Discriminants of Cyclic Homogeneous Inequalities of Three Variables

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Abstract. In this article, we study the structure of the cone of semidefinite forms. It is a closed semialgebraic set but usually is not basic closed semialgebraic set. A discriminant is a defining equation of an irreducible component of algebraic boundary of this cone. We calculate discriminants using new tools — characteristic variety and local cones. A characteristic variety is a semialgebraic subset of a real projective variety on which the family of inequalities is essentially defined as linear functions. Local cone is a subcone of the PSD cone which corresponds to a maximal ideal. This theory works well for a family of polynomials which are invariant under an action of a finite group. After we construct an abstract general theory, we apply it to a family of cyclic homogeneous polynomials of three real variables of degree d. We calculate some discriminants for $d = 3$, 4, 5 and 6, and we show that this theory derives many new results.

Section 0. Introduction.

A study of a PSD cone was derived from Hilbert's 17th problem. About the history of the study, please read [7] §6. Recent important results about PSD cones are found in [2], [8], [22] and [6]. Especially, some studies of algebraic boundaries and discriminants are explained in [2]. But, it seems that the structure of PSD cones is not yet known so well, including the classical case $\mathcal{P}_{3,4} = \Sigma_{3,4}$. In this article, we study such problem using real and complex algebraic geometry. For this purpose, we should generalize the definition of PSD cone. The exact definition will be given in §1, but we present here its idea. Let $\mathcal{H}_{n,d}$ be the vector space of all the homogeneous polynomials of n variables of degree d, and $\mathcal{H} \subset \mathcal{H}_{n,d}$ be a subspace. $\mathcal{H}$ is called a linear system. Let A be a closed semialgebraic subset of $\mathbb{P}_{\mathbb{R}}^{n-1}$. We call

$$\mathcal{P} = \mathcal{P}(A, \mathcal{H}) := \{ f \in \mathcal{H} \mid f(\mathbf{a}) \geq 0 \text{ for all } \mathbf{a} \in A \}$$

to be the PSD cone on A in $\mathcal{H}$. Originally, $\mathcal{P}_{n,2d} := \mathcal{P}(\mathbb{P}_{\mathbb{R}}^{n-1}, \mathcal{H}_{n,2d})$ is called the convex cone of positive semidefinite forms, or shortly the PSD cone ([5], [6], [8], [20], [22]). Since $\mathcal{P}$

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is a closed semialgebraic convex cone (Proposition 1.14), $\mathcal{P}$ has the algebraic boundary $\partial_a\mathcal{P}$. Let φ be a defining polynomial of an irreducible hypersurface component of $\partial_a\mathcal{P}$. We call φ to be a discriminant of $\mathcal{P}$ (cf. [20], [6] p.172). We don't assume that φ is non-negative on $\mathcal{P}$. Let's start from some elementary examples. Cîrtoaje proved the following theorem in 2006.

Theorem 0.1. ([10]) *Let*

$$f(a,b,c) := (a^4+b^4+c^4) + p(a^3b+b^3c+c^3a) + q(ab^3+bc^3+ca^3) \\ + r(a^2b^2+b^2c^2+c^2a^2) - (1+p+q+r)abc(a+b+c).$$

Then, $f(a,\, b,\, c) \geq 0$ for all $a,\, b,\, c \in \mathbb{R}$ if and only if $p^2+pq+q^2 \leq 3r+3$.

In this case, $\mathrm{disc}_4^{c0}(p,q,r) := 3(r+1) - (p^2+pq+q^2)$ is a discriminant, and $\mathrm{disc}_4^{c0} \geq 0$ determine the PSD cone. This PSD cone is a basic closed semialgebraic set. The second example is the following theorem which will be proved in §3.

Theorem 0.2. *Let*

$$f(a,b,c) := (a^3+b^3+c^3) + p(a^2b+b^2c+c^2a) + q(ab^2+bc^2+ca^2) + rabc.$$

Then, $f(a,\, b,\, c) \geq 0$ for all $a \geq 0$, $b \geq 0$, $c \geq 0$ if and only if one of (1) *or* (2) *holds.*

(1) $3+3p+3q+r \geq 0$ *and* $4p^3+4q^3+27 \geq p^2q^2+18pq$.
(2) $3+3p+3q+r \geq 0$ *and* $p \geq 0$ *and* $q \geq 0$.

In this case, $\mathrm{disc}_3^{c+}(p,q) := 4p^3+4q^3+27-p^2q^2-18pq$ is a non-trivial discriminant, and $3+3p+3q+r$ is a trivial discriminant. Note that $\mathcal{P}$ is not a basic closed semialgebraic set, and the signature of $\mathrm{disc}_3^{c+}(p,q)$ is not constant on $\mathcal{P}$ (see Fig. 3.1). It was not easy to calculate discriminants. Schur found the following inequality in the early period of the 20th century.

$$(a^3+b^3+c^3) + 3abc \geq (a^2b+b^2c+c^2a) + (ab^2+bc^2+ca^2).$$

But generalization of Theorem 0.2 is completely new.

In §1 and §2, we introduce some tools to obtain discriminants. We use the similar idea with [6] and [20]. Let X be the image of A by the rational map defined by the linear system $\mathcal{H}$. X is called the 'characteristic variety' of $\mathcal{P}$. Note that $\mathcal{P}$ is the dual cone of the convex set generated by X (Proposition 1.14). In other words, $f \in \mathcal{P}$ is a linear inequality on X. We shall show that the structure of $\partial\mathcal{P}$ is determined by the critical sets of X (Theorem 1.18). Our method to determine $\mathcal{P}$ is summarized in Remark 1.23.

If a finite group G acts on A, and $\mathcal{H}$ is a set of G-invariant polynomials, then there exists the natural dominant rational map $A/G \cdots \to X$. This is an isomorphism under a certain condition (for example Proposition 1.36). This fact helps us to determine the critical sets of X. If $G = \mathbb{Z}/n\mathbb{Z}$, $\mathbb{P}_{\mathbb{R}}^{n-1}/G$ is a real algebraic variety. But for the symmetric group $\mathfrak{S}_n$, the set $\mathbb{P}_{\mathbb{R}}^{n-1}/\mathfrak{S}_n$ is a proper closed subset of the real weighted projective space $\mathbb{P}_{\mathbb{R}}(1,2,\ldots,n)$, and is not a real algebraic variety. To treat critical sets of $\mathbb{P}_{\mathbb{R}}^{n-1}/\mathfrak{S}_n$, we introduce a notion of *semialgebraic varieties* in §1. It is also convenient to consider X to be a semialgebraic variety, for we can apply techniques of scheme theory.

We will show that the local cone $\mathcal{P}_a := \left\{f \in \partial\mathcal{P} \;\middle|\; f(a) = 0\right\} \subset \partial\mathcal{P}$ $(a \in A)$ is useful to calculate discriminants and to determine extremal PSD forms. Especially, if $\mathrm{Bs}\,\mathcal{H} = \emptyset$, then $\partial\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ (Proposition 1.26). For example, in the case Theorem 0.2, $\mathcal{P}_{(0:s:1)}$ is the ray generated by

$$\mathfrak{f}_s(a,b,c) := s^2(a^3+b^3+c^3) - (2s^3-1)(a^2b+b^2c+c^2a) \\ (s^4-2s)(ab^2+bc^2+ca^2) - 3(s^4-2s^3+s^2-2s+1)abc.$$

This $\mathfrak{f}_s$ is an extremal PSD form of $\mathcal{P}$.

If X has many critical subsets, then $\mathcal{P}$ may have many discriminants. The following two theorems are typical examples.

Theorem 0.3. *Let*

$$\begin{aligned} f(a,b,c) &:= (a^4+b^4+c^4)+p(a^3b+b^3c+c^3a)+q(ab^3+bc^3+ca^3) \\ &\quad + r(a^2b^2+b^2c^2+c^2a^2)-(1+p+q+r)abc(a+b+c), \\ \varphi(p,q,r) &:= p^2q^2r^2-4p^3q^3+18p^3qr+18pq^3r-4p^2r^3-4q^2r^3 \\ &\quad -27p^4-27q^4+16r^4-6p^2q^2-80pqr^2 \\ &\quad +144p^2r+144q^2r-192pq-128r^2+256. \end{aligned}$$

Then, $f(a,\ b,\ c)\ge 0$ for all $a\ge 0$, $b\ge 0$ and $c\ge 0$ if and only if one of the following 6 conditions holds.

(1) $r\ge -2$, $p\le -2\sqrt{r+2}$, $p+q\ge 0$, *and* $\varphi(p,q,r)\le 0$.
(2) $r\ge -2$, $q\le -2\sqrt{r+2}$, $p+q\ge 0$, *and* $\varphi(p,q,r)\le 0$.
(3) $r\ge -2$, $-\sqrt{r+4}\le p+q\le 0$, $p\ge -2\sqrt{r+2}$, $q\ge -2\sqrt{r+2}$, *and* $\varphi(p,q,r)\ge 0$.
(4) $r\ge -2$, $p\ge -2\sqrt{r+2}$, $q\ge -2\sqrt{r+2}$, *and* $p+q\ge 0$.
(5) $r\ge 0$, *and* $p^2+pq+q^2\le 3r+3$.
(6) $r\le -2$, $p+q\ge 0$ *and* $\varphi(p,q,r)\le 0$.

In this case, $\mathrm{disc}_4^{c+}(p,q,r):=\varphi(p,q,r)$ and $\mathrm{disc}_4^{c0}(p,q,r)$ are non-trivial discriminants. Note that this PSD cone is not a basic closed semialgebraic set. So, we need to decompose $\mathcal{P}$ into basic semialgebraic sets as the above.

Theorem 0.4. *Let*

$$\begin{aligned} f(a,b,c) &= (a^5+b^5+c^5)+p(a^4b+b^4c+c^4a+ab^4+bc^4+ca^4) \\ &\quad + q(a^3b^2+b^3c^2+c^3a^2+a^2b^3+b^2c^3+c^2a^3) \\ &\quad + rabc(a^2+b^2+c^2)-(1+2p+2q+r+1)abc(ab+bc+ca), \\ d_5(p,q,r) &:= \Big(4(p+1)(p-2)(2p-1)-9q(2p-1)-9r(p+1)\Big)^2 \\ &\quad -\Big((2p-1)^2-3(2q+r+2)\Big)^3. \end{aligned}$$

Then, $f(a,\ b,\ c)\ge 0$ for all $a\ge 0$, $b\ge 0$ and $c\ge 0$ if and only if one of the following 4 conditions holds.

(1) $p\ge -1$, $p+q+1\ge 0$, *and* $2p+r+1>0$.
(2) $p\ge -1$, $p+q+1\ge 0$, $2p+r+1\le 0$, *and* $d_5(p,q,r)\ge 0$.
(3) $-3\le p<-1$, $p+q+1\ge 0$ *and* $d_5(p,q,r)\ge 0$, *and* $(q,r)\ne(-p-1,\ -2p-1)$.
(4) $p\le -3$, $4q\ge (p+1)^2+4$, *and* $d_5(p,q,r)\ge 0$.

In this case, $d_5(p,q,r)$ and $4q-(p+1)^2-4$ are non-trivial discriminants. The number of discriminants will be discussed in §2. After §3, we treat many PSD cones and determine its discriminants and their extremal forms.

In this article, we only treat three variable polynomials. But it is already known that our method is applicable many generalized cases (for example see [4]).

Section 1. Structure of PSD cones.

1.1. Semialgebraic variety.

An abstract generalized definition of semialgebraic (quasi-)varieties is given in [3]. Here we summarize minimum concepts to study algebraic inequalities.

Usually, the symbol $\mathbb{R}$ implies the set of real numbers. But if you regard $\mathbb{R}$ to be any real closed field, all results of this article hold. A definition of real algebraic varieties is given at Definition 3.2.11 in [7]. But in this article, we call such a variety to be an *algebraic quasi-variety.* Note that an algebraic quasi-variety is reduced and has at least one non-singular point but may not be irreducible. If an algebraic quasi-variety is irreducible, we say it is an *algebraic variety.* Note that we can define non-reduced real algebraic quasi-varieties, but we don't need them in this article. (cf. [18])

Let V be a complete real algebraic quasi-variety. For a subset $A \subset V$, the minimal reduced algebraic subset which contain A is called the *Zariski closure* of A and denoted by $\mathrm{Zar}(A)$ or $\mathrm{Zar}_V(A)$. We define $\dim A := \dim \mathrm{Zar}(A)$. On the other hand, topological closure of A in V is denoted by $\mathrm{Cls}_V(A)$ or $\overline{A}$. The interior of A is defined by $\mathrm{Int}(A) := \mathrm{Zar}_V(A) - \mathrm{Cls}_V(\mathrm{Zar}_V(A) - A)$. $\mathrm{Int}(A)$ is also written as A°. The (relative) *boundary* of A in V is defined by $\partial_V A := \mathrm{Cls}_V(A) - \mathrm{Int}(A)$. $\partial A := A - \mathrm{Int}(A)$ is called the *absolute boundary* of A. Note that $\mathrm{Int}(A)$ and ∂A does not depend on the choice of embedding $A \subset V$. $\mathrm{Zar}_V(\partial_V A)$ is denoted by $\partial_a A$, and is called the *algebraic boundary* of A (see [25]).

Note. Some real algebraic varieties have lower dimensional connected components which are subsets of singular locus. Thus definition of $\mathrm{Int}(A)$ may not be good in other theory. But the above definition works well in this article.

Assume that A is a semialgebraic subset of a complete real algebraic quasi-variety V with $\mathrm{Zar}(A) = V$. See Remark 3.2.15 of [7] for the definition of semialgebraic subsets. The structure sheaf $\mathfrak{R}_V$ of V is defined in [7] §2 or [3]. Let U be an affine open subset of V and $B \subset U$ be a non-empty subset. There exists an one-to-one correspondence between the set of maximal ideals of $\mathfrak{R}_V(U)$ and points in U. For a point $x \in U$, we put $S(x) := \mathfrak{R}_V(U) - \mathfrak{m}_x$, here $\mathfrak{m}_x$ is the maximal ideal of $\mathfrak{R}_V(U)$ corresponds to x. $S(B) := \bigcap_{x\in B} S(x)$ is a multiplicatively closed subset of $\mathfrak{R}_V(U)$. We define $\mathfrak{R}_V(B) := S(B)^{-1}\mathfrak{R}_V(U)$. The sheaf of rings $\mathfrak{R}_A$ on A is defined by $\mathfrak{R}_A(W) = \mathfrak{R}_V(W)$ for any Euclidian open subset W of $A \cap U$. Then $\mathfrak{R}_A$ is a coherent sheaf of rings on A (see [3]). A locally ringed space which is isomorphic to $(A, \mathfrak{R}_A)$ is said to be a *semialgebraic quasi-variety.*

If A is irreducible, i.e. V is irreducible, $(A, \mathfrak{R}_A)$ is called a *semialgebraic variety.* In this case, the field of fractions $Q(\mathfrak{R}_A(A))$ is denoted by $\mathrm{Rat}(A)$. Note that $\mathrm{Rat}(A) = \mathrm{Rat}(V)$, and $\dim A = \mathrm{tr.deg}_{\mathbb{R}} \mathrm{Rat}(A)$.

For a semialgebraic quasi-variety A, $\mathrm{Sing}(A) := \{x \in A \mid \mathfrak{R}_{A,x}$ is not a regular local ring $\}$ is called the *singular locus* of A, here $\mathfrak{R}_{A,x}$ is the stalk of $\mathfrak{R}_A$ at x. We denote $\mathrm{Reg}(A) := A - \big(\mathrm{Sing}(A) \cup \partial A\big)$. A regular map between semialgebraic quasi-varieties is defined as a morphism of locally ringed spaces as in [16]. A rational map between semialgebraic varieties is defined by the similar way as complex algebraic varieties.

Example 1.1. We denote

$$\mathbb{P}^n_+ := \big\{(X_0 : \cdots : X_n) \in \mathbb{P}^n_{\mathbb{R}} \bigm| X_iX_j \geq 0 \text{ for all } 0 \leq i < j \leq n\big\}.$$

This is a semialgebraic variety. Usually, any point $(X_0 : \cdots : X_n) \in \mathbb{P}^n_+$ is assumed to be $X_0 \geq 0, \ldots, X_n \geq 0$.

For $A = \mathbb{P}^n_+$ or $A = \mathbb{P}^n_{\mathbb{R}}$, we denote

$$\mathcal{H}_{n+1,d} := \{ f \mid f = f(X_0, \ldots, X_n) \text{ is a homogeneous polynomial of } \deg f = d.\} \cup \{0\}.$$

We usually denote $\mathcal{H}_d := \mathcal{H}_{n+1,d}$ when the index $n+1$ is clear.

We can use results of complex algebraic geometry by the virtue of the following proposition.

Proposition 1.2. (See [3]) *Let A be a semialgebraic variety.*

(1) *Then, there exists a complete complex algebraic variety X with a conjugate anti-holomorphic map $J: X \to X$ such that $\dim A = \dim X$ and that A is a semialgebraic subset of $X(\mathbb{R}) := \{ P \in X \mid J(P) = P \}$.* This X is called a *complex envelope* of A.
(2) *If X and Y are complex envelopes of A, then X and Y are birational equivalent.*
(3) *Let B be a semialgebraic variety, $\varphi: A \to B$ be a regular map, and X, Y be complex envelopes of A, B. Then, there exists a rational map $\tilde{\varphi}: X \cdots \to Y$ such that $\tilde{\varphi}|_A = \varphi$.*

Proposition 1.3. *Let A and B be semialgebraic quasi-varieties and $\varphi: A \to B$ be a regular map. Then $\varphi(A)$ is also a semialgebraic quasi-variety. If A is a semialgebraic variety, then $\varphi(A)$ is also a semialgebraic variety. Here, the structure sheaf of $\varphi(A)$ is defined similarly as [3]Definition 1.4 (see also [16] II Exercise 3.11(d)).*

Proof. This follows from Proposition 2.2.7 of [7]. (See also [3].) □

Example 1.4. Let G be a subgroup of the symmetric group $\mathfrak{S}_{n+1}$. Then, $\mathbb{P}^n_{\mathbb{R}}/G$ and $\mathbb{P}^n_+/G$ are semialgebraic varieties by the above proposition. (See also [21].)

A semialgebraic quasi-variety can be decomposed into a union of non-singular semialgebraic varieties without absolute boundaries as the following way:

Definition 1.5. (Critical decomposition) Let A be a semialgebraic quasi-variety with $\dim A = n$. We shall define $\Delta^i(A)$ ($i = 0, \ldots, n$) by induction on n. If $\dim A = 0$, then $A = \{P_1, \ldots, P_m\}$ where P_i are points. In this case we put $\Delta^0(A) = \{P_1, \ldots, P_m\}$, and put $\Delta^i(A) = \emptyset$ for $i \neq 0$.

Assume that $n = \dim A \geq 1$. Let $Z_1, \ldots, Z_r$ be all the irreducible components of A with $\dim Z_i = n$. Put $A_i := \mathrm{Int}(Z_i - \mathrm{Sing}(A))$, and $\Delta^n(A) := \{A_1, \ldots, A_r\}$. Note that $Z_i \cap Z_j \cap \mathrm{Int}(A) \subset \mathrm{Sing}(A)$ for $i \neq j$.

Let $Y_1, \ldots, Y_k$ be all the irreducible components of A with $\dim Y_j \leq n-1$, and let $B_j := Y_j - (A_1 \cup \cdots \cup A_r)$. Put

$$B := \mathrm{Sing}(A) \cup \partial A \cup B_1 \cup \cdots \cup B_k.$$

Then, we can regard B as a semialgebraic quasi-subvariety of A with the reduced structure (see [3]). Note that $\dim B < \dim A$. Thus we put $\Delta^i(A) := \Delta^i(B)$ for $i \neq n$.

We denote $\Delta(A) := \Delta^0(A) \cup \Delta^1(A) \cup \cdots \cup \Delta^n(A)$, and is called a *critical decomposition* of A. Each element $D \in \Delta(A)$ is called a *critical set* of A. Note that D is a non-singular semialgebraic variety with $\partial D = \emptyset$.

Example 1.6. $\mathbb{P}^2_+$ is isomorphic to a triangle as semialgebraic varieties. Thus, $\Delta^0(\mathbb{P}^2_+) = \{(1:0:0), (0:1:0), (0:0:1)\}$, $\Delta^2(\mathbb{P}^2_+) = \{\mathrm{Int}(\mathbb{P}^2_+)\}$, and $\Delta^1(\mathbb{P}^2_+)$ consists of three open line segments connecting two points in $\Delta^0(\mathbb{P}^2_+)$.

