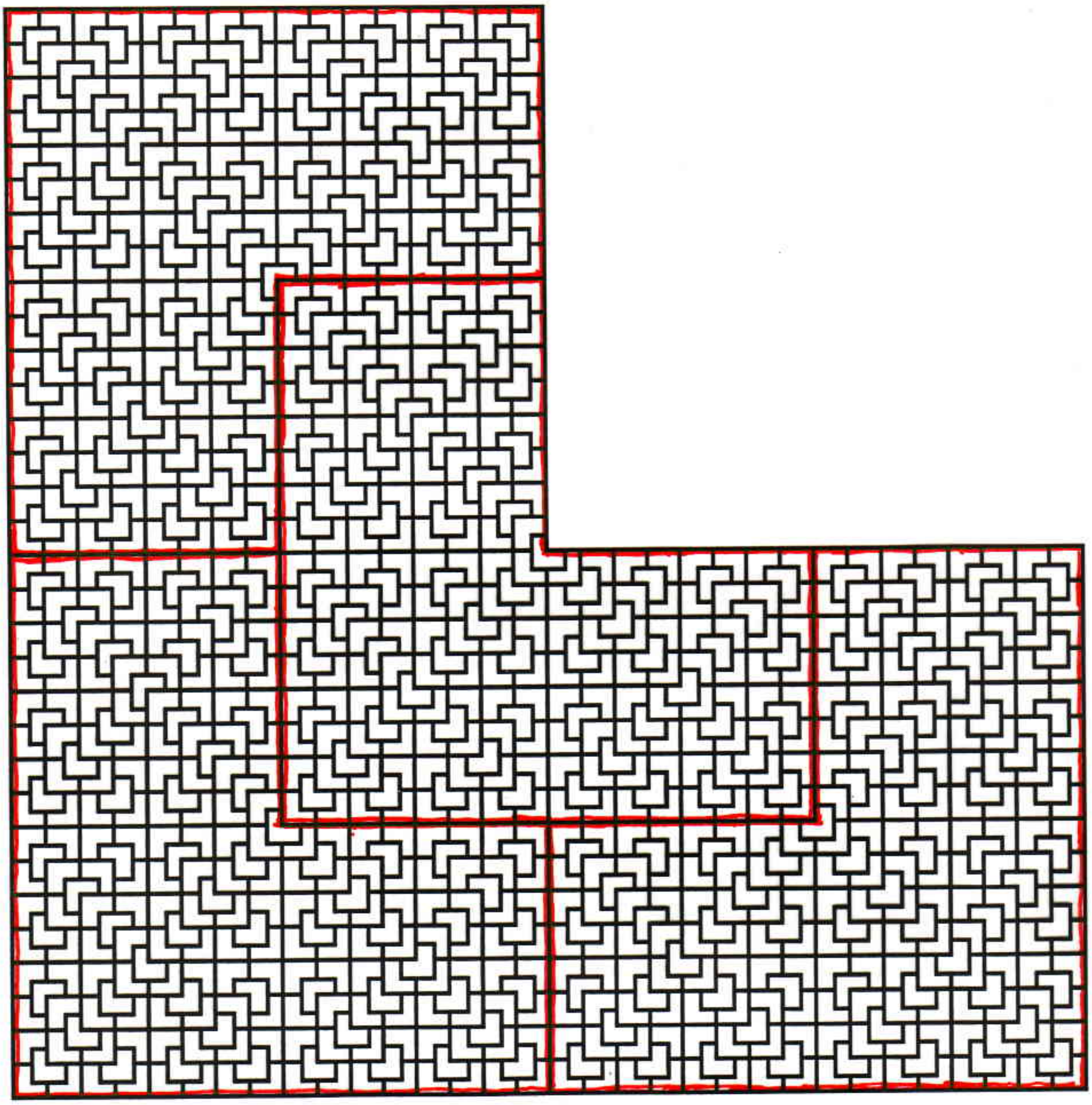
where $\partial_\infty(T) := \bigcap \partial_n(T)$

The Chair Tiling



$$\partial_2(T) = \lambda^2 \, \partial\left(\omega^{-2}(T)\right)$$

The Chair Tiling



$$\partial_4(T) = \lambda^4 \, \partial(\omega^{-4}(T))$$

three possibilities ......

T is of type I $\iff$ $\partial_\infty(T)$ is empty

$\iff$ $[T]_R = [T]_{RAF}$

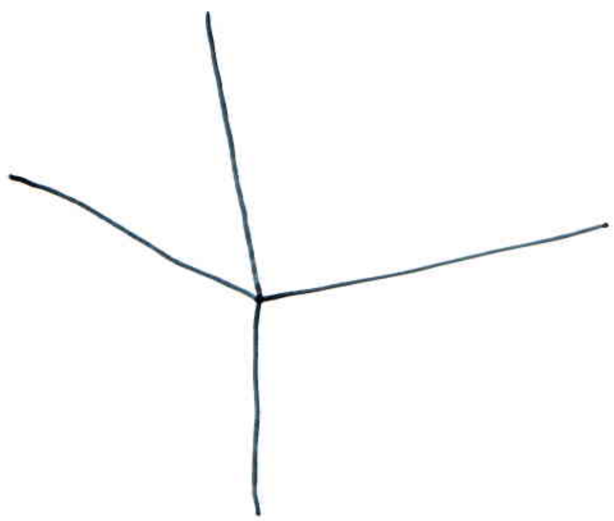
T is of type II $\iff$ $\partial_\infty(T)$ is a line