In our theory, we have to treat homogeneous polynomials on $\mathbb{P}^{n-1}_{\mathbb{R}}$, not on $\mathbb{R}^n$, for we need that A is compact. Thus, we need the following:

Definition 1.7.(Signed linear system) Let A be a semialgebraic quasi-variety, $\mathcal{R}_A^{\mathrm{an}}$ be the sheaf the germs of real analytic functions on A. Assume that there exists an invertible $\mathcal{R}_A$-sheaf $\mathfrak{I}$ and an invertible $\mathcal{R}_A^{\mathrm{an}}$-sheaf $\mathfrak{J}$ such that $\mathfrak{I} \otimes_{\mathcal{R}_A} \mathcal{R}_A^{\mathrm{an}} = \mathfrak{J} \otimes_{\mathcal{R}_A^{\mathrm{an}}} \mathfrak{J}$. For any point $a \in A$, we can take an affine open subset $a \in U \subset A$ such that $\mathfrak{I}|_U = \mathcal{R}_A|_U \cdot e_U^2$ by a certain $e_U \in H^0(U,\, \mathfrak{J})$. Then, for $f \in H^0(A,\, \mathcal{H})$, there exists $g_U \in H^0(U,\, \mathcal{R}_A)$ such that $f|_U = g_U e_U^{\otimes 2}$. We define $\mathrm{sign}(f(a)) \in \{0,\, \pm 1\}$ by $\mathrm{sign}(f(a)) = \mathrm{sign}(g_U(a))$. A finite dimensional subspace $\mathcal{H} \subset H^0(A,\, \mathfrak{I})$ is called a *linear system* on A.

Example 1.8. Let $A = \mathbb{P}_+^n \subset \mathbb{P}_{\mathbb{R}}^n$. Then, $\mathcal{H}_d = \mathcal{H}_{n+1,d}$ is a signed linear system on $\mathbb{P}_+^n$. If d is even, $\mathcal{H}_d$ is also a signed linear system on $\mathbb{P}_{\mathbb{R}}^n$.

Definition 1.9. Let A be a semialgebraic quasi-variety, and $\mathcal{H}$ be a linear system on A.

$$\mathrm{Bs}\,\mathcal{H} := \big\{ x \in A \;\big|\; f(x) = 0 \text{ for all } f \in \mathcal{H} \big\}$$

is called to be the *base locus* of $\mathcal{H}$. Clearly, $\mathrm{Bs}\,\mathcal{H}$ is a semialgebraic closed subset of A.

Assume that $U := A - \mathrm{Bs}\,\mathcal{H} \neq \emptyset$. Let $\{s_0, \ldots, s_N\}$ be a base of $\mathcal{H}$. Then the linear system $\mathcal{H}$ defines a regular map $\Phi : U \longrightarrow \mathbb{P}(\mathcal{H}^\vee)$ by $\Phi(x) = \big(s_0(x) : \cdots : s_N(x)\big)$ for $x \in U$. We denote this Φ by $\Phi_{\mathcal{H}} : A \cdots \to \mathbb{P}(\mathcal{H}^\vee)$.

If $\mathrm{Bs}\,\mathcal{H} = \emptyset$ and $\Phi_{\mathcal{H}} : A \longrightarrow \Phi_{\mathcal{H}}(A)$ is an isomorphism, we say $\mathcal{H}$ is *very ample*.

Proposition 1.10. *Let $G \subset \mathfrak{S}_{n+1}$, and $\pi\colon \mathbb{P}_{\mathbb{R}}^n \longrightarrow \mathbb{P}_{\mathbb{R}}^n/G$ is the natural surjection. Let $\mathcal{H}_d^G := \big\{ f \in \mathcal{H}_{n+1,d} \;\big|\; \sigma(f) = f$ for all $\sigma \in G \big\}$. Then,*
(1) *$\mathbb{P}_{\mathbb{R}}^n/G$ is a normal semialgebraic variety.*
(2) *If $d \in \mathbb{N}$ is a multiple of $\#G$, then $\pi(\mathcal{H}_d^G)$ is a very ample linear system on $\mathbb{P}_{\mathbb{R}}^n/G$.*

Proof. (1) follow from Proposition 1.2 and the theorem that if X is non-singular complex algebraic variety and a finite group G ants on X, then X/G is normal.

(2) Extend π to $\pi\colon \mathbb{P}_{\mathbb{C}}^n \to \mathbb{P}_{\mathbb{C}}^n/G$. Let H be a hyperplane of $\mathbb{P}_{\mathbb{C}}^n$, and $D := \sum_{\sigma \in G} \sigma(H)$. Then $\pi_* D$ is a very ample divisor in $\mathbb{P}_{\mathbb{C}}^n/G$. Thus, $\pi_* \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^n}(d)^G$ is a very ample invertible sheaf on $\mathbb{P}_{\mathbb{C}}^n/G$. So, $\pi(\mathcal{H}_d^G)$ is a very ample linear system on $\mathbb{P}_{\mathbb{R}}^n/G$. □

To observe G-invariant polynomials on A, it is useful to consider quotient spaces A/G (cf. [24]).

1.2. PSD Cone.

Definition 1.11. (PSD cone) Let A be a semialgebraic quasi-variety, and $0 \neq \mathcal{H}$ be a signed linear system on A. A closed convex cone

$$\mathcal{P}(A,\, \mathcal{H}) := \big\{ f \in \mathcal{H} \;\big|\; f(a) \geq 0 \text{ for all } a \in A \big\}$$

is called the *PSD cone* on A in $\mathcal{H}$.

Assume that $U := A - \mathrm{Bs}\,\mathcal{H} \neq \emptyset$. Note that $\mathcal{P}(U,\, \mathcal{H}) = \mathcal{P}(A,\, \mathcal{H})$, since $f(a) = 0$ for all $f \in \mathcal{H}$ and $a \in \mathrm{Bs}\,\mathcal{H}$. We denote

$$X(A,\, \mathcal{H}) := \mathrm{Cls}_{\mathbb{P}(\mathcal{H})}(\Phi_{\mathcal{H}}(U))$$

and we call $X(A,\, \mathcal{H})$ to be the *characteristic variety* of $\mathcal{P}(A,\, \mathcal{H})$.

Let $\mathbb{R}_+ := \big\{ \alpha \in \mathbb{R} \;\big|\; \alpha \geq 0 \big\}$, and $\{s_0, \ldots, s_N\}$ be a base of $\mathcal{H}$. Put

$$\widetilde{X}(A,\, \mathcal{H}) := \bigcup_{x \in A} \mathbb{R}_+ \cdot \big(s_0(x), \cdots, s_N(x)\big) \subset \mathbb{R}^{N+1},$$

and let $\mathfrak{C}(A, \mathfrak{H})$ be the closure of the convex cone generated by $\widetilde{X}(A, \mathfrak{H})$. $\mathfrak{C}(A, \mathfrak{H})$ is called the *characteristic cone* of $\mathfrak{P}(A, \mathfrak{H})$.

Example 1.12. Let $G \subset \mathfrak{S}_{n+1}$, and d is a positive multiple of $\#G$. Then, $X(\mathbb{P}^n_+, \mathfrak{H}^G_d) \cong \mathbb{P}^n_+/G$, and $X(\mathbb{P}^n_{\mathbb{R}}, \mathfrak{H}^G_d) \cong \mathbb{P}^n_{\mathbb{R}}/G$, by Proposition 1.10.

Proposition 1.13. *Let A be a semialgebraic quasi-variety, $\mathfrak{H}$ be a signed linear system on A, and $A - \mathrm{Bs}\,\mathfrak{H} \neq \emptyset$. Put $X := X(A, \mathfrak{H})$, and let $\mathfrak{H}_1$ be the set of linear polynomials on $\mathbb{P}(\mathfrak{H}^\vee)$ including 0. Then*

$$\Phi^*_{\mathfrak{H}} : \mathfrak{P}(X, \mathfrak{H}_1) \xrightarrow{\cong} \mathfrak{P}(A, \mathfrak{H})$$

is a linear bijective map.

Proof. $\Phi^*_{\mathfrak{H}} : \mathfrak{H}_1 \longrightarrow \mathfrak{H}$ is a linear bijective map as a result of complex algebraic geometry. Take $f \in \mathfrak{H}$. There exists a unique linear homogeneous polynomial g on $\mathbb{P}(\mathfrak{H}^\vee)$ such that $f = \Phi^*_{\mathfrak{H}}(g)$. $f(a) \geq 0$ for all $a \in A$ if and only if $g(P) \geq 0$ for all $P \in X$. Thus we have the conclusion. □

Proposition 1.14. (Semialgebraicity Theorem) *Let A be a semialgebraic quasi-variety, and $\mathfrak{H}$ be a signed linear system on A such that $A - \mathrm{Bs}\,\mathfrak{H} \neq \emptyset$. Then,*
(1) *$\mathfrak{P}(A, \mathfrak{H})$ is a semialgebraic closed convex cone in the Euclidian space $\mathfrak{H}$.*
(2) *$\mathfrak{P}(A, \mathfrak{H})$ is the dual convex cone of the characteristic cone $\mathfrak{C}(A, \mathfrak{H})$.*

Proof. (1) By Proposition 1.13, we may assume $A = X \subset \mathbb{P}^N_{\mathbb{R}}$ and $\mathfrak{H} = \mathfrak{H}_1$.

Step 1. We consider the case A is a basic semialgebraic subset: $A = \{x \in \mathbb{P}^N_{\mathbb{R}} \mid f_i(x) \geq 0$ for $i = 1,\ldots, r\}$, where $f_i \in \mathbb{R}[x_0,\ldots, x_N]$ are homogeneous polynomials of even degrees. Put $B := \{(x, y) \in \mathbb{P}^N_{\mathbb{R}} \times \mathbb{R}^{N+1} \mid f_i(x) \geq 0\ (\forall i)$, and $x \cdot y < 0\}$, where $x \cdot y = x_0 y_0 + \cdots + x_N y_N$. B is also a semialgebraic set. Let $\pi_2 : \mathbb{P}^N_{\mathbb{R}} \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}$ be the second projection $\pi_2(x, y) = y$. $\pi_2(B) = \{y \in \mathbb{R}^{N+1} \mid x \cdot y < 0$ for $\exists x \in A\}$ is also semialgebraic by Tarski-Seidenberg theorem. Thus, $\mathfrak{P}(A, \mathfrak{H}) = \mathbb{R}^{N+1} - \pi_2(\widetilde{C})$ is semialgebraic.

Step 2. We consider general semialgebraic quasi-variety $A \subset \mathbb{P}^N_{\mathbb{R}}$. There exists basic semialgebraic subset $B_1,\ldots, B_k \subset \mathbb{P}^N_{\mathbb{R}}$ such that $A = B_1 \cup \cdots \cup B_k$. Then, $\mathfrak{P}(A, \mathfrak{H}) = \mathfrak{P}(B_1, \mathfrak{H}) \cap \cdots \cap \mathfrak{P}(B_k, \mathfrak{H})$. Thus $\mathfrak{P}(A, \mathfrak{H})$ is semialgebraic.

(2) Similar. □

Definition 1.15. (Face component) Let C be a semialgebraic closed convex set of $\mathbb{R}^N$ or $\mathbb{P}^N_{\mathbb{R}}$ with $\dim C = N$. Take $D \in \Delta^{N-1}(C)$ as Definition 1.5. Then $\overline{D}$ is called a *face component* of C or of ∂C. The defining equation of the hypersurface $\mathrm{Zar}(D)$ is called a *discriminant* of D or of C and denoted by $\mathrm{disc}(D)$.

What we should do is to determine all the face components of $\mathfrak{P}(X, \mathfrak{H}_1)$. The following proposition clarifies the geometric meaning of equality conditions of algebraic inequalities. A convex cone $C \subset \mathbb{R}^N$ is said to be *non-degenerate* if $\dim C = N$.

Proposition 1.16. (Boundary Theorem) *Let A be a compact semialgebraic quasi-variety, and $0 \neq \mathfrak{H}$ be a signed linear system on A. Assume that $\mathfrak{P} := \mathfrak{P}(A, \mathfrak{H}) \subset \mathfrak{H}$ is non-degenerate. Let $f \in \mathfrak{P}$.*
(1) *If $f(a) = 0$ for a certain $a \in A - \mathrm{Bs}\,\mathfrak{H}$, then $f \in \partial\mathfrak{P}$.*
(2) *If $f \in \partial\mathfrak{P}$, then there exists $a \in A$ such that $f(a) = 0$.*

Proof. We can reduce to the case A is irreducible, and $\mathcal{H} \subset \operatorname{Rat}(A)$, since $\mathcal{P}(A_1 \cup A_2, \mathcal{H}) = \mathcal{P}(A_1, \mathcal{H}) \cap \mathcal{P}(A_2, \mathcal{H})$.

(1) Since $a \notin \operatorname{Bs}\mathcal{H}$, there exists $g \in \mathcal{P}$ such that $g(a) > 0$. Then for all $\varepsilon > 0$, $f(a) - \varepsilon g(a) < 0$. This means $f - \varepsilon g \notin \mathcal{P}$. Thus $f \in \partial\mathcal{P}$.

(2) Assume that $f \in \mathcal{P}$ satisfies $f(a) > 0$ for all $a \in A$. Then, $f \pm \varepsilon g \in \mathcal{P}$ for any $g \in \mathcal{P}$, and $0 < \varepsilon \ll 1$. Thus $f \notin \partial\mathcal{P}$. □

Definition 1.17. (Dual variety) Let $\mathbb{P} = \mathbb{P}^N_{\mathbb{R}}$ and $\mathbb{P}^\vee$ be the set of all the hyperplanes in $\mathbb{P}$. Assume that $D \subset \mathbb{P}$ is a non-singular semialgebraic variety with $\partial D = \emptyset$ (i.e. $\Delta(D) = \{D\}$). For $x \in D$, let $T_{D,x} := T_{\operatorname{Zar}(D),x} \subset \mathbb{P}$ be the tangent space of $\operatorname{Zar}(D)$ at x. Then,

$$D^\vee := \left\{ H \in \mathbb{P}^\vee \;\middle|\; H \supset T_{D,x} \text{ for a certain } x \in D \right\}$$

is called the *dual variety* of D. Since D is irreducible and non-singular, $D^\vee$ is irreducible. Thus $D^\vee$ is a semialgebraic variety. Note that $D^\vee$ may have singularities.

Theorem 1.18. *Let $X \subset \mathbb{P} = \mathbb{P}^n$ be a closed semialgebraic quasi-variety, $\mathcal{P} := \mathcal{P}(X, \mathcal{H}_1)$, and $\pi : (\mathcal{H}_1 - \{0\}) \to \mathbb{P}(\mathcal{H}_1)$ be the natural surjection. Put $\mathbb{P}(\mathcal{P}) := \pi(\mathcal{P} - \{0\}) \subset \mathbb{P}(\mathcal{H}_1)$. Note that $\mathbb{P}(\mathcal{H}_1) = \mathbb{P}^\vee$, since $\mathbb{P} = \mathbb{P}(\mathcal{H}_1^\vee)$. Then,*

$$\partial\mathbb{P}(\mathcal{P}) \subset \bigcup_{D \in \Delta(X)} D^\vee.$$

Proof. Take $0 \neq f \in \partial\mathcal{P} \subset \mathcal{H}_1$. Let $H_f \subset \mathbb{P} = \mathbb{P}(\mathcal{H}_1^\vee)$ be the hyperplane corresponds to f. Since $\operatorname{Bs}\mathcal{H}_1 = \emptyset$, $f(x) = 0$ for a certain $x \in X$. There exists $D \in \Delta(X)$ such that $x \in D$. Since $f(y) \geq 0$ for all $y \in D$, we conclude that $H_f \supset T_{D,x}$. Thus $H_f \in D^\vee$. □

Note. If $D \in \Delta(X)$ satisfies $X \cap \operatorname{Int}(\mathbb{P}(\mathcal{C}(X, \mathcal{H}_1))) \neq \emptyset$, then $D^\vee \not\subset \mathcal{P}$ by Proposition 1.14(2).

Definition 1.19. Use the same notation with Theorem 1.18. For $D \in \Delta(X)$, we denote

$$\mathcal{F}(D) := \operatorname{Cls}_{\mathcal{H}_1}(\pi^{-1}(D^\vee) \cap \partial\mathcal{P}).$$

Note that $\mathcal{F}(D)$ and $D^\vee$ have the same discriminants.

Example 1.20. Let d be a positive even integer, $\mathcal{H}_d := \mathcal{H}_{n,d}$, $\mathcal{P}_{n,d} := \mathcal{P}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathcal{H}_d)$, and $X_{n,d} := X(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathcal{H}_d)$. Since $\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(d)$ is very ample, we have $X_{n,d} \cong \mathbb{P}^{n-1}_{\mathbb{R}}$, $\Delta^{n-1}(X_{n,d}) = \{X_{n,d}\}$, and $\Delta^i(X_{n,d}) = \emptyset$ for $i \neq n-1$. Thus $\partial\mathcal{P}_{n,d}$ is irreducible. That is, the defining equation of $\operatorname{Zar}(\partial\mathcal{P}_{n,d})$ is irreducible. (Compare with Chapter 4 and 5 of [6].)

Remark 1.21. (1) Under the same assumption with Theorem 1.18, assume that $X = X(A, \mathcal{H})$, $P \in A$, and $x := \Phi_{\mathcal{H}}(P) \in D \in \Delta^r(X)$. Let $B := \Phi_{\mathcal{H}}^{-1}(D) \subset A$. Assume that $B \to D$ is a finite unramified morphism, and there exists a local coordinate system $(t_1, \ldots, t_r)$ at a certain neighborhood of P in B.

Let $\{s_0, \ldots, s_N\}$ be a base of $\mathcal{H}$. Identify $p_0 s_0 + \cdots + p_N s_N \in \mathcal{H}$ with $(p_0, \ldots, p_N) \in \mathbb{R}^{N+1}$. We take $(p_0 : \cdots : p_N)$ as a homogeneous coordinate system of $\mathbb{P}(\mathcal{H})$. Then,

$$T_{D,x} := \left\{ \left(\cdots : s_i(P) + \sum_{j=1}^{r} v_j \frac{\partial s_i}{\partial t_j}(P) : \cdots \right) \in \mathbb{P}^n_{\mathbb{R}} \;\middle|\; (v_1, \ldots, v_r) \in \mathbb{R}^r \right\}.$$

Thus, $\mathrm{disc}(D) = \mathrm{disc}_D(p_0,\dots, p_N)$ of $\mathrm{Zar}(D^\vee)$ can be obtained eliminating $t_1,\dots, t_r$ from the system of equations

$$\sum_{i=0}^{N} p_i s_i(P(t_1,\dots,t_r)) = 0 \quad \text{and} \quad \sum_{i=0}^{N} p_i \frac{\partial s_i}{\partial t_j}(P(t_1,\dots,t_r)) = 0 \quad (j = 1,\dots, r).$$

Here $P(t_1,\dots, t_r)$ is the function which represent the coordinate of P.

(2) Especially, consider the special case $\dim D = N-1$. Then, $\mathrm{Zar}(D)$ is a hypersurface of $\mathbb{P}^N_{\mathbb{R}}$ defined by a certain irreducible polynomial $h(x_0,\dots, x_N)$. Let $h_i = \dfrac{\partial h}{\partial x_i}$. Then

$$T_{D,x} = \bigl\{(x_0 : \cdots : x_N) \in \mathbb{P}^N_{\mathbb{R}} \bigm| h_0(P)x_0 + \cdots + h_N(P)x_N = 0\bigr\}.$$

Thus, $\mathrm{disc}(D) = \mathrm{disc}_D(p_0,\dots, p_N)$ of $\mathrm{Zar}(D^\vee)$ can be obtained eliminating $x_0,\dots, x_N$ from the system of equations $p_0x_0 + \cdots + p_Nx_N = 0$ and $p_i = h_i(x_0,\dots, x_N)$ $(i = 0,\dots, N)$.

(3) Assume that $D \in \Delta^0(X)$, and $x = D = (b_0 : \cdots : b_N)$. Then, $\mathrm{Zar}(P^\vee)$ is the hyper plane defined by $b_0p_0 + \cdots + b_Np_N = 0$. Thus, $\mathrm{disc}(D) = b_0p_0 + \cdots + b_Np_N$.

Lemma 1.22. *Let V be a non-singular complete real algebraic variety, and A be an open subset of V such that $\mathrm{Int}(\mathrm{Cls}_V(A)) = A$. If $\partial_a A$ is a union of hypersurfaces of V, then $\mathrm{Cls}_V(A)$ is a semialgebraic subset of V.*

Proof. We regard $\partial_a A$ to be a reduced divisor D. There exists a composition of blowing ups $\varphi\colon Y \longrightarrow V$ such that $\mathrm{Sing}(Y) = \emptyset$ and φ^*D is a normal crossing divisor. It is enough to show that $\overline{\varphi^{-1}(A)}$ is a semialgebraic subset of Y.

Take a point $P \in Y$. Choose an affine open subset $P \in W \subset Y$, and take a distance function d on W. For $\varepsilon > 0$ $(\in \mathbb{R})$, let $B_p(\varepsilon) := \bigl\{Q \in W \bigm| d(P,Q) \le \varepsilon\bigr\}$. Since Y is compact, it is enough to prove that for any $P \in \overline{\varphi^{-1}(A)}$, there exists $\varepsilon > 0$ such that $\overline{\varphi^{-1}(A)} \cap B_p(\varepsilon)$ is a semialgebraic set. Since D is normal crossing, we can choose an analytic local coordinate system $(x_1,\dots, x_n)$ at $P \in U$ such that $D \cap B_P(\varepsilon) = \bigl(V(x_1) \cup \cdots \cup V(x_m)\bigr) \cap B_P(\varepsilon)$. Note that x_i is not always a polynomial but an analytic function.

Let f be the defining polynomial of $(\mathrm{Supp}\, D) \cap B_P(\varepsilon)$. Since D is reduced, we may assume that $f = x_1 \cdots x_m$. Let $\mathbf{s} = (s_1,\dots, s_m) \in \{\pm 1\}^m$, and let $Q_{\mathbf{s}}$ be the subset of $B_P(\varepsilon)$ defined by $s_1x_1 \ge 0,\dots, s_mx_m \ge 0$. $(\partial Y) \cap B_P(\varepsilon)$ is a union of some $Q_{\mathbf{s}}$. Thus, it is enough to prove every $Q_{\mathbf{s}}$ is a semialgebraic set. We may assume $f \ge 0$ on $Q_{(1,\dots,1)}$. Then $s_1 \cdots s_m f \ge 0$ on $Q_{(s_1,\dots,s_m)}$.

Let $(y_1,\dots, y_n)$ be an algebraic coordinate system on $B_P(\varepsilon)$. We may assume $x_i = y_i + g_i(y_1,\dots, y_n)$ for $1 \le i \le m$, where g_i is a power series with $\mathrm{ord}\, g_i \ge 2$. Let $V \subset B_P(\varepsilon)$ be the set defined by $x_1 = \cdots = x_m = 0$. V is an algebraic variety, since V is defined by $y_1 = \cdots y_m = 0$ if ε is sufficiently small. In other word, V is a subvariety of $\mathrm{Sing}(D) \cap B_p(\varepsilon)$.

Since $s_1 \cdots s_m f \ge 0$ on $Q_{(s_1,\dots,s_m)}$, we have

$$Q_{\mathbf{s}} = \bigl\{\mathbf{y} \in B_p(\varepsilon) \bigm| s_1 \cdots s_m f(y) \ge 0,\, s_1y_1 + \cdots + s_my_m \ge 0\bigr\}.$$

Therefore $Q_{\mathbf{s}}$ is a basic semialgebraic set. □

Remark 1.23. Let A be a semialgebraic quasi-variety, and $0 \ne \mathcal{H}$ be a signed linear system on A. Theoretically, we can determine $\mathcal{P} = \mathcal{P}(A, \mathcal{H})$ by the following algorithm.

Step 1. Determine $X = X(A, \mathcal{H})$ and $\Delta(X)$, using algebraic geometry.

Step 2. For every $D \in \Delta(X)$, calculate the dual variety $\mathrm{Zar}(D^\vee) = \mathrm{Zar}(D)^\vee$, according to Remark 1.21. Usually, we need a computer here.

Step 3. $S := \bigcup_{D \in \Delta(X)} \mathrm{Zar}(D)^\vee$ cuts $\mathcal{H}$ into blocks $B_1, \ldots, B_k$ such that $\partial B_i \subset S$ and $\mathrm{Int}(B_i) \cap S = \emptyset$. Find out convex cone $\mathcal{P}$ which is a union of some $\overline{B_i}$. If there exists $f \in B_i$ such that $f(a) < 0$ for a certain $a \in A$, then $\mathrm{Int}(B_i) \cap \mathcal{P} = \emptyset$. Contrary, if there exists $0 \neq f \in B_i$ such that $f \in \mathcal{P}$, then $\overline{B_i} \subset \mathcal{P}$. Note that each block B_i is semialgebraic by Lemma 1.22.

Step 4. Obtained $\mathcal{P}$ may not be a basic semialgebraic set. In such a case, find out a nice decomposition of $\mathcal{P}$ into basic semialgebraic subsets, since we want to describe $\mathcal{P}$ by a system of inequalities.

1.3. Local cone.

In Remark 1.23, every step is not so easy, when $\dim \mathcal{H}$ is not small. We need some more tools to execute the above steps. The following idea is useful to reduce to lower dimensional case and to find extremal inequalities.

Definition 1.24. (Local Cone) Let A be a semialgebraic quasi-variety, and $\mathcal{H}$ be a signed linear system on A. For a subset $I \subset A$, we put

$$\mathcal{H}_I := \left\{ f \in \mathcal{H} \;\middle|\; f(a) = 0 \text{ for all } a \in I \right\}, \quad \mathcal{P}_I := \mathcal{P} \cap \mathcal{H}_I = \mathcal{P}(A, \mathcal{H}_I).$$

We call $\mathcal{P}_I$ to be the *local cone* at I. If $I = \{a\}$, we denote $\mathcal{P}_{\{a\}}$ by $\mathcal{P}_a$.

Proposition 1.25. (1) *$\mathcal{P}_I$ is a semialgebraic closed convex cone in $\mathcal{H}_I$.*
(2) $\mathcal{P}_I = \mathcal{P}(A, \mathcal{H}_I)$.
(3) *Let I and J be subsets of A. Then $(\mathcal{P}_I)_J = \mathcal{P}_{(I \cup J)}$.*

Proof. Trivial. □

Let $C \subset \mathbb{R}^n$ be a closed convex cone, and $0 \neq f \in C$. We say f is *extremal* in C if g, $h \in C$ and $f = g + h$ then g and h are multiples of f.

Proposition 1.26. (Local Cone Theorem)
Let A be a compact semialgebraic quasi-variety, and $0 \neq \mathcal{H}$ be a signed linear system on A. Then,
(1) $\partial \mathcal{P} \subset \bigcup_{a \in A} \mathcal{P}_a$.
(2) *If $I \not\subset \mathrm{Bs}\,\mathcal{H}$, then $\mathcal{P}_I \subset \partial \mathcal{P}$.*
(3) *Assume that A is irreducible and $A - \mathrm{Bs}\,\mathcal{H} \neq \emptyset$. Let $\mathcal{U} := \bigcup_{a \in A - \mathrm{Bs}\,\mathcal{H}} \mathcal{P}_a$. Then $\overline{\mathcal{U}} = \partial \mathcal{P}$.*
(4) *Let $0 \neq f \in \mathcal{P}_a$. f is extremal in $\mathcal{P}$ if and only if f is extremal in $\mathcal{P}_a$.*

Proof. (1) Take $0 \neq f \in \partial \mathcal{P}$. Then, $f(a) = 0$ for a certain $a \in A$ by Proposition 1.16. Thus $f \in \mathcal{P}_a$.

(2) Take $f \in \mathcal{P}_I$. There exists $a \in I - \mathrm{Bs}\,\mathcal{H}$ such that $f(a) = 0$, since $I \not\subset \mathrm{Bs}\,\mathcal{H}$. Then $f \in \partial \mathcal{P}$ by Proposition 1.16.

(3) By (2), $\mathcal{U}$ contains a non empty open subset of $\partial \mathcal{P}$. Since $\mathrm{Bs}\,\mathcal{H}$ is a Zariski closed subset of A, $\dim(\partial \mathcal{P} - \mathcal{U}) < \dim \partial \mathcal{P}$. Thus $\overline{\mathcal{U}} = \partial \mathcal{P}$.

(4) If f is extremal in $\mathcal{P}$, it is clear that f is extremal in $\mathcal{P}_a$.

Assume that $f \in \mathcal{P}_a$ is not extremal in $\mathcal{P}$. Then, there exist g, $h \in \mathcal{P} - \mathbb{R}_+ \cdot f$ such that $f = g + h$. Since $g(a) \geq 0$, $h(a) \geq 0$ and $g(a) + h(a) = f(a) = 0$, we have $g(a) = h(a) = 0$. Thus g, $h \in \mathcal{P}_a$. Therefore, f is not extremal in $\mathcal{P}_a$. □

Proposition 1.27. (Face Component Theorem) *Let X be a closed semialgebraic subset of $\mathbb{P}^N_{\mathbb{R}}$ such that X is not included in any proper linear subspace of $\mathbb{P}^N_{\mathbb{R}}$. Assume that $\mathcal{P} := \mathcal{P}(X, \mathcal{H}_1)$ is non-degenerate in $\mathcal{H}_1$. Take $x \in D \in \Delta^r(X)$.*
(1) $\dim \mathcal{P}_x \leq N - r$.
(2) $\mathcal{F}(D) = \mathrm{Cls}\left(\bigcup_{x \in D} \mathcal{P}_x\right)$.

Proof. For $f \in \mathcal{H} := \mathcal{H}_1$, let H_f be the hyperplane in $\mathbb{P}(\mathcal{H})$ defined by $f = 0$.

(1) Since $\mathcal{P}$ is non-degenerate, $\dim(U \cap \mathcal{P}) = N+1$ for any Euclidean open neighborhood U of P. Note that $\dim T_{D,x} = \dim D = r$, since D is non-singular. The condition $T_{D,x} \subset H_f$ means that f passes through independent $r+1$ points. Thus $\dim \mathcal{P}_x = \dim \mathcal{P} - (r+1) \leq N - r$.

(2) $\supset$ is clear. We prove $\subset$. Take $f \in D^\vee \subset \mathrm{Int}(\mathcal{F}(D))$. Then, $f(x) = 0$ for a certain $x \in D$. That is, $f \in \mathcal{P}_x$. □

Not that $\dim \mathcal{H}_x = \dim \mathcal{H} - 1 = N$. If $r \geq 0$, $\dim \mathcal{P}_x \leq N - r$. Thus $\mathcal{P}_x$ is degenerate in $\mathcal{H}_x$.

Remark 1.28. We provide an algorithm to obtain the base of $\mathrm{Zar}(\mathcal{P}_P)$. This algorithm helps us to find extremal inequalities. We use the same symbols as Remark 1.21. Assume that $\mathcal{P} = \mathcal{P}(A, \mathcal{H})$ and $X = X(A, \mathcal{H})$ satisfy the condition of Proposition 1.27. Let $D \in \Delta^r(X)$ and $P \in B = \Phi_{\mathcal{H}}^{-1}(D)$. Assume that $B \to D$ is a finite unramified morphism, and there exists a local coordinate system $(t_1, \ldots, t_r)$ at a certain neighborhood of P in B. Take $f = \sum_{i=0}^{N} p_i s_i \in \mathcal{P}_P$. Since $f \in \mathcal{H}_P$, we have

$$\sum_{i=0}^{N} p_i s_i(P(t_1, \ldots, t_r)) = 0. \tag{1.29}$$

Since $(p_0, \ldots, p_N)$ is perpendicular to $T_{D, \Phi_{\mathcal{H}}(P)}$, we have

$$\sum_{i=0}^{N} p_i \frac{\partial s_i}{\partial t_j}(P(t_1, \ldots, t_r)) = 0 \tag{1.30}$$

for $j = 1, \ldots, r$. Consider (1.29), (1.30) as a system of linear equations on $(p_0, \ldots, p_N)$. Then, its solution space is just $\mathrm{Zar}(\mathcal{P}_P)$ if $\mathcal{P}_P \neq 0$. Note that (1.29), (1.30) are the same system of equations with Remark 1.21(1).

1.4. Relative theorems.

Proposition 1.31. (Relative theorem for $\mathcal{H}$) *Let A be a compact semialgebraic quasi-variety, $\mathcal{H}$ be signed linear system on A, and $\mathcal{P} := \mathcal{P}(A, \mathcal{H})$. For a subspace $0 \neq \mathcal{H}' \subset \mathcal{H}$, let $\mathcal{P}' := \mathcal{P}(A, \mathcal{H}')$. Then,*
(1) *$\mathcal{P}' = \mathcal{P} \cap \mathcal{H}'$ and $\partial \mathcal{P}' \subset \partial \mathcal{P} \cap \mathcal{H}'$.*
(2) *If $\mathrm{Bs}\,\mathcal{H}' = \emptyset$ and $\mathcal{P}$ is non-degenerate, then $\mathcal{P}'$ is non-degenerate.*
(3) $\mathcal{P}'_x = \mathcal{P}_x \cap \mathcal{H}'$ for any $x \in A$.
(4) *Let $0 \neq f \in \mathcal{P}'$. If f is an extremal element of $\mathcal{P}$, then f is an extremal element of $\mathcal{P}'$.*

Proof. (1), (3) and (4) are clear.

(2) If $\mathfrak{P}'$ degenerates, then its dual cone $\mathfrak{C}(A,\, \mathfrak{H}')$ contains a line L. Let $\rho : \mathfrak{C}(A,\, \mathfrak{H}) \longrightarrow \mathfrak{C}(A,\, \mathfrak{H}')$ be the map induced from the inclusion map $\mathfrak{H}' \subset \mathfrak{H}$. Since ρ is a linear projection, $\rho^{-1}(L)$ contains a line. This implies that $\mathfrak{P}$ degenerates. □

Proposition 1.32. (Relative theorem for A) *Let A be a compact semialgebraic variety, and $B \subset A$ be a closed semialgebraic subvariety such that $\mathrm{Zar}_A(B) = A$. Let $\mathfrak{H}$ be a signed linear system on A. Let $Y := X(B,\, \mathfrak{H})$, $X := X(A,\, \mathfrak{H})$, $D_A := \mathrm{Reg}(X) \in \Delta(X)$, and $D_B := \mathrm{Reg}(Y) \in \Delta(Y)$, $\mathfrak{P} := \mathfrak{P}(A,\, \mathfrak{H})$, and $\mathfrak{P}' := \mathfrak{P}(B,\, \mathfrak{H})$. Assume that $\mathfrak{P}$ is non-degenerate and $\dim \mathfrak{F}(D_B) = \dim \mathfrak{P}' - 1$. Then $\mathrm{Zar}(\mathfrak{F}(D_A)) = \mathrm{Zar}(\mathfrak{F}(D_B))$.*

Proof. Since $B \subset A$, we have $\mathfrak{P}' \supset \mathfrak{P}$. Put $N := \dim \mathfrak{H} - 1$. Since $N = \dim \mathfrak{P} \leq \dim \mathfrak{P}' = \dim \mathfrak{F}(D_B) \leq \dim \mathfrak{F}(D_A) \leq N$, these agree. Since $B \subset A$, we have $D_B \subset D_A$, $D_A^\vee \subset D_B^\vee$, and $\mathfrak{F}(D_A) \subset \mathfrak{F}(D_B)$. Since $D_A^\vee$ and $D_B^\vee$ are irreducible, and $\dim \mathfrak{F}(D_B) = \dim \mathfrak{F}(D_A) = N$, we have $\mathrm{Zar}(\mathfrak{F}(D_A)) = \mathrm{Zar}(\mathfrak{F}(D_B))$. □

Proposition 1.33. (Closure Theorem) *Let $C \subset \mathbb{R}^m$ be a semialgebraic closed convex cone.*

(1) *Let F be a face component of C, and let $P,\, Q,\, R \in \mathbb{R}^m$ are distinct points such that Q is in the interior of the line segment PR. If $P \in C$, $P \notin F$ and $Q \in F$, then $R \notin C$.*

(2) *Assume that C contains no lines. Let $F_0,\, F_1, \ldots,\, F_r$ be face components of C such that $\partial C = F_0 \cup F_1 \cup \cdots \cup F_r$. Then*

$$\partial F_0 = (F_1 \cup F_2 \cup \cdots \cup F_r) \cap F_0.$$

Proof. (1) If $R \in C$, then $P,\, R \in F$.

(2) Trivial. □

1.5. The cases $A = \mathbb{P}^n_{\mathbb{R}}$ and $A = \mathbb{P}^n_+$.

Note that a local cone $\mathfrak{P}_a$ degenerate and $\mathrm{Bs}\, \mathfrak{H}_a \ni a$. To apply Proposition 1.16, 1.27 and 1.31, we need the following:

Proposition 1.34.(Non-degeneracy Theorem) *Assume that '$d \in \mathbb{N}$ and $A = \mathbb{P}^n_+$', or, '$d \in 2\mathbb{N}$ and $A = \mathbb{P}^n_{\mathbb{R}}$'. Assume that $\mathfrak{H} \subset \mathfrak{H}_{n+1,d}$, and $\mathrm{Bs}\, \mathfrak{H} = \emptyset$. Then $\mathfrak{P}(A,\, \mathfrak{H})$ is non-degenerate.*

Proof. (1) First, we consider the case $\mathfrak{H} = \mathfrak{H}_{n+1,d}$. We denote the coordinate system of $\mathbb{P}^n_{\mathbb{R}}$ by $(a_0 : \cdots : a_n)$. Let $s_0 := \sum_{i=0}^{n} a_i^d \in \mathfrak{H}$. Note that $s_0 \in \mathfrak{P}(A,\, \mathfrak{H})$.

Assume that $\mathfrak{P}$ is degenerate. Then, its dual cone $\mathfrak{C}(A,\, \mathfrak{H})$ contains a line L passing through the origin O. There exists two points $P,\, Q \in L \cap \widetilde{X}(A,\, \mathfrak{H})$ such that O is contained in a line segment PQ. Let $p,\, q \in A$ be points correspond to $P,\, Q$. Then $s_0(p)s_0(q) \leq 0$. Since $s_0 \geq 0$, $s_0(p) = s_q(q) = 0$. This implies $p = q = 0$ and $P = Q$. A contradiction.

(2) The general case follows from Proposition 1.31. □

Let A be a non-singular semialgebraic variety. Consider the case a finite group G acts on A. Let $\pi \colon A \longrightarrow A/G$ be the natural surjection. We denote

$$A^G := \{\mathbf{a} \in A \mid \sigma(\mathbf{a}) = \mathbf{a} \text{ for all } \sigma \in G\}.$$

Note that $\mathrm{Sing}(A/G) \subset \pi(A^G)$.

Example 1.35. Let $A = \mathbb{P}^n_{\mathbb{R}}$ and $G = \mathbb{Z}/(n+1)\mathbb{Z}$. Then, $A^G = \{\mathbf{1}\}$, where $\mathbf{1} := (1 : 1 : \cdots : 1)$. $\operatorname{Sing}(\mathbb{P}^n_{\mathbb{R}}/G) = \{\pi(\mathbf{1})\}$.

Proposition 1.36. *Let $A = \mathbb{P}^n_{\mathbb{R}}$ or $A = \mathbb{P}^n_{+}$, and $G \subset \mathfrak{S}_{n+1}$. Put $g := \#G$, and $X^G_d := X(A, \mathfrak{H}^G_d)$.*
(1) *If $d = kg + 2m$ ($k \geq 1$, $m \geq 0$) and $\operatorname{Bs}\mathfrak{H}^G_d = \emptyset$, then $A/G \cong X^G_d$.*
(2) *If $A = \mathbb{P}^n_{+}$, $d \geq g$ and $\operatorname{Bs}\mathfrak{H}^G_d = \emptyset$, then $A/G \cong X^G_d$.*

Proof. For $f \in \mathfrak{H}^G_d$, we put $V(f) := \{x \in A \mid f(x) = 0\}$. Note that $\Phi_{\mathfrak{H}^G_d} : A \longrightarrow X^G_d$ factors as $A \xrightarrow{\pi} A/G \xrightarrow{\Psi^G_d} X^G_d$.

(1) We may assume that $A = \mathbb{P}^n_{\mathbb{R}}$. If $d = kg$, then $\mathbb{P}^n_{\mathbb{R}}/G \cong X^G_d$, by Proposition 1.10.

Consider the case $d = kg + 2m$. Let $S_2 := a_0^2 + \cdots + a_n^2$. We define an injection

$$\iota : \mathfrak{H}^G_{kg} \xrightarrow{\times S_2^m} \mathfrak{H}^G_d$$

by $\iota(f) = fS_2^m$. Since $V(S_2^m) = V(S_2) = \emptyset$, there exists the regular map $\rho : X^G_d \longrightarrow X^G_{kg}$. Note that $\Phi_{\mathfrak{H}^G_{kg}} = \rho \circ \Phi_{\mathfrak{H}^G_d}$. Since Ψ^G_{kg} is an isomorphism, Ψ^G_d is also an isomorphism.

(2) Consider the case $d = n + 1 + l$. Let $S_1 := a_0 + \cdots + a_n$. We define an injection $\iota : \mathfrak{H}^G_g \xrightarrow{\times S_1^{d-g}} \mathfrak{H}^G_d$ by $\iota(f) = fS_1^{d-g}$. Since $A \cap V(S_1^{d-g}) = \emptyset$, there exists the regular map $\rho : X^G_d \longrightarrow X^G_g$. The left part is similar as (1). ∎

Section 2. Cyclic and Symmetric inequalities.

2.1. Cyclic inequalities of three variables.

Consider typical problems:
(1) Probe that $f(a, b, c) \geq 0$ for all $a, b, c \in \mathbb{R}$.
(2) Probe that $f(a, b, c) \geq 0$ for all $a \geq 0$, $b \geq 0$, $c \geq 0$.