$\iff$ $[T]_R = [T]_{RAF} \cup [T']_{RAF}$

T is of type III $\iff$ $\partial_\infty(T)$ is a union of half lines

$\iff$ $[T]_R = [T]_{RAF} \cup [T_1]_{RAF} \cup \cdots \cup [T_n]_{RAF}$

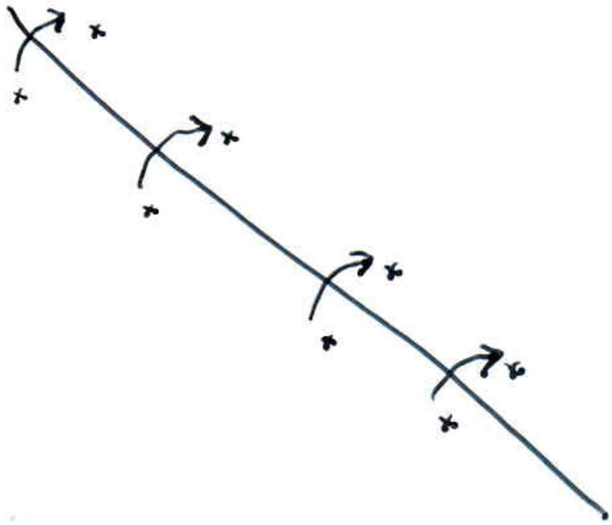
type II



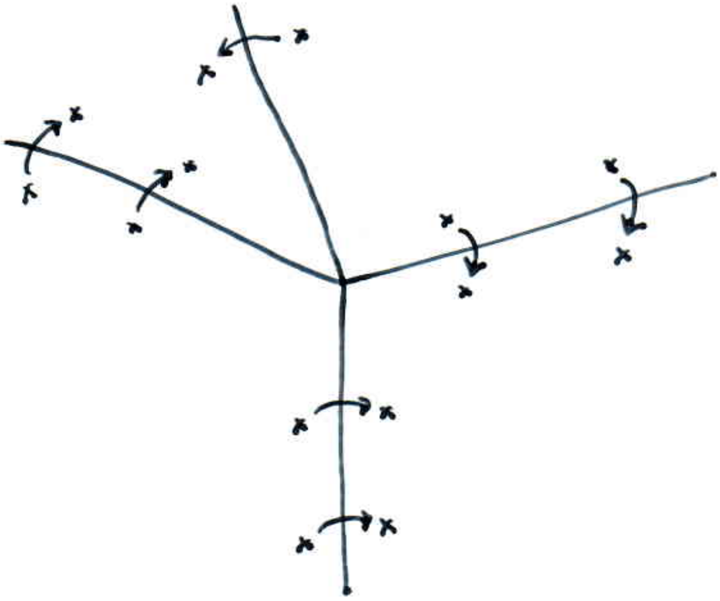
type III

We have only finitely many type III orbits, because we have finitely many polygons and each polygon has finitely many corners.

We can define a glueing map $\beta : B \to B^*$



type II

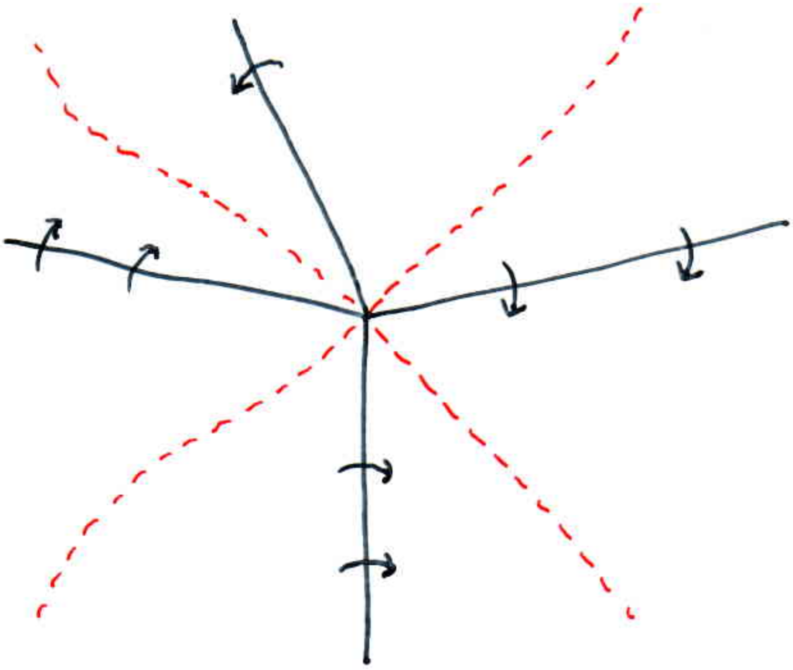


type III

so that $R = R_{AF} \vee \{(x, \beta(x)) \mid x \in B\}$

But we cannot apply the absorption thm. to this situation.

We need to cut type III orbits



We can construct a subrelation $R'_{AF} \subset R_{AF}$ such that the absorption theorem applies to R'_{AF} and $\beta : B \to B^*$

$$\rightsquigarrow \quad \widetilde{R} := R'_{AF} \vee \{(x, \beta(x)) \mid x \in B\}$$

is affable

But, $\widetilde{R} \subsetneq R$

R can be written as a "finite extension" of $\widetilde{R}$

i.e. $\exists\ (T_1, T_1'), \cdots, (T_n, T_n')$

s.t. $R = \widetilde{R} \vee \{(T_i, T_i') \mid i = 1, 2, \cdots, n\}$

because we have only finitely many type III orbits.

By applying the absorption thm. again, we can show R is affable.