We shall study the case that f is a cyclic homogeneous polynomial of degree d. Let $G = \mathbb{Z}/3\mathbb{Z}$, and $A = \mathbb{P}^2_{\mathbb{R}}$ or $A = \mathbb{P}^2_{+}$. We denote the homogeneous coordinate system of A by $(a : b : c)$. Let

$$\mathfrak{H}^c_d := (\mathfrak{H}_{3,d})^G.$$

Problem (1) is study of $\mathfrak{P}(\mathbb{P}^2_{\mathbb{R}}, \mathfrak{H}^c_d)$, and (2) is of $\mathfrak{P}(\mathbb{P}^2_{+}, \mathfrak{H}^c_d)$. We shall denote

$$\mathfrak{P}^c_d := \mathfrak{P}(\mathbb{P}^2_{\mathbb{R}}, \mathfrak{H}^c_d), \quad \mathfrak{P}^{c+}_d := \mathfrak{P}(\mathbb{P}^2_{+}, \mathfrak{H}^c_d).$$

By Proposition 1.34, these are non-degenerate. The following are typical elements of $\mathfrak{H}^c_d$.

$$S_{i,j,k}(a, b, c) := a^i b^j c^k + b^i c^j a^k + c^i a^j b^k$$
$$S_{i,j}(a, b, c) := a^i b^j + b^i c^j + c^i a^j, \quad S_i(a, b, c) := a^i + b^i + c^i, \quad U(a, b, c) := abc,$$
$$T_{i,j,k}(a, b, c) := S_{i,j,k}(a, b, c) + S_{j,i,k}(a, b, c), \quad T_{i,j}(a, b, c) := T_{i,j,0}(a, b, c).$$

We usually omit (a, b, c) if the variables are a, b, c, e.g. $S_i = a^i + b^i + c^i$, $T_{i,j} = T_{i,j}(a, b, c)$.

Take the set I_d of indices (i, j, k) such that $B_d := \{S_{i,j,k} \mid (i, j, k) \in I_d\}$ form a basis of $\mathfrak{H}^c_d$. Let $N := \dim \mathfrak{H}^c_d - 1$ and

$$|(i, j, k)| := \max\{|i - j|, |i - k|, |j - k|\}.$$

Align all the elements of B_d as $s_0, \ldots, s_N$ so that $s_0 = a^d + b^d + c^d$, and that $s_N = S_{i,j,k}$ with the minimum $|(i, j, k)|$. For example, we can choose $I_3 = \{(3, 0, 0), (2, 1, 0), (1, 2, 0), (1, 1, 1)\}$, and $I_4 = \{(4, 0, 0), (3, 1, 0), (1, 3, 0), (2, 2, 0), (2, 1, 1)\}$.

Proposition 2.1. (1) $\dim \mathcal{H}_d^c = \lceil (d+1)(d+2)/6 \rceil$.
(2) $\mathrm{Bs}\, \mathcal{H}_d^c = \emptyset$ *for* $d \geq 3$.

Proof. (1) follows from $\# I_d = \lceil (d+1)(d+2)/6 \rceil$.

(2) By Proposition 1.10(2), $\mathrm{Bs}\, \mathcal{H}_{3k}^c = \emptyset$. $\mathrm{Bs}\, \mathcal{H}_4^c \subset V(S_4) = \emptyset$. By $\iota : \mathcal{H}_d^c \xrightarrow{\times S_2} \mathcal{H}_{d+2}^c$, we have $\mathrm{Bs}\, \mathcal{H}_{d+2}^c \subset \mathrm{Bs}\, \mathcal{H}_d^c$. ◻

Here, we summarize some results of high school algebra — special cases of Muirhead inequality (see [19]).

Proposition 2.2. (1) *For all* a, b, $c \in \mathbb{R}$ *and* $-1 \leq k \leq 2$, *the following hold:*
$2S_4 \geq T_{3,1}$, $S_4 \geq S_{2,2} \geq US_1$,
$2S_6 \geq T_{4,2} \geq 2US_3$, $T_{4,2} \geq UT_{2,1}$, $T_{4,2} \geq 6U^2$.
$S_2 + kS_{1,1} \geq 0$.
(2) *For* $a \geq 0$, $b \geq 0$, $c \geq 0$, *the following hold:*
$2S_3 \geq T_{2,1} \geq 6U$,
$3S_4 \geq T_{3,1} \geq 2S_{2,2} \geq 2US_1$,
$2S_5 \geq T_{4,1} \geq T_{3,2} \geq 2US_2 \geq 2US_{1,1}$,
$2S_6 \geq T_{5,1} \geq T_{4,2} \geq 2US_3 \geq UT_{2,1} \geq 6U^2$,
$2S_{3,3} \geq UT_{2,1} \geq 6U^2$.

All of the above inequalities $f(a,b,c) \geq 0$ satisfy the condition $f(a,a,a) = 0$. We will find that this equality condition has special meaning. So, let

$$\mathcal{H}_d^{c0} := \left\{ f \in \mathcal{H}_d^c \;\middle|\; f(1,1,1) = 0 \right\}, \quad \mathcal{P}_d^{c0} := \mathcal{P}(\mathbb{P}_{\mathbb{R}}^2, \mathcal{H}_d^{c0}), \quad \mathcal{P}_d^{c0+} := \mathcal{P}(\mathbb{P}_{+}^2, \mathcal{H}_d^{c0}).$$

For $f \in \mathcal{H}_d^c$, we say f is *monic* if the coefficient of S_d in f is equal to 1. We say f lies on *infinity* if the coefficient of S_d in f is equal to 0.

Proposition 2.3. (1) $\dim \mathcal{H}_d^{c0} = \lceil (d+1)(d+2)/6 \rceil - 1$.
(2) $\mathrm{Bs}\, \mathcal{H}_d^{c0} = \{(1:1:1)\}$ *for* $d \geq 3$.

Proof. Easy exercise. ◻

Proposition 2.4. *Let* $G = \mathbb{Z}/3\mathbb{Z}$, $\pi \colon \mathbb{P}_{\mathbb{R}}^2 \to \mathbb{P}_{\mathbb{R}}^2/G$ *be the natural surjection, and* $L := \left\{ (0:s:1) \in \mathbb{P}_{+}^2 \;\middle|\; s > 0 \right\}$. *Then,*
(1) $\mathbb{P}_{\mathbb{R}}^2/G$ *is a normal real algebraic surface, and* $\Delta^0(\mathbb{P}_{\mathbb{R}}^2/G) = \{\pi(1:1:1)\}$, $\Delta^1(\mathbb{P}_{\mathbb{R}}^2/G) = \emptyset$, $\Delta^2(\mathbb{P}_{\mathbb{R}}^2/G) = \{\mathrm{Reg}(\mathbb{P}_{\mathbb{R}}^2/G)\}$.
(2) $\mathbb{P}_{+}^2/G$ *is a normal semialgebraic surface, and* $\Delta^0(\mathbb{P}_{+}^2/G) = \{\pi(1:1:1), \pi(0:0:1)\}$, $\Delta^1(\mathbb{P}_{+}^2/G) = \{\pi(L)\}$, $\Delta^2(\mathbb{P}_{+}^2/G) = \{\mathrm{Reg}(\mathbb{P}_{+}^2/G)\}$.

Proof. (1) Through the study of cyclic quotient singularity, it is well known that $\mathbb{P}_{\mathbb{R}}^2/G$ is a complete real algebraic variety, and have unique singular point $\pi(1:1:1)$. Thus we obtain (1).

(2) follows from Example 1.4 and (1). Note that $\mathrm{Zar}(\pi(L))$ has a unique singular point at $\pi(0:0:1)$. ◻

Next we shall study characteristic varieties. Put

$$\Phi_d^c := \Phi_{\mathcal{H}_d^c}, \quad X_d^c := X(\mathbb{P}_{\mathbb{R}}^2, \mathcal{H}_d^c), \quad X_d^{c+} := X(\mathbb{P}_{+}^2, \mathcal{H}_d^c).$$

By Proposition 1.36, $X_d^{c+} \cong \mathbb{P}_{+}^2/G$ for $d \geq 3$, and $X_d^c \cong \mathbb{P}_{\mathbb{R}}^2/G$ for $d = 3$ and $d \geq 5$. In these cases, $\Delta(X_d^c)$ and $\Delta(X_d^{c+})$ are given by the above proposition. We shall study X_4^c.

Proposition 2.5. *Let $P := \Phi_4^c(1:1:1) \in X_4^c$. Then $\Delta^0(X_4^c) = \{P\}$, $\Delta^1(X_4^c) = \emptyset$, and $\Delta^2(X_4^c) = \{\ (X_4^c - \{P\})\}$.*

Proof. $\Phi_4^c : \mathbb{P}^2_{\mathbb{R}} \cdots \to X_4^c$ can be extended to a rational map $\Phi_{\mathbb{C}} : \mathbb{P}^2_{\mathbb{C}} \cdots \to \mathbb{P}(\mathcal{H}_4^c \otimes_{\mathbb{R}} \mathbb{C})$, and $\pi\colon \mathbb{P}_{\mathbb{R}} \to \mathbb{P}_{\mathbb{R}}/G$ can be extended to $\pi_{\mathbb{C}}\colon \mathbb{P}_{\mathbb{C}} \to \mathbb{P}_{\mathbb{C}}/G$, here $G = \mathbb{Z}/3\mathbb{Z}$. Let $\omega = \dfrac{-1+\sqrt{-3}}{2}$, $\mathbf{1} = (1:1:1)$, $Q_1 := (\omega : \omega^2 : 1)$ and $Q_2 := (\omega^2 : \omega : 1) \in \mathbb{P}^2_{\mathbb{C}}$. Then $(\mathbb{P}^2_{\mathbb{C}})^G = \{\mathbf{1},\, Q_1,\, Q_2\}$, and $\mathrm{Bs}(\mathcal{H}_4^c \otimes_{\mathbb{R}} \mathbb{C}) = \{Q_1,\, Q_2\}$. Thus the locus of indeterminacy of $\Phi_{\mathbb{C}} : \mathbb{P}^2_{\mathbb{C}} \cdots \to (X_4^c)_{\mathbb{C}}$ is $\{Q_1,\, Q_2\}$. Let $L_{\mathbb{C}} := \{(a:b:c) \in \mathbb{P}^2_{\mathbb{C}} \mid a+b+c=1\}$, $L := L_{\mathbb{C}} \cap \mathbb{P}^2_{\mathbb{R}} = V(S_1)$, $P := \pi(\mathbf{1})$, $P_1 := \pi_{\mathbb{C}}(Q_1)$, $P_2 := \pi_{\mathbb{C}}(Q_2) \in \mathbb{P}^2_{\mathbb{C}}/G$, $C_{\mathbb{C}} := \pi_{\mathbb{C}}(L_{\mathbb{C}}) \subset \mathbb{P}^2_{\mathbb{C}}/G$ and $C := \pi(L) \subset \mathbb{P}^2_{\mathbb{R}}/G$. Note that $Q_1,\, Q_2 \in L_{\mathbb{C}}$, and $\mathrm{Sing}(\mathbb{P}^2_{\mathbb{C}}/G) = \{P,\, P_1,\, P_2\}$. Since every function in $\mathcal{H}_4^c$ is constant on L, $\Phi_4^c(L)$ is a point. On the other hand, it is easy to see that $\Psi_4^c : (\mathbb{P}^2_{\mathbb{R}}/G - C) \longrightarrow (X_4^c - \Phi_4^c(L))$ is an isomorphism, as the proof of Proposition 1.36. Let $\psi : \widetilde{X} \longrightarrow \mathbb{P}^2_{\mathbb{C}}/G$ be the blowing up at P_1 and P_2, and $C_{\tilde{X}}$ be the strict transform of $C_{\mathbb{C}}$. Then, $C_{\tilde{X}}$ is an exceptional curve of the first kind, and $\widetilde{X} \longrightarrow (X_4^c)_{\mathbb{C}}$ is the contraction of $C_{\tilde{X}}$. Regarding $\mathbb{P}^2_{\mathbb{R}}/G \subset \widetilde{X}$, we conclude that $\Psi_4^c : \mathbb{P}^2_{\mathbb{R}}/G \longrightarrow X_4^c$ is a smooth contraction of C. Thus, X_4^c is a complete real algebraic variety, which has unique isolated singular point of A_1-type. ∎

By the above two propositions, we have:

Proposition 2.6. *Assume $d \geq 3$. Let $P := \Phi_d^c(1:1:1)$, $O := \Phi_d^c(0:0:1)$, and $C_d := \{\Phi_d^c(0:s:1) \mid s > 0\}$.*

(1) *The boundary of $\mathcal{P}_d^c = \mathcal{P}(\mathbb{P}^2_{\mathbb{R}},\, \mathcal{H}_d^c)$ consists of two face components $\mathcal{F}_d^c := \mathcal{F}(\mathrm{Reg}(X_d^c))$ and $\mathcal{F}(P)$. Moreover, $\mathcal{F}(P) = \mathcal{P}_d^{c0}$.*
(2) *The boundary of $\mathcal{P}_d^{c+}$ consists of at most four face components $\mathcal{F}_d^{c+} := \mathcal{F}(\mathrm{Reg}(X_d^{c+}))$, $\mathcal{E}_d^{c+} := \mathcal{F}(C_d)$, $\mathcal{F}(O)$, and $\mathcal{F}(P)$. Moreover, $\mathcal{F}(P) = \mathcal{P}_d^{c0+}$.*
(3) *If $\dim \mathcal{F}_d^{c+} = \dim \mathcal{H}_d^c - 1$, then $\mathrm{disc}(\mathrm{Reg}(X_d^c)) = \mathrm{disc}(\mathrm{Reg}(X_d^{c+}))$.*

Proof. (1) By the definition, $\mathcal{F}(P) = \mathcal{P}_d^{c0}$. It is a subset of a hyperplane in $\mathcal{H}_d^{c0} \subset \mathcal{H}_d^c$ by Remark 1.21(3). $\mathcal{P}_d^c$ is non-degenerate by Proposition 1.34. Thus, $\mathcal{F}_d^c \cup \mathcal{P}_d^{c0}$ must enclose a non-degenerate convex cone. Thus $\mathcal{F}_d^c$ is a face component of $\mathcal{P}_d^c$.

(2) can be proved similarly.

(3) follows from Proposition 1.32. ∎

Definition 2.7. We say $\mathcal{F}_d^c$ is the *main component* of $\mathcal{P}_d^c$, and $\mathrm{disc}(\mathrm{Reg}(X_d^c))$ is the *main discriminant* of $\mathcal{P}_d^c$. If $\dim \mathcal{F}_d^{c+} = \dim \mathcal{H}_d^c - 1$, then we say $\mathcal{F}_d^{c+}$ is the *main component* of $\mathcal{P}_d^{c+}$, and $\mathrm{disc}(\mathrm{Reg}(X_d^{c+}))$ is the *main discriminant* of $\mathcal{P}_d^c$. Otherwise, if $\dim \mathcal{F}_d^{c+} < \dim \mathcal{H}_d^c - 1$, then we say $\mathcal{P}_d^{c+}$ has no main component. $\mathcal{E}_d^{c+}$ is called an *edge component* of $\mathcal{P}_d^{c+}$, and $\mathrm{disc}(C_d)$ is called an *edge discriminant* of $\mathcal{P}_d^{c+}$.

$\mathrm{disc}(P)$ is the linear polynomial corresponding to $f(1,1,1) = 0$. $\mathrm{disc}(O)$ is the linear polynomial corresponding to f being at infinity, i.e. the coefficient of S_d in f is zero. By Proposition 1.32 and 1.34, we have:

Proposition 2.8. *Assume that $\mathcal{P}_d^{c+}$ has the main component. Then $\mathcal{P}_d^{c+}$ and $\mathcal{P}_d^c$ have the same main discriminant.*

Proposition 2.9. (Edge Discriminant Theorem) *Assume that $d \geq 3$ and $\mathcal{P}_d^{c0+}$ has an edge discriminant. Then, it agrees with the edge discriminant of $\mathcal{P}_d^{c+}$.*

Proof. Let $\{s_0,\dots, s_{N-1}\}$ be a base of $\mathcal{H}_d^{c0}$, and let $\mathrm{disc}_d^{c0+}(p_0,\dots,p_{N-1})$ be the edge discriminant of $\mathcal{P}_d^{c0+}$ corresponding $s_0,\dots, s_{N-1}$. Since disc_d^{c0+} exists, we have $\dim \mathcal{E}_d^{c0+} = N-2$, and $\dim(\mathcal{P}_d^{c0+})_P \leq N-3$ for $P \in C_d$. Take $s_N = S_{i,j,k} \in \mathcal{H}_d^c - \mathcal{H}_d^{c0}$ so that $|(i,j,k)|$ is minimum. Then, s_N is a multiple of $U = abc$. Thus $s_N(0,s,1) = 0$ and $s_N \in (\mathcal{P}_d^{c+})_P$. Since $\mathrm{Zar}((\mathcal{P}_d^{c+})_P) = \mathrm{Zar}\left((\mathcal{P}_d^{c0+})_P + \mathbb{R}_+ \cdot s_N\right)$, we conclude that $\mathrm{Zar}(\mathcal{E}_d^{c+})$ is the cone with the base $\mathrm{Zar}(\mathcal{E}_d^{c0+})$ and the vertex s_N at infinity. Let $\mathrm{disc}_d^{c+}(p_0,\dots,p_N)$ be the edge discriminant of $\mathcal{P}_d^{c+}$ corresponding $s_0,\dots, s_N$. By the above discussion, $\mathrm{disc}_d^{c+}(p_0,\dots,p_N) = \mathrm{disc}_d^{c0+}(p_0,\dots,p_{N-1})$. □

This proof implies the following:

Proposition 2.10. (Variables of the edge discriminant) *Let $d \geq 3$. Choose a basis $s_0,\dots, s_N$ of $\mathcal{H}_d^c$ so that each s_i is of the form $s_i = S_{j,k,l}$ for some $j \geq k \geq l \geq 0$. If s_i is a multiple of $U = abc$, i.e. $l \geq 1$, then p_i does not appear in $\mathrm{disc}_d^{c+}(p_0,\dots, p_N)$.*

2.2. Symmetric inequalities of three variables.

To study symmetric inequality, we shall determine $\Delta(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3)$ and $\Delta(\mathbb{P}^2_+/\mathfrak{S}_3)$.

Let's start from $\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3$. Let $\pi\colon \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3$ be the natural surjection. We can choose a fundamental domain of π as

$$A_F := \left\{(s:t:1) \in \mathbb{P}^2_{\mathbb{R}} \;\middle|\; s+t+1 \geq 0,\ s \leq t \leq 1\right\}.$$

$\pi(A_F) = \mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3$ and the restriction map $\pi\colon A_F^\circ \longrightarrow \pi(A_F^\circ)$ is an isomorphism as a semialgebraic variety.

Recall that $\mathbb{P}^2_{\mathbb{C}}/\mathfrak{S}_3 \cong \mathbb{P}_{\mathbb{C}}(1,2,3)$, since $\mathbb{C}[a,\ b,\ c]^{\mathfrak{S}_3} = \mathbb{C}[S_1,\ S_{1,1},\ U]$. So, we usually take the homogeneous coordinate system $(x_0 : x_1 : x_2)$ corresponding to $(S_1 : S_{1,1} : U)$. The weighted projective plane $\mathbb{P}_{\mathbb{C}}(1,2,3)$ has isolated singularities at $(0:1:0)$ and $(0:0:1)$. Let $\pi_{\mathbb{C}} : \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}}/\mathfrak{S}_3$ be the natural surjection. Since $\pi_{\mathbb{C}}^{-1}(0:0:1)$ is imaginal points, $(0:0:1) \notin \mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3$. Note that $\pi_{\mathbb{C}}^{-1}(0:1:0) \cap A_F = \{(-1:0:1)\}$.

For $(a:b:c) \in \mathbb{P}^2_{\mathbb{R}}$,

$$27U^2 - 18S_1S_{1,1}U + 4S_1^3U + 4S_{1,1}^3 - S_1^2S_{1,1}^2 = -(a-b)^2(b-c)^2(c-a)^2 \leq 0.$$

Thus $\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3$ is the closed semialgebraic subset of $\mathbb{P}_{\mathbb{R}}(1,2,3)$ defined by

$$27x_2^2 - 18x_0x_1x_2 + 4x_0^3x_2 + 4x_1^3 - x_0^2x_1^2 \leq 0. \tag{2.11}$$

Next we consider $\partial(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3)$.

Proposition 2.12. *Let $L_F^b := \left\{(s:1:1) \in \mathbb{P}^2_{\mathbb{R}} \;\middle|\; s \in \mathbb{R}\cup\{\infty\},\ s \neq 1,\ s \neq -2\right\}$. Then $\Delta^2(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3) = \left\{\pi(A_F^\circ)\right\}$, $\Delta^1(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3) = \left\{\pi(L_F^b))\right\}$, and $\Delta^0(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3) = \left\{\pi(1:1:1),\right.$ $\left.\pi(-1:0:1)\right\}$.*

Proof. Note that the edge $\left\{(s:s:1) \in A_F \;\middle|\; -1/2 \leq s \leq 1\right\}$ is transported to $\left\{(s:1:1) \in \mathbb{P}^2_{\mathbb{R}} \;\middle|\; s \leq -2 \text{ or } s \geq 1\right\}$ by a suitable element of $\mathfrak{S}_3$. It is easy to see that $\pi(\overline{L_F^b})$ agree with the algebraic curve on $\mathbb{P}_{\mathbb{R}}(1,2,3)$ defined by $27x_2^2 - 18x_0x_1x_2 + 4x_0^3x_2 + 4x_1^3 - x_0^2x_1^2 = 0$, and $\mathrm{Sing}(\pi(\overline{L_F^b})) = \left\{\pi(1:1:1)\right\}$.

Let $L_1 := \left\{(s:-s-1:1) \in \mathbb{P}^2_{\mathbb{R}} \;\middle|\; -2 \leq s < -1\right\}$ and $L_2 := \left\{(s:-s-1:1) \in \mathbb{P}^2_{\mathbb{R}} \;\middle|\; -1 < s \leq -1/2\right\}$. Since $\pi(1/s : -1/s-1:1) = \pi(s:-s-1:1)$, $\pi(L_1) = \pi(L_2)$ is the open line segment connecting $(0:0:1)$ and $(0:-3:-2)$. Since $\mathrm{Sing}(\mathbb{P}_{\mathbb{C}}(1,2,3)) \cap \pi(L_1) = \emptyset$, we have $\pi(L_1) \subset \mathrm{Reg}(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3)$. $\pi(-1:0:1) = (0:0:1) \in \mathrm{Sing}(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3)$, and $\pi(-2:1:1) = (0:-3:-2) \notin \mathrm{Sing}(\partial(\mathbb{P}^2_{\mathbb{R}}/\mathfrak{S}_3))$. Thus, we have the conclusion. □

Next we study $\mathbb{P}^2_+/\mathfrak{S}_3$. One of the fundamental domains of $\pi\colon \mathbb{P}^2_+ \to \mathbb{P}^2_+/\mathfrak{S}_3$ is

$$A_{F+} := \{(s:t:1) \in \mathbb{P}^2_+ \mid 0 \le s \le t \le 1\}.$$

$\mathbb{P}^2_+/\mathfrak{S}_3$ is the semialgebraic subset of $\mathbb{P}_{\mathbb{R}}(1,2,3)$ defined by (2.11) and $x_1/x_0^2 \ge 0$, $x_2/x_0^3 \ge 0$. Thus we have the following proposition (cf. Fig.2.1).

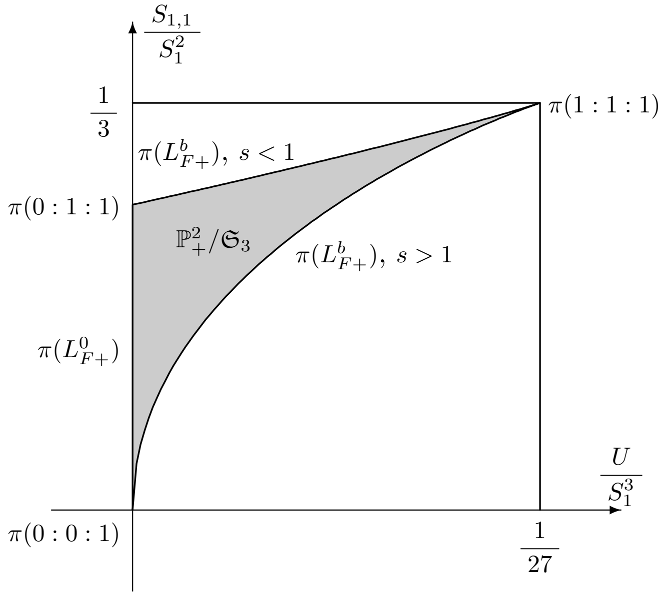

Fig.2.1. Graph of $\mathbb{P}^2_+/\mathfrak{S}_3$

Proposition 2.13. *Let*

$$L^b_{F+} := \{(s:1:1) \in \mathbb{P}^2_{\mathbb{R}} \mid 0 < s < 1 \text{ or } 1 < s < \infty\},$$
$$L^0_{F+} := \{(0:s:1) \in \mathbb{P}^2_{\mathbb{R}} \mid 0 < s < 1\}.$$

Then $\Delta^2(\mathbb{P}^2_+/\mathfrak{S}_3) = \{\pi(A^\circ_{F+})\}$, $\Delta^1(\mathbb{P}^2_+/\mathfrak{S}_3) = \{\pi(L^b_{F+}), \pi(L^0_{F+})\}$, $\Delta^0(\mathbb{P}^2_+/\mathfrak{S}_3) = \{\pi(0:0:1), \pi(0:1:1), \pi(1:1:1)\}$.

We also study $\operatorname{Bs}\mathcal{H}^s_d$ and $\operatorname{Bs}\mathcal{H}^{s0}_d$ where

$$\mathcal{H}^s_d := \{f \in \mathcal{H}_d \mid f \text{ is symmetric.}\},$$
$$\mathcal{H}^{s0}_d := \{f \in \mathcal{H}_d \mid f(1,1,1) = 0 \text{ and } f \text{ is symmetric.}\}.$$

Proposition 2.14. (1) *If $d \ge 4$ is even, then* $\operatorname{Bs}\mathcal{H}^s_d = \emptyset$ *and* $\operatorname{Bs}\mathcal{H}^{s0}_d = \{(1:1:1)\}$.

(2) *If $d \ge 3$ is odd, then* $\operatorname{Bs}\mathcal{H}^s_d \cap A_F = \{(-1:0:1)\}$. *If $d \ge 5$ is odd, then* $\operatorname{Bs}\mathcal{H}^{s0}_d \cap A_F = \{(1:1:1), (-1:0:1)\}$.

(3) *Let* $L = \{(a:b:c) \in \mathbb{P}^2_{\mathbb{R}} \mid a+b+c = 0\}$. *For $d \ge 3$, $\Phi^s_d(L)$ is a point if and only if $d = 3$, 4, 5 or 7.*

Proof. (1), (2) Consider an injection $\mathcal{H}^s_d \xrightarrow{\times S_2} \mathcal{H}^s_{d+2}$. Then $\operatorname{Bs}\mathcal{H}^s_{d+2} \subset \operatorname{Bs}\mathcal{H}^s_d \cup V(S_2) = \operatorname{Bs}\mathcal{H}^s_d$. Since $\operatorname{Bs}\mathcal{H}^s_4 = \emptyset$, we have $\operatorname{Bs}\mathcal{H}^s_d = \emptyset$ for even $d \ge 4$. The proof of $\operatorname{Bs}\mathcal{H}^{s0}_d = \{(1:1:1)\}$ is similar.

If $d \ge 5$ is odd, then $\operatorname{Bs}\mathcal{H}^s_d \subset \operatorname{Bs}\mathcal{H}^s_3$ and $\operatorname{Bs}\mathcal{H}^{s0}_d \subset \operatorname{Bs}\mathcal{H}^{s0}_5$. It is easy to see $\operatorname{Bs}\mathcal{H}^s_3 \cap A_F = \{(-1:0:1)\}$, and $\operatorname{Bs}\mathcal{H}^{s0}_5 \cap A_F = \{(1:1:1), (-1:0:1)\}$.

Take fundamental symmetric polynomials $\sigma_1 := S_1$, $\sigma_2 := S_{1,1}$ and $\sigma_3 := U$. Note that $\mathcal{H}^s_d = (\mathbb{R}[\sigma_1, \sigma_2, \sigma_3])_d$ and $S_1(-1:0:1) = 0$, $U(-1:0:1) = 0$, $S_{1,1}(-1:0:1) = -1$. If d

is odd, then any monomials $\sigma_1^k\sigma_2^l\sigma_3^m$ with $k+2l+3m=d$ are multiple of σ_1 or σ_3. Thus $f(-1,0,1)=0$ for any $f\in\mathcal{H}_d^s$.

(3) In $\{\sigma_1^k\sigma_2^l\sigma_3^m \mid k+2l+3m=d\}$, the number of elements of the form $\sigma_2^l\sigma_3^m$ is at most one if and only if $d=3$, 4, 5 or 7. ◻

Section 3. Cubic cyclic inequalities.

3.1. The PSD cone $\mathcal{P}_3^{c+}$.

Cubic cyclic inequalities are studied in [1] and [2]. Let $A:=\mathbb{P}_+^2$ with the homogeneous coordinate system $(a:b:c)$, $G:=\mathbb{Z}/3\mathbb{Z}$, $\mathcal{H}_d^c:=(\mathcal{H}_{3,d})^G$, and $\mathcal{P}_d^{c+}:=\mathcal{P}(A,\mathcal{H}_d^c)$. Note that $\mathcal{P}_d^{c+}$ is non-degenerate by Proposition 1.34.

Consider the case $d=3$. We choose a base $s_0,\ldots, s_3$ of $\mathcal{H}_3^c$ as $s_0:=S_3-3U$, $s_1:=S_{2,1}-3U$, $s_2:=S_{1,2}-3U$, $s_3:=U=abc$. Note that $\{s_0,\ s_1,\ s_2\}$ is a base of $\mathcal{H}_3^{c0}$. We execute the algorithm of Remark 1.23.

Step 1: Eliminate a, b, c from $x_i=s_i(a,b,s)$. Then, we obtain that $X_3^{c+}:=\Phi_3^c(\mathbb{P}_+^2)\cong\mathbb{P}_+^2/G$ is a cubic semialgebraic surface in $\mathbb{P}_\mathbb{R}^3:(x_0:\cdots:x_3)$ defined by

$$F_3:=x_1^3-x_0x_1x_2+x_2^3+x_0^2x_3-3x_0x_1x_3+9x_1^2x_3-3x_0x_2x_3-9x_1x_2x_3+9x_2^2x_3=0$$

and $x_ix_3\geq 0$ $(0\leq i\leq 2)$. Let $P_3:=(0:0:0:1)$, $O_3:=(1:0:0:0)$, and $C_3:=\big(X_3^{c+}\cap\{x_3=0\}\big)-\{P\}$. Then, $\Delta^0(X_3^{c+})=\{P_3,\ O_3\}$, $\Delta^1(X_3^{c+})=\{C_3\}$, and $\Delta^2(X_3^{c+})=\{\mathrm{Reg}(X_3^{c+})\}$.

On the other hand, $X_3^{c0+}:=X(\mathbb{P}_+^2,\ \mathcal{H}_3^{c0})$ is the domain on $\mathbb{P}_\mathbb{R}^2$ enclosed by $C_3\cup\{O_3\}$, and $\Delta^0(X_3^{c0+})=\{O_3\}$, $\Delta^1(X_3^{c0+})=\{C_3\}$, $\Delta^2(X_3^{c0+})=\{\mathrm{Reg}(X_3^{c0+})\}$.

Step 2: Identify $p_0s_0+\cdots+p_3s_3\in\mathcal{H}_3^c$ with $(p_0,\ldots, p_3)\in\mathbb{R}^4$. By Remark 1.21(3), $\mathrm{disc}(O_3)=p_0$, $\mathrm{disc}(P_3)=p_0+p_1+p_2+3p_3$. Let's calculate $\mathrm{Zar}(C_3^\vee)$ by the algorithm in Remark 1.21(2). $\mathrm{Zar}(C_3)$ is defined by $x_3=0$ and $x_1^3+x_2^3-x_0x_1x_2=0$. Thus, $\mathrm{disc}(C_3)=4p_0p_1^3+4p_0p_2^3+27p_0^4-p_1^2p_2^2-18p_0^2p_1p_2$.

We denote local cones of $\mathcal{P}_3^{c+}$, $\mathcal{P}_3^{c0+}$ at the point $(0:s:1)\in\mathbb{P}_+^2$ by $\mathcal{L}_{0,s}^{c+}$, $\mathcal{L}_{0,s}^{c0+}$. Note that if they are not 0, then $\dim\mathcal{L}_{0,s}^{c+}\leq 2$, and $\dim\mathcal{L}_{0,s}^{c0+}=1$ by Proposition 1.27(1). By the algorithm of Remark 1.28, we know that $\mathcal{L}_{0,s}^{c0+}$ is generated by

$$\mathfrak{f}_s(a,b,c):=s^2S_3-(2s^3-1)S_{2,1}+(s^4-2s)S_{1,2}-3(s^4-2s^3+s^2-2s+1)U.$$

This is extremal in $\mathcal{P}_3^{c+}$ by Proposition 1.26(4). As a limit $s\to+\infty$, we put $\mathfrak{f}_\infty(a,b,c):=s_2'=S_{1,2}-3U$.

Theorem 3.1. (Structure of $\mathcal{P}_3^{c+}$)

(1) *$\mathcal{P}_3^{c+}$ has no main component.*
(2) *$\mathcal{L}_{0,s}^{c+}=\mathbb{R}_+\cdot\mathfrak{f}_s+\mathbb{R}_+\cdot U$ for $s>0$.*
(3) *$\mathcal{F}(O_3)=\mathbb{R}_+\cdot U+\mathbb{R}_+\cdot(S_{2,1}-3U)+\mathbb{R}_+\cdot(S_{1,2}-3U)$.*

Proof. (1) Let Y be the cone whose vertex is P_3 and whose base is $C_3\cup O_3$. Since X_3^c is a cubic surface with an A_1-singular point P_3, it is easy to see that $\mathrm{Reg}(X_3^{c+})\subset\mathrm{Int}(Y)$. Thus, $\mathrm{Reg}(X_3^{c+})^\vee$ is included the exterior of $\mathcal{P}_3^{c+}$. Thus $\mathcal{F}_3^{c+}$ is not a face component.

(2) $f=\alpha\mathfrak{f}_s+\beta U$ satisfies $f(0,s,1)=0$. Thus $f\in\mathcal{L}_{0,s}^{c+}$ if $\alpha\geq 0$ and $\beta\geq 0$. Note that $\dim\mathcal{L}_{0,s}^{c+}=2$. Contrary, assume $f\in\mathcal{L}_{0,s}^{c+}$. Then $\beta=f(1,1,1)\geq 0$ and $\alpha=(1/s^2)f(0,0,1)\geq 0$.

(3) Put $\mathcal{F} = \mathcal{F}(O_3) = \mathcal{L}_{0,0}^{c+}$. Then $\partial\mathcal{F} = \big(\mathcal{E}_3^{c+} \cup \mathcal{P}_3^{c0+}\big) \cap \mathcal{F}$ by Proposition 1.33(2). Since

$$\mathcal{E}_3^{c+} \cap \mathcal{F} = \big(\mathbb{R}_+ \cdot \mathfrak{f}_0 + \mathbb{R}_+ \cdot U\big) \cup \big(\mathbb{R}_+ \cdot \mathfrak{f}_\infty + \mathbb{R}_+ \cdot U\big),$$
$$\mathcal{P}_3^{c0+} \cap \mathcal{F} = \mathcal{L}_{0,0}^{c0+} = \mathbb{R}_+ \cdot \mathfrak{f}_0 \cup \mathbb{R}_+ \cdot \mathfrak{f}_\infty,$$

we have $\mathcal{F} = \mathbb{R}_+ \cdot U + \mathbb{R}_+ \cdot (S_{2,1} - 3U) + \mathbb{R}_+ \cdot (S_{1,2} - 3U)$. □

Remark 3.2. (1) Let $\Psi_3^c \colon X_3^{c+} \to X_3^{c0+}$ be the projection defined by $\Psi_3^c(x_0 : x_1 : x_2 : x_3) = (x_0 : x_1 : x_2)$. Then Ψ_3^c is a birational map and is continuous, since $\deg_{x_3} F_3 = 1$. But Ψ_3^c is not regular at P_3. In fact, $\Phi_3^c(P_3) = (3 : 1 : 1) \notin \mathrm{Sing}(X_3^{c0+})$.

(2) We have an analytic poof of $\mathfrak{f}_s \in \mathcal{P}_3^{c+}$ as

$$\mathfrak{f}_s(a, b, c) = \mathfrak{f}_s(a, 1 - k(1 - a), 1)$$
$$= (1 - a)^2 \Big\{a(1 - ks)^2(k + s^2) + \big(1 + (1 - k)s^2\big)(1 - k - s)^2\Big\} \geq 0.$$

But we don't need such a proof.

We don't need Step 3 of Remark 1.23, since the convex set is unique. Step 4 is easy. Thus we have:

Proof of Theorem 0.2. Let $\mathrm{disc}_3^{c+}(p, q, r) = \mathrm{disc}_3^{c0+}(p, q) := \mathrm{disc}(C_3)(1, p, q, r) = 4p^3 + 4q^3 + 27 - p^2q^2 - 18pq$. Figure 3.1 is the graph of $\mathrm{disc}_3^{c0+}(p, q) = 0$.

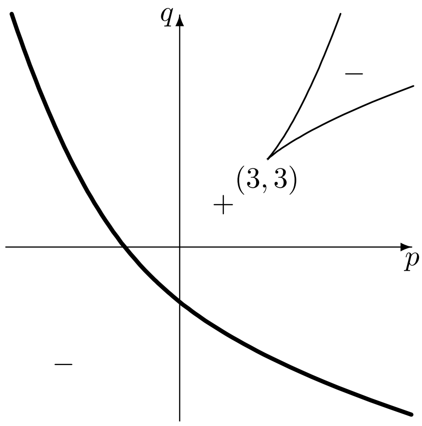

Fig. 3.1. Graph of $\mathrm{disc}_3^{c0+}(p, q) = 0$

Thus $\mathcal{P}_3^{c+}$ stands over the union of the following (1') and (2').

(1') $4p^3 + 4q^3 + 27 \geq p^2q^2 + 18pq$.

(2') $p \geq 0$ and $q \geq 0$.

Take $f = S_3 + pS_{2,1} + qS_{1,2} + rU \in \mathcal{H}_3^c$. Note that $f \notin \mathcal{F}(O)$. $3 + 3p + 3q + r = f(1, 1, 1) \geq 0$ for $f \in \mathcal{P}_3^{c+}$.

It is easy to see that if (1) or (2) holds, then $f \in \mathcal{P}_3^{c+}$. □

Corollary 3.3. *An extremal element of $\mathcal{P}_3^{c+}$ is of the form $\alpha\mathfrak{f}_s$ ($\alpha > 0$, $s \in [0, \infty]$) or βU ($\beta > 0$).*

Corollary 3.4. *Let $G = \mathfrak{S}_3$, $\mathcal{H}_3^s := \mathcal{H}_3^G$ and $\mathcal{P}_3^{s+} := \mathcal{P}(\mathbb{P}_+^3, \mathcal{H}_3^s)$. Then, $\mathcal{P}_3^{s+}$ is a three dimensional triangular cone whose three edges are $\mathbb{R}_+ \cdot (T_{2,1} - 6U)$, $\mathbb{R}_+ \cdot (S_3 + 3U - T_{2,1})$, and $\mathbb{R}_+ \cdot U$.*

Proof. This follows from $\mathcal{P}_3^{s+} = \mathcal{P}_3^{c+} \cap \mathcal{H}_3^G$. $\square$

Corollary 3.5. *Let $G = \mathfrak{S}_3$, $\mathcal{H}_3^{s0} := (\mathcal{H}_3^{c0})^G$, $\mathcal{P}_3^{s0+} := \mathcal{P}(\mathbb{P}_+^3, \mathcal{H}_3^{s0})$. Then, $\mathcal{P}_3^{s0+}$ is the fan on $\mathbb{R}$ with two edges $\mathbb{R}_+ \cdot (S_3 + 3U - T_{2,1})$ and $\mathbb{R}_+ \cdot (T_{2,1} - 6U)$.*

Note that $S_3 + 3U \geq T_{2,1}$ is Shur's inequality of degree 3.

3.2. Structure of X_d^{c0} and X_d^{c0+}.

Let $\Phi_d^{c0} := \Phi_{\mathcal{H}_d^{c0}}$, $X_d^{c0} := \Phi_d^{c0}(\mathbb{P}_\mathbb{R}^2)$, and $X_d^{c0+} := \Phi_d^{c0}(\mathbb{P}_+^2)$.

Proposition 3.6. *If $d \geq 3$, then $\Phi_d^{c0}(1:1:1)$ is a point. In other word, the rational map $\Phi_d^{c0} : \mathbb{P}_\mathbb{R}^2 \cdots \to X_d^{c0}$ can be extended to $(1:1:1)$ as a continuous map. Moreover, $\Phi_d^{c0}(1:1:1)$ is a non-singular point of X_d^{c0}.*

Proof. We consider on the ground field $\mathbb{C}$. Let $\rho_1 : Y' \longrightarrow \mathbb{P}_\mathbb{C}^2$ be the blowing up at $(1:1:1)$, and let $E_1' := \rho_1^{-1}(1:1:1)$.

To begin with, we study the case $d = 3$. We take $s_0 := S_3 - 3U$, $s_1 := S_{2,1} - 3U$, $s_2 := S_{1,2} - 3U$ as a base of $\mathcal{H}_3^{c0} \otimes_\mathbb{R} \mathbb{C}$. Note that

$$s_0(1+t, 1+ut+vt^2, 1) = 3(u^2-u+1)t^2 + (3(2u-1)v + (u^3+1))t^3 + O(t^4),$$
$$s_1(1+t, 1+ut+vt^2, 1) = (u^2-u+1)t^2 + ((2u-1)v + u)t^3 + O(t^4),$$
$$s_2(1+t, 1+ut+vt^2, 1) = (u^2-u+1)t^2 + ((2u-1)v + u^2)t^3 + O(t^4),$$

here $O(t^4)$ is a sum of terms whose degree of t is not less than 4. Let $\zeta_6 = (1+\sqrt{-3})/2$. For roots $u = \zeta_6, \zeta_6^5$ of $u^2 - u + 1 = 0$, let y_2, y_3 be the points on E_1' corresponding to the vector $(1:u:1)$ at the point $(1:1:1)$. Note that $\mathrm{Bs}(\rho_1^* \mathcal{H}_3^{c0} \otimes_\mathbb{R} \mathbb{C}) = \{y_2, y_3\}$. Let $\rho_2 : \widetilde{\mathbb{P}_\mathbb{C}^2} \longrightarrow Y'$ be the blowing up at y_2, y_3, and let $E_2 := \rho_2^{-1}(y_2)$, $E_3 := \rho_2^{-1}(y_3)$, and E_1 be the strict transform of E_1', and $\rho := \rho_2 \circ \rho_1$.

Then $\mathrm{Bs}(\rho^* \mathcal{H}_3^{c0} \otimes_\mathbb{R} \mathbb{C}) = \emptyset$. Let $\widetilde{\Phi_3^{c0}} = \Phi_{\rho^* \mathcal{H}_3^{c0} \otimes_\mathbb{R} \mathbb{C}} : \widetilde{\mathbb{P}_\mathbb{C}^2} \longrightarrow \mathbb{P}_\mathbb{C}^2$. Note that $\widetilde{\Phi_3^{c0}}(E_1) = (3:1:1) =: Q$. For $u = \zeta_6$ and ζ_6^5, let L_2, L_3 be the line passing through Q and $((u^3+1) : u : u^2)$. Then, $\widetilde{\Phi_3^{c0}}(E_2) = L_2$ and $\widetilde{\Phi_3^{c0}}(E_2) = L_3$. Since $L_2 \cap \mathbb{P}_\mathbb{R}^2 = \{Q\}$ and $L_3 \cap \mathbb{P}_\mathbb{R}^2 = \{Q\}$, we conclude that $\Phi_3^{c0}(1:1:1) = Q$ over the field $\mathbb{R}$. Thus $\Phi_3^{c0} : \mathbb{P}_\mathbb{R}^2 \longrightarrow \mathbb{P}_\mathbb{R}^2$ can be extended to $(1:1:1)$ as a continuous map.

Consider the case $d \geq 4$. We define the injection $\iota : \mathcal{H}_3^{c0} \longrightarrow \mathcal{H}_d^{c0}$ by $\iota_h(f) = S_1^{s-3} f$. Then $\mathrm{Bs}(\rho^* \mathcal{H}_d^{c0} \otimes_\mathbb{R} \mathbb{C}) \subset \mathrm{Bs}(\rho^* \mathcal{H}_3^{c0} \otimes_\mathbb{R} \mathbb{C}) \cup V(S_1)$. Since $\mathrm{Bs}(\mathcal{H}_d^{c0} \otimes_\mathbb{R} \mathbb{C}) = \{(1:1:1)\}$, we have $\mathrm{Bs}(\rho^* \mathcal{H}_d^{c0} \otimes_\mathbb{R} \mathbb{C}) = \emptyset$. Let $\widetilde{\Phi_d^{c0}} := \Phi_{\rho^* \mathcal{H}_d^{c0} \otimes_\mathbb{R} \mathbb{C}} : \widetilde{\mathbb{P}_\mathbb{C}^2} \longrightarrow \mathbb{P}((\mathcal{H}_d^{c0})^\vee \otimes_\mathbb{R} \mathbb{C})$, and $(X_d^{c0})_\mathbb{C} := \widetilde{\Phi_d^{c0}}(\widetilde{\mathbb{P}_\mathbb{C}^2})$.

For $s = S_4 - US_1$, $S_{3,1} - US1$, $S_{2,2} - US_1$, $S_{1,3} - US_1 \in \mathcal{H}_4^{c0}$ and $s = S_5 - US_{1,1}$, $S_{4,1} - US_{1,1}$, $S_{3,2} - US_{1,1}$, $S_{2,3} - US_{1,1}$, $S_{1,4} - US_{1,1}$, $US_1 - US_{1,1} \in \mathcal{H}_5^{c0}$, the following holds:

$$s(1+t, 1+ut, 1) = c_0(u^2-u+1)t^2 + \sum_{i \geq 3} h_i(u) t^i. \tag{3.7}$$

We denote $d = 3k + e$, here $e \in \{3, 4, 5\}$. Then $\mathcal{H}_d^{c0} = \mathcal{H}_{3k}^c \cdot \mathcal{H}_e^{c0}$. Thus, (3.7) holds for any $s \in \mathcal{H}_d^{c0}$. Therefore, $\widetilde{\Phi_d^{c0}}(E_1)$ is a point.

Since $\mathrm{Bs}(\rho^* \mathcal{H}_3^{c0} \otimes_\mathbb{R} \mathbb{C}) = \emptyset$, there exists the natural regular map $\psi : (X_d^{c0})_\mathbb{C} \longrightarrow (X_d^{c0})_\mathbb{C}$. Let $Q' := \widetilde{\Phi_d^{c0}}(E_1)$ and $L_i' := \widetilde{\Phi_d^{c0}}(E_i)$ $(i = 2, 3)$. Then, $\psi(L_i') = L_i$. Since $L_i \cap X_3^{c0} = \{Q\}$, we have $L_i' \cap X_d^{c0} = \{Q'\}$. Thus $\Phi_d^{c0}(1:1:1) = Q$.

Since Q is a non-singular point of $(X_3^{c0})_{\mathbb{C}} = \mathbb{P}^2_{\mathbb{C}}$, Q' is also a non-singular point of $(X_d^{c0})_{\mathbb{C}}$. □

Proposition 3.8. *Let $G := \mathbb{Z}/3\mathbb{Z}$ and let $\pi : \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}/G$ be the natural surjection. Decompose Φ_d^c and Φ_d^{c0} as*

$$\Phi_d^c : \mathbb{P}^2_{\mathbb{R}} \xrightarrow{\pi} \mathbb{P}^2_{\mathbb{R}}/G \xrightarrow{\Psi_d^c} X_d^c, \quad \Phi_d^{c0} : \mathbb{P}^2_{\mathbb{R}} \xrightarrow{\pi} \mathbb{P}^2_{\mathbb{R}}/G \xrightarrow{\Psi_d^{c0}} X_d^{c0}.$$

Let $\varphi_d : X_d^c \cdots\to X_d^{c0}$ be the rational map induced by the inclusion $\mathcal{H}^{c0}_{3,d} \xrightarrow{\subset} \mathcal{H}^c_{3,d}$.

(1) *If $d = 3$ or $d \geq 5$, then $\Psi_d^{c0} : \mathbb{P}^2_{\mathbb{R}}/G \longrightarrow X_d^{c0}$ and $\varphi_d : X_d^c \longrightarrow X_d^{c0}$ are birational bijective continuous map.*

(2) *If $d \geq 3$, then $\Phi_d^{c0}(1:1:1)$ is an interior point of X_d^{c0}.*

Proof. Let $A = \mathbb{P}^2_{\mathbb{R}} - \{(1:1:1)\}$, $W_d^c := \Phi_d^c(A)$, and $W_d^{c0} := \Phi_d^{c0}(A)$. We have observed that (1) and (2) hold for $d = 3$. Moreover, $\Psi_3^{c0} : A/G \longrightarrow W_3^{c0}$ and $\varphi_3 : W_3^c \longrightarrow W_3^{c0}$ are isomorphisms.

Consider the case $d = 6$. Since $\mathcal{H}_6^{c0} = \mathcal{H}_3^c \cdot \mathcal{H}_3^{c0}$, we have $\operatorname{Bs}\mathcal{H}_6^{c0} = \emptyset$. A birational bijective continuous map Ψ_3^{c0} factors as $\Phi_3^{c0} : \mathbb{P}^2_{\mathbb{R}}/G \xrightarrow{\Psi_6^{c0}} X_6^{c0} \longrightarrow X_3^{c0}$. Thus $\varphi_6 : W_6^c \longrightarrow W_6^{c0}$ is an isomorphism, and $\varphi_6 : X_6^c \cdots\to X_6^{c0}$ is a bijective continuous map.

Consider the cases $d = 5$ and $d \geq 7$. By $\mathcal{H}^c_{d-2} \xrightarrow{\times S_2} \mathcal{H}^c_d$, we have $\operatorname{Bs}\mathcal{H}_d^{c0} = \emptyset$. For $e = 3$ or 6, there exists a birational continuous bijection $\Phi_e^{c0} : \mathbb{P}^2_{\mathbb{R}}/G \xrightarrow{\Psi_d^{c0}} X_d^{c0} \longrightarrow X_e^{c0}$. Thus $\varphi_d : W_d^c \longrightarrow W_d^{c0}$ is an isomorphism, and $\varphi_d : X_d^c \cdots\to X_d^{c0}$ is a birational bijective continuous map.

(2) Since $\Psi_3^{c0} : \mathbb{P}^2_+/G \longrightarrow X_3^{c0+}$ is a composition of $\Psi_d^{c0} : \mathbb{P}^2_+/G \longrightarrow X_d^{c0+}$ and $X_d^{c0+} \to X_3^{c0+}$, we know that $\Phi_d^{c0}(1:1:1)$ lies interior of X_d^{c0}. □

Corollary 3.9. *Let $O_d := \Phi_d^{c0}(0:0:1)$, $L := \{(0:s:1) \in \mathbb{P}^2_+ \mid s > 0\}$, and $C_d := \Phi_d^{c0}(L)$. Then,*

(1) *If $d \geq 3$, then $X_d^{c0+} \cong X_3^{c0+}$, and $\Delta^0(X_d^{c0+}) = \{O_d\}$, $\Delta^1(X_d^{c0+}) = \{C_d\}$, $\Delta^2(X_d^{c0+}) = \{\operatorname{Reg}(X_d^{c0+})\}$.*

(2) *If $d \geq 3$ is odd, then $X_d^{c0} \cong \mathbb{P}^2_{\mathbb{R}}$, and $\Delta^0(X_d^{c0}) = \Delta^1(X_d^{c0}) = \emptyset$, $\Delta^2(X_d^{c0}) = \{X_d^{c0}\}$.*

Section 4. Quartic cyclic inequalities.

4.1. The PSD cone $\mathcal{P}_4^{c0}$.

Hilbert proved that every element in $\mathcal{P}_4 := \mathcal{P}(\mathbb{P}_{\mathbb{R}}, \mathcal{H}_4)$ can be presented as a sum of squares of quadric polynomials in [17], and this is the source of Hilbert's 17th problem (see also [22]). Quartic cyclic inequalities ware studied in [8], [10], [12] and [2]. Cîrtoaje proved that every element $f(a,b,c) \in \partial\mathcal{P}_4^{c0}$ is of the form

$$f(a,b,c) = \lambda \sum_{\text{cyclic}} (2a^2 - b^2 - c^2 + pab - (p+q)bc + qca)^2 \quad (\lambda \geq 0).$$

In this section, we solve some open problems presented in above articles.

We want to study $\mathcal{P}_4^c$. But it is hard because of Conjecture 4.8. So, we start from $\mathcal{P}_4^{c0} = \mathcal{P}(\mathbb{P}^2_{\mathbb{R}}, \mathcal{H}_4^{c0})$, and $\mathcal{P}_4^{c0+} = \mathcal{P}(\mathbb{P}^2_{\mathbb{R}}, \mathcal{H}_4^{c0})$, here $\mathcal{H}_4^{c0} := \{f \in \mathcal{H}_4^c \mid f(1,1,1) = 0\}$.

We execute Step 1 of Remark 1.23. We choose $s_0 := S_4 - US_1$, $s_1 := S_{3,1} - US_1$, $s_2 := S_{1,3} - US_1$, $s_3 := S_{2,2} - US_1$ as a base of $\mathcal{H}_4^{c0}$, and $s_0, \ldots, s_3$, $s_4 := US_1$ as a base of

$\mathcal{H}_4^c$. Eliminate a, b, c from the above using the computer software Mathematica. Then we have:

Proposition 4.1. (1) *X_4^{c0} is a quadric surface in $\mathbb{P}^3_{\mathbb{R}}$ defined by*

$$F_4^{c0} := (x_1 + x_2)^2 + 3(x_1 - x_2)^2 + (x_0 - 2x_3)^2 - x_0^2.$$

Thus $\Delta^0(X_4^{c0}) = \Delta^1(X_4^{c0}) = \emptyset$, $\Delta^2(X_4^{c0}) = \{X_4^{c0}\}$, and $\partial\mathcal{P}_4^{c0}$ has the unique face component $\mathcal{F}(X_4^{c0})$.

(2) *X_4^c is the semialgebraic subset of $\mathbb{P}^4_{\mathbb{R}}$ defined by F_4^{c0} and*

$$F_4^c := (x_1 + x_2 + 3x_4)^2 - (x_0 + 2x_3 + 3x_4)(x_3 + 3x_4).$$

(3) *The inclusion $\mathcal{H}_4^{c0} \overset{\subset}{\longrightarrow} \mathcal{H}_4^c$ induce a rational map $\varphi_4^c : X_4^c \longrightarrow X_4^{c0}$ which is a continuous map and a bijection.*

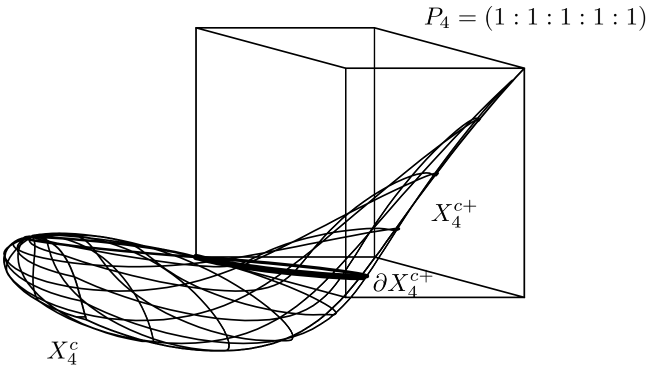

Fig 4.1. Projection of X_4^c

Proof. (1) and (2) are easy. Remember Proposition 2.5.

(3) φ_4^c is defined by $\varphi(x_0 : x_1 : x_2 : x_3 : x_4) = (x_0 : x_1 : x_2 : x_3)$. Since $\deg_{x_4}\big((x_1 + x_2 + 3x_4)^2 - (x_0 + 2x_3 + 3x_4)(x_3 + 3x_4)\big) = 1$, $\varphi_4^c : X_4^c \longrightarrow X_4^{c0}$ is a bijective continuous map. □

Next we execute Step 2 of Remark 1.23. Let $\mathcal{L}_{s,t}^c$ and $\mathcal{L}_{s,t}^{c0}$ be the local cones of $\mathcal{P}_4^c$ and $\mathcal{P}_d^{c0}$ at the point $(s : t : 1) \in \mathbb{P}^2_{\mathbb{R}}$. Then $\dim \mathcal{L}_{s,t}^c \le 2$ and $\dim \mathcal{L}_{s,t}^{c0} = 1$ for any $(s, t) \in \mathbb{P}^2_{\mathbb{R}}$ by Proposition 1.27(1). Execute the calculation of Remark 1.21(2) and Remark 1.28 using computer. Then we have that $\partial\mathcal{P}_4^{c0}$ is the quadric surface defined by

$$\mathrm{disc}_4^{c0} = 3(p_0p_3 + p_0^2) - (p_1^2 + p_1p_2 + p_2^2),$$

and that the generator of $\mathcal{L}_{s,t}^{c0}$ is the following $\mathfrak{g}_{s,t}^A$. Definition of $\mathfrak{g}_{s,t}^A$ is somewhat complicated:

$$\begin{aligned}
\mathfrak{g}_{p,q}^X(a,b,c) &:= S_4 + pS_{3,1} + qS_{1,3} \\
&\qquad + \left(\frac{p^2+pq+q^2}{3} - 1\right) S_{2,2} - \left(p + q + \frac{p^2+pq+q^2}{3}\right) US_1 \\
\mathfrak{g}_\infty^X(a,b,c) &:= S_{2,2} - US_1, \\
\mathfrak{p}(s,t) &:= -\frac{2S_{3,1}(s,t,1) - S_{1,3}(s,t,1) - S_{2,1,1}(s,t,1)}{S_{2,2}(s,t,1) - S_{2,1,1}(s,t,1)} \\
\mathfrak{g}_{s,t}^A(a,b,c) &:= \mathfrak{g}_{\mathfrak{p}(s,t),\mathfrak{p}(t,s)}^X(a,b,c)
\end{aligned}$$

where $\mathfrak{p}(1,1) := -2$, $\mathfrak{p}(0,0) = \infty \in \mathbb{P}^1_{\mathbb{R}}$, and $\mathfrak{g}^A_{0,0} := \mathfrak{g}^X_\infty$. Note that $\mathfrak{p}(s,t) = \mathfrak{p}(1/t, s/t) = \mathfrak{p}(t/s, 1/s)$. We don't need Step 3 and 4 for $\mathcal{P}^{c0}_4$, and we have:

Theorem 4.2. (Structure of $\mathcal{P}^{c0}_4$, cf. [2], [10]) *The four dimensional convex cone $\mathcal{P}^{c0}_4$ has the following structure*:

(1) *For $f = p_0S_4 + p_1S_{3,1} + p_2S_{1,3} + p_3S_{2,2} - 3(p_0 + p_1 + p_2 + P_3)US_1 \in \mathcal{H}^{c0}_4$, $f \in \mathcal{P}^{c0}_4$ if and only if* $\operatorname{disc}^{c0}_4(p_0, p_1, p_2, p_3) \geq 0$.

(2) $\mathcal{L}^{c0}_{s,t} = \mathbb{R}_+ \cdot \mathfrak{g}^A_{s,t}$.

(3) *An extremal element in $\mathcal{P}^{c0}_4$ is of the form $\alpha\mathfrak{g}^X_{p,q}$ or $\alpha\mathfrak{g}^X_\infty$ ($\exists\alpha > 0$).*

4.2. The PSD cone $\mathcal{P}^{c0+}_4$.

Study $X^{c0+}_4 := X(\mathbb{P}^2_+, \mathcal{H}^{c0}_4) \subset X^{c0}_4$. Let $O_4 := \Phi^{c0}_4(0:0:1) = (1:0:0:0)$, and $C_4 := \{\Phi^{c0}_4(0:s:1) \mid s > 0\}$. By Corollary 3.9, $\Delta^0(X^{c0+}_4) = \{O_4\}$, $\Delta^1(X^{c0+}_4) = \{C_d\}$, and $\Delta^2(X^{c0+}_4) = \{\operatorname{Reg}(X^{c0+}_4)\}$. The main discriminant of $\mathcal{P}^{c0+}_4$ is $\operatorname{disc}^{c0}_4$ by Proposition 2.8. $\operatorname{disc}(O_4) = p_0$ by Remark 1.21(3).

Let's calculate the edge discriminant. By Remark 1.21(2), $\operatorname{Zar}(C^\vee_4)$ can obtained by eliminating s from

$$\sum_{i=0}^{3} p_i s_i(0,s,1) = 0 \quad \text{and} \quad \sum_{i=0}^{3} p_i \frac{\partial s_i}{\partial s}(0,s,1) = 0. \tag{4.3}$$

A computer gives a solution

$$\begin{aligned}\operatorname{disc}^{c+}_4 &= 256p_0^6 - 27p_0^2p_1^4 - 192p_0^4p_1p_2 - 6p_0^2p_1^2p_2^2 - 4p_1^3p_2^3 - 27p_0^2p_2^4 \\ &+ 144p_0^3p_1^2p_3 + 18p_0p_1^3p_2p_3 + 144p_0^3p_2^2p_3 + 18p_0p_1p_2^3p_3 - 128p_0^4p_3^2 \\ &- 80p_0^2p_1p_2p_3^2 + p_1^2p_2^2p_3^2 - 4p_0p_1^2p_3^3 - 4p_0p_2^2p_3^3 + 16p_0^2p_3^4.\end{aligned}$$

Note that this agree with the edge discriminant of $\mathcal{P}^{c+}_4$, by Proposition 2.9. If we solve (4.3) as Remark 1.28, we find the following $\mathfrak{h}_s$:

$$\begin{aligned}\mathfrak{h}_s &:= S_{3,1} + s^2S_{1,3} - 2sS_{2,2} - (s-1)^2US_1, \\ \mathfrak{h}_\infty &:= S_{1,3} - US_1.\end{aligned}$$

Note that $\mathfrak{h}_0 = S_{3,1} - US_1$, and $\mathfrak{g}^A_{0,0} = \mathfrak{g}^X_\infty = S_{2,2} - US_1$. Put $\mathfrak{g}^A_{0,+\infty} := \mathfrak{g}^X_\infty$ formally. We denote the local cones of $\mathcal{P}^{c+}_4$ and $\mathcal{P}^{c0+}_d$ at $(s:t:1) \in \mathbb{P}^2_{\mathbb{R}}$ by $\mathcal{L}^{c+}_{s,t}$ and $\mathcal{L}^{c0+}_{s,t}$.

Theorem 4.4. (Structure of $\mathcal{P}^{c0+}_4$, cf. [2]) *The four dimensional convex cone $\mathcal{P}^{c0+}_4$ has the following structure*:

(1) $\mathcal{L}^{c0+}_{s,t} = \mathcal{L}^{c0}_{s,t} = \mathbb{R}_+ \cdot \mathfrak{g}^A_{s,t}$ *for $s > 0$, $t > 0$, and*

$$\mathcal{F}^{c0+}_4 = \mathbb{R}_+ \cdot \left(\left\{\mathfrak{g}^X_{p,q} \mid 9(p+q)^2 - (p-q)^2 \geq 6^2,\ p+q \leq 0\right\} \cup \left\{\mathfrak{g}^X_\infty\right\}\right).$$

(2) *For $s > 0$, $\mathcal{L}^{c0+}_{0,s} = \mathbb{R}_+ \cdot \mathfrak{g}^A_{0,s} + \mathbb{R} \cdot \mathfrak{h}_s$.*

(3) *An extremal element in $\mathcal{P}^{c0+}_4$ is one of the forms $\alpha\mathfrak{g}^A_{s,t}$ (s, $t \in \mathbb{R}_+$) or $\alpha\mathfrak{h}_s$ ($s \in [0,\infty]$, $\alpha > 0$).*

Proof. (1) Since $\dim \mathcal{L}^{c0+}_{s,t} = 1$ and $\mathcal{L}^{c0+}_{s,t} \supset \mathcal{L}^{c0}_{s,t} \neq 0$, we have $\mathcal{L}^{c0+}_{s,t} = \mathcal{L}^{c0}_{s,t}$ for $s > 0$, $t > 0$. Especially, $\mathfrak{g}^A_{s,t}$ is extremal in $\mathcal{P}^{c0+}_4$.

(2) Note that $\dim \mathcal{L}^{c0+}_{0,s} \leq 2$ for $s > 0$. Let a, b, $c \in \mathbb{R}_+$, and $s > 0$. Then

$$\begin{aligned}\mathfrak{h}_s(a,b,c) &= s^2(S_{1,3} - US_1) - 2s(S_{2,2} - US_1) + (S_{3,1} - US_1) \\ &= (S_{1,3} - US_1)\left(s - \frac{S_{2,2} - US_1}{S_{1,3} - US_1}\right)^2 + \frac{US_1(S_2 - S_{1,1})^2}{S_{1,3} - US_1} \geq 0.\end{aligned}$$

A computation using Mathematica shows us $\mathfrak{h}_s(1,1,1) = \mathfrak{h}_s(0,s,1) = \mathfrak{h}_s(0,0,1) = 0$. Thus, $\mathfrak{h}_s \in \boldsymbol{\mathcal{L}}^{c0+}_{0,s} \cap \boldsymbol{\mathcal{L}}^{c0+}_{0,0} \subset \partial \boldsymbol{\mathcal{P}}^{c0+}_4$. Since $\dim\left(\boldsymbol{\mathcal{L}}^{c0+}_{0,s} \cap \boldsymbol{\mathcal{L}}^{c0+}_{0,0}\right) \le 1$, we have $\boldsymbol{\mathcal{L}}^{c0+}_{0,s} \cap \boldsymbol{\mathcal{L}}^{c0+}_{0,0} = \mathbb{R}_+ \cdot \mathfrak{h}_s$. Thus, $\mathfrak{h}_s$ is extremal in $\boldsymbol{\mathcal{P}}^{c0+}_4$. Since $\mathfrak{g}^A_{s,t}$ is also extremal, we have $\boldsymbol{\mathcal{L}}^{c0+}_{0,s} = \mathbb{R}_+ \cdot \mathfrak{g}^A_{0,s} + \mathbb{R} \cdot \mathfrak{h}_s$.

(3) follows from (1) and (2). □

Now, we can prove Theorem 0.3. We should perform Step 3 and 4 of Remark 1.23.

Proof of Theorem 0.3. For each vector subspace $V \subset \boldsymbol{\mathcal{H}}^{c0}_4$, let $\breve{V}$ be the set of all the monic polynomials in V. Note that $\varphi(p,q,r) = \mathrm{disc}^{c+}_4(1,p,q,r)$. Let $(x,\, y,\, z)$ be the coordinate system of $\breve{\boldsymbol{\mathcal{H}}}^{c0}_4 = \mathbb{R}^3$. By Theorem 4.4,

$$\breve{\boldsymbol{\mathcal{F}}}^{c0+}_4 = \left\{ (x,y,z) \in \breve{\boldsymbol{\mathcal{H}}}^{c0}_4 \;\middle|\; \begin{array}{l} x^2 + xy + y^2 = 3z + 3, \\ 9(x+y)^2 - (x-y)^2 \ge 6^2,\ x + y \le 0 \end{array} \right\}.$$

$\breve{\boldsymbol{\mathcal{E}}}^{c0+}_4$ is a subset of the sextic rational surface $S := \left\{ (x,y,z) \in \breve{\boldsymbol{\mathcal{H}}}^{c0}_4 \;\middle|\; \varphi(x,\, y,\, z) = 0 \right\}$. To observe the surfaces $\partial \breve{\boldsymbol{\mathcal{P}}}^{c0+}_4$, we draw the section of it by the surface $V_r := \left\{ (x,y,z) \in \breve{\boldsymbol{\mathcal{H}}}^{c0}_4 \;\middle|\; z = r \right\}$. Let $P_r := \boldsymbol{\mathcal{P}}^{c0+}_4 \cap V_r$, $C_r := (\partial \boldsymbol{\mathcal{P}}^{c0+}_4) \cap V_r$, $F_r := \boldsymbol{\mathcal{F}}^{c0}_4 \cap V_r$, $F^+_r := \boldsymbol{\mathcal{F}}^{c0+}_4 \cap V_r$, and $E_r := S \cap V_r$. Note that F_r is an ellipse defined by $3(x+y)^2 + (x-y)^2 = 12(r+1)$. F^+_r is an arc of the ellipse F_r with the ends

$$(x,\, y) = \left(\frac{\pm 3\sqrt{r} - \sqrt{r+4}}{2},\ \frac{\mp 3\sqrt{r} - \sqrt{r+4}}{2} \right), \tag{4.5}$$

since $9(x+y)^2 - (x-y)^2 \ge 6^2$. E_r is an irreducible sextic rational curve which does not depend on r. The rational curve E_r has a parametrization

$$x = \frac{1}{2}\left(\frac{1}{s^3} - \frac{r}{s} - 3s \right), \quad y = \frac{1}{2}\left(s^3 - rs - \frac{3}{s} \right) \quad s \in \mathbb{R} - \{0\}$$

If $r > 0$, E_r tangents to F_r at the points given by (4.5). In the case $r > 6$, we obtain the graph as in Figure 4.2.

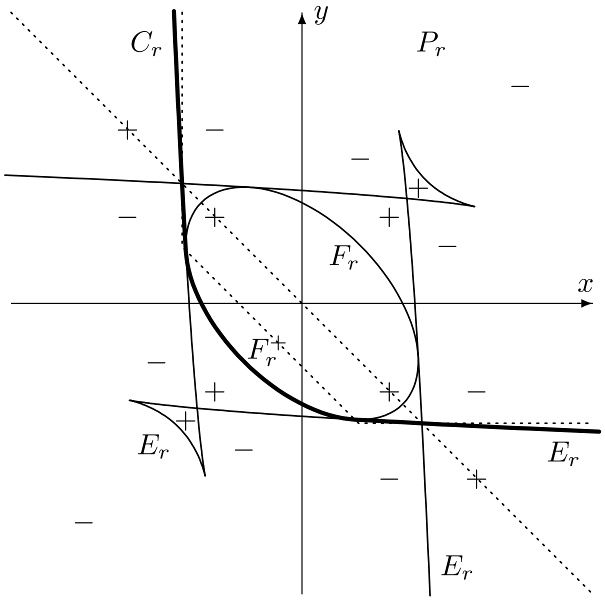

Fig.4.2. $r > 6$

The curve E_r has two branches which are symmetric with respect to the origin and the line $x + y = 0$. Each branch has two cusps at $s = \pm\sqrt{\dfrac{r \pm \sqrt{r^2 - 36}}{6}}$, and a node at $x = y = \pm 2\sqrt{r-2}$. The intersections of two branches $(x,\, y) = (\pm 2\sqrt{r+2},\, \mp 2\sqrt{r+2}\,)$ are also nodes of E_r. The boundary C_r of P_r is displayed by thick curve in Figure 4.2. Thus, in the case $r > 6$, we conclude that C_r can be represented as the union of the domain defined by (1), (2), (3), (4) and (5) of Theorem.

In the case $0 \le r \le 6$, the graph becomes as in Figure 4.3. E_r does not have cusps if $r < 6$. By the similar observation as the above case, we conclude that P_r can be represented as the union of the domain defined by (1), (2), (3), (4) and (5).

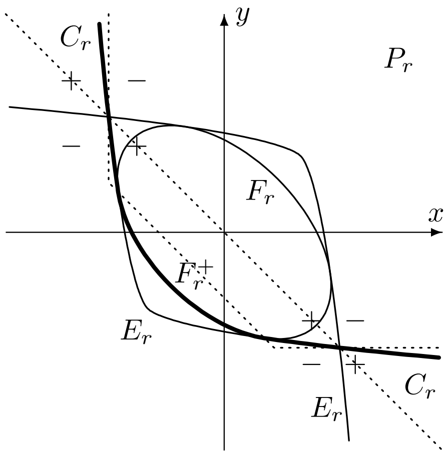

Fig.4.3. $0 \le r \le 6$

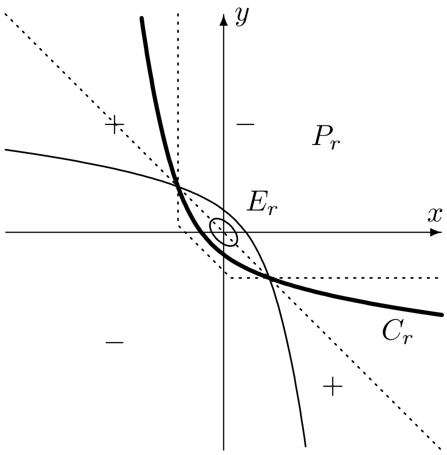

Fig.4.4. $-1 \le r < 0$

In the case $-1 \le r < 0$ (Fig.4.4), the ellipse F_r does not touch to E_r. Thus C_r agrees to one of the branches of E_r, and P_r can be represented as the union of the domain defined by (1), (2), (3) and (4).

In the case $-2 \le r < -1$ (Fig.4.5), the ellipse F_r does not appear. Thus, P_r can be represented as the union of the domain defined by (1), (2), (3) and (4).

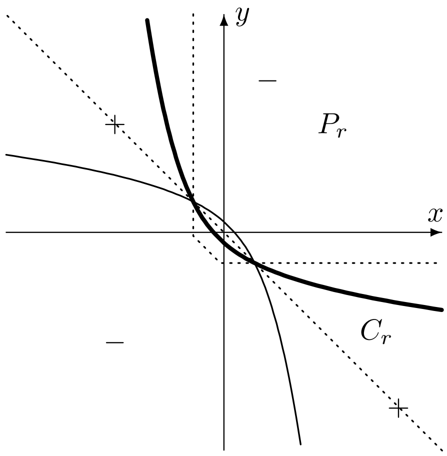

Fig.4.5. $-2 < r < -1$

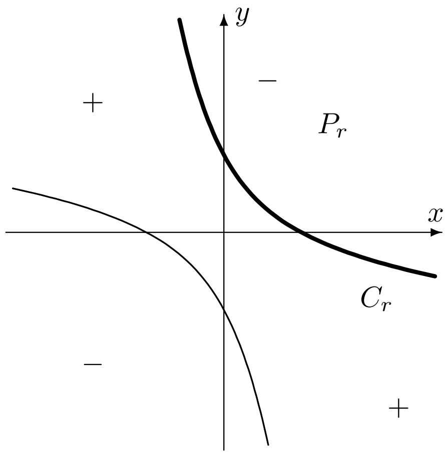

Fig.4.6. $r < -2$

In the case $r < -2$ (Fig.4.6), E_r has no singularities. Thus, P_r is the domain defined by (6). $\square$

Remark 4.6. Since $\mathrm{disc}_4^{c+}(1, p, p, q) = (4r - p^2 - 8)^2(r + 2p + 2)(r - 2p + 2)$, the curve $p = q$, $4r = p^2 + 8$ is also a zero locus of disc_4^{c+} in $\mathfrak{P}_4^{c0+}$. Points on this curve correspond to inequalities $(S_2 - tS_{1,1})^2 \geq 0$.

4.3. The PSD cone $\mathfrak{P}_4^c$.

By Proposition 2.6, $\partial\mathfrak{P}_4^c$ has just two face components. One is the main component $\mathfrak{F}_4^c$, and the other is $\mathfrak{P}_4^{c0}$. As a base of $\mathfrak{H}_4^c$, we choose $s_0 = S_4 - US_1$, $s_1 := S_{3,1} - US_1$, $s_2 := S_{1,3} - US_1$, $s_3 := S_{2,2} - US_1$, $s_4 := US_1$. The main discriminant is more complicated. So, we present it in inhomogeneous form for $f = s_0 + ps_1 + qs_2 + rs_3 + vs_4 \in \mathfrak{H}_4^c$.

$$\begin{aligned}\mathrm{disc}_4^c(1, p, q, r, v) = &\big(3(r + 1) - (p^2 + pq + q^2)\big)(2p + 2q + r + 5)^3 \\ &- v\Big(p^4 + q^4 + 34p^3q + 34pq^3 + 39p^2q^2 \\ &+ 2(p + q)(5p^2 + 7pq + 5q^2)r - (2p^2 + pq + 2q^2)r^2 \\ &+ 86p^3 + 86q^3 - 12(v - 16)(p^2q + pq^2) - (v - 84)(p^2 + q^2)r \\ &+ (v + 18)pqr - 22(p + q)r^2 + 8r^3 - 57(v - 2)(p^2 + q^2) \\ &+ (v^2 - 63v + 51)pq - 2(13v + 126)(p + q)r + 2(3v - 106)r^2 \\ &+ 2(7v^2 + 3v - 139)(p + q) + 8(19v - 70)r \\ &- (v^3 + 20v^2 - 162v + 388)\Big)\end{aligned}$$

We shall complete Step 1 and 2 of Remark 1.23.

Theorem 4.7. (Structure of $\mathfrak{P}_4^c$) *For $-1/2 \leq k \leq 1$, let*

$$\begin{aligned}\mathfrak{e}_k^X(a, b, c) &:= \big(k(a^2 + b^2 + c^2) - (ab + bc + ca)\big)^2 \\ &= k^2S_4 - 2kT_{3,1} + (2k^2 + 1)S_{2,2} - (2k - 2)US_1, \\ k(s, t) &= \frac{S_{1,1}(s, t, 1)}{S_2(s, t, 1)} = \frac{st + s + t}{s^2 + t^2 + 1} \in [-1/2,\, 1], \\ \mathfrak{e}_{s,t}^A(a, b, c) &:= \mathfrak{e}_{k(s,t)}^X(a, b, c).\end{aligned}$$

(1) For $(s, t) \in \mathbb{R}^2 - \{(1, 1)\}$, $\mathfrak{L}_{s,t}^c = \big\{\alpha\mathfrak{g}_{s,t}^A + \beta\mathfrak{e}_{s,t}^A \;\big|\; \alpha,\, \beta \in \mathbb{R}_+.\big\}$
(2) If $s > 0$, $t > 0$, and $(s, t) \neq (1, 1)$, then $\mathfrak{L}_{s,t}^{c+} := \mathfrak{L}_{s,t}^c$.
(3) An extremal element in $\mathfrak{P}_4^c$ is of the form $\alpha\mathfrak{g}_{p,q}^X$ $(\alpha > 0)$ or $\alpha\mathfrak{g}_\infty^X$ $(\alpha > 0)$ or $\alpha\mathfrak{e}_k^X$ $(-1/2 \leq k \leq 1,\ \alpha > 0)$.

Proof. (1), (2) By Proposition 1.27, $\dim\mathfrak{L}_{s,t}^c = N - 2 \leq 2$ if $(s,\, t) \neq (1,\, 1)$. If $s > 0$ and $t > 0$, then $\dim\mathfrak{L}_{s,t}^{c+} = 2$ and $\mathfrak{L}_{s,t}^c \subset \mathfrak{L}_{s,t}^{c+}$. Let $f_{s,t,\alpha,\beta} := \alpha\mathfrak{g}_{s,t}^A + \beta\mathfrak{e}_{s,t}^A$. Since $\mathfrak{g}_{s,t}^A \in \mathfrak{L}_{s,t}^c$, and $\mathfrak{e}_{s,t}^A \in \mathfrak{L}_{s,t}^c$, we have $f_{s,t,\alpha,\beta} \in \mathfrak{L}_{s,t}^c$, if $\alpha \geq 0$ and $\beta \geq 0$.

Since $f_{s,t,\alpha,0} \in \partial\mathfrak{P}_4^c$, we have $f_{s,t,\alpha,-\beta} \notin \mathfrak{P}_4^c$ by Proposition 1.33(1). If $s > 0$ and $t > 0$, then $f_{s,t,\alpha,-\beta} \notin \mathfrak{P}_4^{c+}$.

Assume that $\alpha < 0$. There exists $(s', t') \neq (s, t)$ such that $k(s', t') = k(s, t)$ and $\mathfrak{g}_{s,t}^A(s', t', 1) > 0$. Since $\mathfrak{e}_{s,t}^A(s', t', 1) = 0$, $f_{s,t,\alpha,1}(s', t', 1) < 0$. Thus $f_{s,t,\alpha,1} \notin \mathfrak{P}_4^c$. If $s > 0$ and $t > 0$, then $f_{s,t,\alpha,1} \notin \mathfrak{P}_4^{c+}$.

Thus $\mathfrak{L}_{s,t}^c$ and $\mathfrak{L}_{s.t}^{c+}$ are generated by $\mathfrak{g}_{s,t}^A$ and $\mathfrak{e}_{s,t}^A$.

(3) follows from (1). ∎

We now complete Steps 1, 2, and 3 of Remark 1.23 for $\mathcal{P}^c_4$. Regretfully, we can't give complete proof for Step 4. But graphical observation leads to the following:

Conjecture 4.8. For $f = S_4 + pS_{3,1} + qS_{1,3} + rS_{2,2} + (v - 1 - 2p - r)US_1 \in \mathcal{H}^c_4$, $f(a, b, c) \geq 0$ for all a, b, $c \in \mathbb{R}$, if and only if $v \geq 0$ and one of the following hold:
(1) $v = 0$ and $\mathrm{disc}^{c0}_4(1, p, q, r) \geq 0$.
(2) $0 < v \leq 27$ and $\mathrm{disc}^c_4(1, p, q, r, v) \geq 0$ and $4r + 4(u + 2\sqrt{3u} + 1) \geq (p + q)^2$.
(3) $v > 27$ and $\mathrm{disc}^c_4(1, p, p, r, v) \geq 0$ and $r \geq \dfrac{(p+q)^2}{16} + 2$.

The domain $\mathrm{disc}^c_4 \geq 0$ consists of some blocks. $\mathrm{Sing}(V(\mathrm{disc}^c_4))$ is complicated. What we should prove is that $4r + 4(u + 2\sqrt{3u} + 1) \geq (p + q)^2$ and $r \geq (p + q)^2/16 + 2$ with $\mathrm{disc}^{c0}_4(1, p, q, r) \geq 0$ cut off the correct $\mathcal{P}^c_4$. These two inequalities are not determinants.

By the way, Theorem 4.7 resolves the following conjecture. The sufficiency part was proved by Cîrtoaje and Zhou in [13]. The necessity part was their conjecture.

Theorem 4.9. (Cîrtoaje-Zhou Conjecture) *For $f = S_4 + pS_{3,1} + qS_{1,3} + rS_{2,2} + vUS_1 \in \mathcal{H}^c_4$, let*

$$\begin{aligned}
\alpha_f &:= 1 + p + q + r + v = \frac{1}{3} f(1, 1, 1),\\
\beta_f &:= 6 + 3p + 3q + 2r + v,\\
\gamma_f &:= 2(1 + p + q),\\
\delta_f &:= 2 + 2r - v - (p^2 + pq + q^2 + p + q),\\
\varphi_f(x) &:= 2\sqrt{\alpha_f}x^3 - \beta_f x^2 + \gamma_f \sqrt{\alpha_f}x + \delta_f.
\end{aligned}$$

Then, $f(a, b, c) \geq 0$ for all a, b, $c \in \mathbb{R}$, if and only if one of (1) *and* (2) *holds.*
(1) *$\alpha_f \geq 0$ and there exists a $x \in (-\sqrt{3}, \sqrt{3})$ such that $\varphi_f(x) \geq 0$.*
(2) *f is of the form $f = (S_2 - \kappa S_{1,1})^2$ ($\exists \kappa \in \mathbb{R}$). That is, f satisfies $p = q$, $p^2 - 4p = 4r$ and $p^2 + 2p = 2v$.*

Moreover, f which does not satisfies (2) *belongs to $\mathcal{P}^{c0}_4$ if and only if the solution of $\varphi_f(x) \geq 0$ in $x \in (-\sqrt{3}, \sqrt{3})$ is unique.*

Proof. (If part) It is clear that if f satisfies (2) then $f \in \mathcal{P}^c_4$. We assume that f satisfies (1). Fix an $x_0 \in (-\sqrt{3}, \sqrt{3})$ with $\varphi_f(x_0) \geq 0$, and let

$$\begin{aligned}
p_1 &:= \frac{3p - 2\sqrt{\alpha_f}x_0 + 2x_0^2}{3 - x_0^2}, \quad q_1 := \frac{3q - 2\sqrt{\alpha_f}x_0 + 2x_0^2}{3 - x_0^2},\\
r_1 &:= \frac{3r - \alpha_f + 2\sqrt{\alpha_f}x_0 - 3x_0^2}{3 - x_0^2}, \quad v_1 := \frac{3v - 2\alpha_f + 2\sqrt{\alpha_f}x_0}{3 - x_0^2},\\
f_1(a, b, c) &:= S_4 + p_1S_{3,1} + q_1S_{1,3} + r_1S_{2,2} + v_1US_1,\\
e_1(a, b, c) &:= \mathfrak{e}^X_{x_0/(x_0 - \sqrt{\alpha_f})}(a, b, c).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&3f = (3 - x_0^2)f_1 + (x_0 - \sqrt{\alpha_f})^2 e_1 \geq (3 - x_0^2)f_1,\\
&f_1(1, 1, 1) = 1 + p_1 + q_1 + r_1 + v_1 = 0,\\
&\frac{p_1^2 + p_1q_1 + q_1^2}{3} - 1 - r_1 = -\frac{3\varphi_f(x_0)}{(3 - x_0^2)^2} \leq 0.
\end{aligned}$$

Note that $f_1 \in \mathcal{P}_4^{c0}$. Since $\mathfrak{g}_{p_1,q_1}^X$, $\mathfrak{g}_\infty^X$, $e_1 \in \mathcal{P}_4^c$, we have

$$f = \frac{3-x_0^2}{3}\mathfrak{g}_{p_1,q_1}^X + \frac{\varphi_f(x_0)}{3-x_0^2}\mathfrak{g}_\infty^X + \frac{(x_0-\sqrt{\alpha_f}\,)^2}{3}e_1 \in \mathcal{P}_4^c.$$

(Only if part) We shall prove that if $f \in \mathcal{P}_4^c$ does not satisfy (2) then f satisfies (1). Since $f \in \mathcal{P}_4^c$, we have $1+p+q+r+v = \dfrac{1}{3}f(1,1,1) \geq 0$, Note that

$$\varphi_f(\sqrt{3}\,) = -4\left(\sqrt{\alpha_f} - \frac{\sqrt{3}}{4}(p+q+4)\right)^2 - \frac{(p-q)^2}{4} \leq 0,$$

$$\varphi_f(-\sqrt{3}\,) = -4\left(\sqrt{\alpha_f} + \frac{\sqrt{3}}{4}(p+q+4)\right)^2 - \frac{(p-q)^2}{4} \leq 0.$$

Case 1: We treat the case $f \in \mathcal{L}_{s,t}^c$.

Let $p_2 := \mathfrak{p}(s,t)$, $q_2 := \mathfrak{p}(t,s)$, $r_2 := \dfrac{p_2^2+p_2q_2+q_2^2}{3} - 1$, and $v_2 := -p_2 - q_2 - \dfrac{p_2^2+p_2q_2+q_2^2}{3}$. We assume that $-1/2 \leq k < 1$, and $f = (\alpha_2\mathfrak{g}_{p_2,q_2}^X + \beta_2\mathfrak{e}_k^X)/(\alpha_2+\beta_2k^2)$ (Not always $f \in \mathcal{L}_{s,t}^c$). Then

$$p = \frac{\alpha_2p_2 - 2\beta_2k}{\alpha_2+\beta_2k^2}, \quad q = \frac{\alpha_2q_2-2\beta_2k}{\alpha_2+\beta_2k^2},$$
$$r = \frac{\alpha_2r_2+\beta_2(2k^2+1)}{\alpha_2+\beta_2k^2}, \quad v = \frac{\alpha_2v_2-\beta_2(2k-2)}{\alpha_2+\beta_2k^2}.$$

Using Mathematica, very complicated calculation show us that

$$\varphi_f\left(\frac{\sqrt{\alpha_f}k}{k-1}\right) = 0, \quad \text{and} \quad \left|\frac{\sqrt{\alpha_f}k}{k-1}\right| = \left|-\frac{\sqrt{3\beta_2}k}{\sqrt{\alpha_2+\beta_2k^2}}\right| < \sqrt{3}.$$

Moreover, if $k = \dfrac{S_{1,1}(s,t,1)}{S_2(s,t,1)}$, that is, if $f \in \mathcal{L}_{s,t}^c$, then $x = \dfrac{\sqrt{\alpha_f}k}{k-1}$ is a multiple root of the cubic equation $\varphi_f(x) = 0$. Since $\varphi_f(\sqrt{3}\,) \leq 0$ and $\varphi_f(-\sqrt{3}\,) \leq 0$, we have

$$\left\{x \in (-\sqrt{3},\,\sqrt{3}\,) \;\middle|\; \varphi_f(x) \geq 0\right\} = \left\{\frac{\sqrt{\alpha_f}k}{k-1}\right\}.$$

Case 2: Consider the case $f \in \mathcal{P}_4^{c0}$.

In this case, $3\alpha_f = f(1,1,1) = 0$, and $\varphi_f(x) = -(5+2p+2q+r)x^2+(3+3r-p^2-pq-q^2)$. Since $\varphi_f(0) = \mathrm{disc}_4^{c0}(1,p,q,r) \geq 0$ and $\varphi_f(\sqrt{3}\,) < 0$, there exists $0 \leq x_1 < \sqrt{3}$ such that $\varphi_f(x_1) = 0$. Note that if $f \notin \partial\mathcal{P}_4^{c0}$, then $\varphi_f(0) > 0$.

Case 3: Consider the general case $f \in \mathcal{P}_4^c$.

In the convex cone $\mathcal{P}_4^c$, we take a line ℓ which passes through a point f and a point on the half line $\mathbb{R}_+ \cdot \mathfrak{g}_\infty^X$. Consider the intersection of ℓ and $\partial\mathcal{P}_4^c$. Then there exists the unique $\delta \geq 0$ such that $f_0 := f - \delta\mathfrak{g}_\infty^X \in \partial\mathcal{P}_4^c$. Equivalently

$$f_0 = S_4 + pS_{3,1} + qS_{1,3} + (r-\delta)S_{2,2} + (v+\delta)US_1.$$

Thus $\varphi_{f_0}(x) = \varphi_f(x) - \delta(3-x^2)$. By Proposition 2.6, $f_0 \in \mathcal{L}_{s,t}$ or $f_0 \in \mathcal{P}_4^{c0}$.

Case 3-1: Consider the case $f_0 \in \mathcal{L}_{s,t}^c$.

Then, there exists the unique $x_0 \in (-\sqrt{3},\ \sqrt{3}\,)$ such that, $\varphi_{f_0}(x_0) = 0$ by the result of the Case 1. Then $\varphi_f(x_0) = \varphi_{f_0}(x_0) + \delta(3 - x_0^2) \geq 0$. Thus (1) holds. Note that if $f \notin \partial\mathcal{P}_4^c$ then $\varphi_f(x_0) > 0$. Thus there exists many $x \in (-\sqrt{3},\ \sqrt{3}\,)$ such that $\varphi_f(x) \geq 0$.

Case 3-2: Consider the case $f_0 \in \mathcal{P}_4^{c0}$ and $f_0 \notin \mathcal{F}_4^{c0}$.

Then, by the result of the Case 2, we have $\varphi_f(0) \geq \varphi_{f_0}(0) > 0$. $\square$

4.4. The PSD cone $\mathcal{P}_4^{c+}$.

By Proposition 2.6, $\mathcal{P}_4^{c+} = \mathcal{P}(\mathbb{P}_+^2,\ \mathcal{H}_4^c)$ has at most four face components: the main component $\mathcal{F}_4^{c+}+$, the edge component $\mathcal{E}_4^{c+}$, $\mathcal{P}_4^{c0+}$ and $\mathcal{F}(O_4)$. All these four exist as the following theorem.

Theorem 4.10. *The five dimensional cone $\mathcal{P}_4^{c+}$ satisfies the following:*

(1) $\mathcal{F}_4^{c+} = \operatorname{Cls}\left(\bigcup_{(s,t)\in\mathbb{R}_+^2-\{(1,1)\}} \mathcal{L}_{s,t}^c \right)$ *is a face component.*
(2) $\mathcal{E}_4^{c+} = \left\{ \alpha_1 \mathfrak{e}_{0,s}^A + \alpha_2 \mathfrak{g}_{0,s}^A + \alpha_3 \mathfrak{h}_s + \alpha_4 US_1 \;\middle|\; s \geq 0,\ \alpha_i \geq 0 \right\}$ *is a face component.*
(3) *The main discriminant of $\mathcal{F}_4^{c+}$ is disc_4^c, and the edge discriminant is disc_4^{c+}.*
(4) *An extremal element of $\mathcal{P}_4^{c+}$ is of the form $\alpha\mathfrak{g}_{p,q}^X$ $(9(p+q)^2 - (p-q)^2 \geq 6^2$, $p+q \leq 0$, $\alpha > 0)$ or $\alpha\mathfrak{g}_\infty^X$ $(\alpha > 0)$ or $\alpha\mathfrak{h}_s$ $(s \in [0,\infty]$, $\alpha > 0)$ or $\alpha\mathfrak{e}_k^X$ $(0 \leq k \leq 1$, $\alpha > 0)$ or αUS_1 $(\alpha > 0)$.*

Proof. (1) follows from Theorem 4.7(2).

(2) Since $f = \alpha_1 \mathfrak{e}_{0,s}^A + \alpha_2 \mathfrak{g}_{0,s}^A + \alpha_3 \mathfrak{h}_s + \alpha_4 US_1$ satisfies $f(0,s,1) = 0$, we have $f \in \mathcal{L}_{0,s}^{c+}$ for $s \geq 0$, $\alpha_i \geq 0$.

We shall prove the converse. Since $\dim \mathcal{L}_{0,s}^{c+} = 3$ for $s > 0$, any element $f \in \mathcal{L}_{0,s}^{c+}$ can be represented as $f = \alpha_1 \mathfrak{e}_{0,s}^A + \alpha_2 \mathfrak{g}_{0,s}^A + \alpha_3 \mathfrak{h}_s + \alpha_4 US_1$ by certain $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \in \mathbb{R}$. We shall show that we can choose $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \in \mathbb{R}_+$. Since

$$(s^2+1)^2 \mathfrak{e}_{0,s}^A + 3s\mathfrak{h}_s = s^2 \mathfrak{g}_{0,s}^A + 3(s^2 - s + 1)^2 US_1,$$

we can assume that (i)"$\alpha_2 \geq 0$ and $\alpha_4 = 0$" or (ii) "$\alpha_2 = 0$ and $\alpha_4 \geq 0$".

Consider the case (i). Assume that $f = \alpha_1 \mathfrak{e}_{0,s}^A + \alpha_2 \mathfrak{g}_{0,s} + \alpha_3 \mathfrak{h}_s \in \mathcal{P}_4^{c+}$, and $\alpha_2 \geq 0$. Since $f(1,1,1) = \alpha_1 \mathfrak{e}_{0,s}^A(1,1,1)$, we have $\alpha_1 \geq 0$. Since

$$\begin{aligned}
\mathfrak{e}_{0,s}^A(x,s,1) &= \frac{x^2(s^3+s^2+s+1-sx)^2}{(s^2+1)^2}, \\
\mathfrak{g}_{0,s}^A(x,s,1) &= \frac{x^2\big(s^6 - s^5 + s^3 - s + 1 - (2s^4 - s^3 - s^2 + 2s)x + s^2x^2\big)}{s^2}, \\
\mathfrak{h}_s(x,s,1) &= x\big(s^5 - s^4 + s^3 + s^2 - s + 1 - (3s^3 - 2s^2 + 3s)x + (s^2+s)x^2\big),
\end{aligned}$$

we have $\displaystyle\lim_{x\to 0} \frac{\mathfrak{e}_{0,s}^A(x,s,1)}{\mathfrak{h}_s(x,s,1)} = 0$ and $\displaystyle\lim_{x\to 0} \frac{\mathfrak{g}_{0,s}^A(x,s,1)}{\mathfrak{h}_s(x,s,1)} = 0$. Thus $\displaystyle 0 \leq \lim_{x\to 0} \frac{f(x,s,1)}{\mathfrak{h}_s(x,s,1)} = \alpha_3$.

Consider the case (ii). Assume that $f = \alpha_1 \mathfrak{e}_{0,s}^A + \alpha_3 \mathfrak{h}_s + \alpha_4 US_1 \in \mathcal{P}_4^{c+}$ and $\alpha_4 \geq 0$. Since $f(0,0,1) = \alpha_1 \mathfrak{e}_{0,s}^A(0,0,1)$, we have $\alpha_1 \geq 0$. Since $\mathfrak{e}_{0,s}^A(0,1/s,1) = 0$, $US_1(0,1/s,1) = 0$, $\displaystyle\mathfrak{h}_s(0,1/s,1) = \frac{(s-1)^2(s+1)^2}{s^3} > 0$, and $f(0,1/s,1) = \alpha_3 h_s(0,1/s,1)$, we have $\alpha_3 \geq 0$.

Thus we have

$$\mathcal{L}_{0,s}^{c+} = \left\{ \alpha_1 \mathfrak{e}_{0,s}^A + \alpha_2 \mathfrak{g}_{0,s}^A + \alpha_3 \mathfrak{h}_s + \alpha_4 US_1 \;\middle|\; \alpha_i \geq 0 \right\}$$

for $s > 0$. The left part is easy.

(3) The main discriminant is disc_4^c by Proposition 2.8, and the edge discriminant is disc_4^{c+} by Proposition 2.9.

(4) is clear. □

Thus, we complete Step 1 and 2 of Remark 1.23, and we determined all the extremal inequalities. We don't yet succeed in Step 3 and 4.

4.5. The PSD cones $\mathcal{P}_4^s$.

Let $\mathcal{H}_d^s$ and $\mathcal{H}_d^{s0}$ be the same with Proposition 2.14. The aim of this subsection is to prove the following theorem.

Theorem 4.11. *Take $f = S_4 + pT_{3,1} + rS_{2,2} + (v - 1 - 2p - r)US_1 \in \mathcal{H}_4^s$. Then, $f(a, b, c) \geq 0$ for all a, b, $c \in \mathbb{R}$, if and only if $v \geq 0$ and one of the (1), (2) and (3) holds.*

(1) *$v = 0$ and $r \geq p^2 - 1$.*

(2) *$0 < v \leq 27$, $\mathrm{disc}_4^c(1, p, p, r, v) \geq 0$, and $r \geq p^2 - (v + 2\sqrt{3v} + 1)$.*

(3) *$v > 27$, $\mathrm{disc}_4^c(1, p, p, r, v) \geq 0$, and $r \geq \dfrac{p^2}{4} + 2$.*

Put $G := \mathfrak{S}_3$, $\Phi_d^s = \Phi_{\mathcal{H}_d^s}$, $\Phi_d^{s0} = \Phi_{\mathcal{H}_d^{s0}}$, $\mathcal{P}_d^s = \mathcal{P}(\mathbb{P}_{\mathbb{R}}^2, \mathcal{H}_d^s)$, and $\mathcal{P}_d^{s0} = \mathcal{P}(\mathbb{P}_{\mathbb{R}}^2, \mathcal{H}_d^{s0})$. We have to study of $\mathcal{P}_4^{s0}$ before $\mathcal{P}_4^s$. We choose $s_0 = S_4 - US_1$, $s_1 = T_{3,1} - 2US_1$, $s_2 = S_{2,2} - US_1$ as a base of $\mathcal{H}_4^{s0}$, and we choose s_0, s_1, s_2 and $s_3 = US_1$ as a base of $\mathcal{H}_4^s$. We shall execute Step 1 of Remark 1.23. Let $X_4^s = X(\mathbb{P}_{\mathbb{R}}^2, \mathcal{H}_4^s)$, and $X_4^{s0} = X(\mathbb{P}_{\mathbb{R}}^2, \mathcal{H}_4^{s0})$. As a result of elimination, we obtain

$$X_4^{s0} = \{(x_0 : x_1 : x_2) \in \mathbb{P}_{\mathbb{R}}^2 \mid F_4^{s0}(x_0, x_1, x_2) \leq 0\},$$
$$X_4^s = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}_{\mathbb{R}}^3 \mid F_4^{s0}(x_0, x_1, x_2) \leq 0 \text{ and } F_4^s(x_0, x_1, x_2, x_3) = 0\},$$

here

$$F_4^{s0}(x_0, x_1, x_2) := x_1^2 + (x_0 - 2x_2)^2 - x_0^2,$$
$$F_4^s(x_0, x_1, x_2, x_3) := (x_1 + 3x_3)^2 - (x_0 + 2x_2 + 3x_3)(x_2 + 3x_3) = 0.$$

Using Proposition 2.13, 2.14, and the above, we obtain the following:

$$\Delta^0(X_4^{s0}) = \emptyset,\ \Delta^1(X_4^{s0}) = \{\Phi_4^{s0}(\overline{L_F^b})\},\ \Delta^2(X_4^{s0}) = \{\Phi_4^{s0}(A^\circ)\},$$
$$\Delta^0(X_4^s) = \{\Phi_4^{s0}(\mathbf{1})\},\ \Delta^1(X_4^s) = \{\Phi_4^s(L_F^b)\},\ \Delta^2(X_4^s) = \{\Phi_4^s(A_F^\circ)\},$$

where $\mathbf{1} = (1 : 1 : 1)$, $L_F^b := \{(s : 1 : 1) \in \mathbb{P}_{\mathbb{R}}^2 \mid s \neq 1\}$, and $A_F^\circ := \{(s : t : 1) \in \mathbb{P}_{\mathbb{R}}^2 \mid s + t + 1 > 0,\ s < t < 1\}$. Since X_4^{s0} lies on a plane and $\mathrm{Zar}(X_4^s)$ is a ruled surface (conic cone), their dual varieties have lower dimensions. Thus, $\mathcal{P}_4^{s0}$ and $\mathcal{P}_4^s$ have no main component. Note that $\mathcal{F}(\Phi_4^{s0}(\mathbf{1})) = \mathcal{P}_4^{s0}$. Since $\mathcal{P}_4^{s0}$ has only one face component, we immediately have the following proposition.

Proposition 4.12. (cf. [2], [10]) (1) *For $f = pS_4 + qT_{3,1} + rS_{2,2} - (p + 2q + r)US_1 \in \mathcal{H}_4^{s0}$, $f(a, b, c) \geq 0$ for all a, b, $c \in \mathbb{R}$ if and only if $p \geq 0$ and $p(r + q) \geq r^2$.*

(2) *The extremal elements of $\mathcal{P}_4^{s0}$ is a positive multiple of $\mathfrak{g}_{p,p}^X$ ($\exists p \in \mathbb{R}$) or $S_{2,2} - US_1$.*

We shall study the dual variety of $\Phi_4^s(L_F^b)$. Put $\mathcal{F}_4^s := \mathcal{F}(\Phi_4^s(L_F^b))$. By the algorithm of Remark 1.21(1), we obtain that the discriminant of $\mathcal{F}_4^s$ agrees with $\mathrm{disc}_4^c(p_0, p_1, p_1, p_2, p_3)$.

The dual variety of $\mathrm{Zar}(X_4^s)$ is defined by $8p_0^2 + p_1^2 - 4p_1p_2 = 0$ and $3p_0 - 3p_1 - 3p_2 + p_3 = 0$. Its extremal inequality at $(s : t : 1) \in A = \mathbb{P}_{\mathbb{R}}^2$ is $\mathfrak{e}_{s,t}^A$. Thus we can determine all the extremal inequalities of $\mathcal{P}_4^s$ as the following:

Proposition 4.13. *Let $\boldsymbol{\mathfrak{L}}^s_{s,t}$ be the local cone of $\boldsymbol{\mathfrak{P}}^s_4$ at $(s:t:1) \in A = \mathbb{P}^2_{\mathbb{R}}$.*
(1) *If $s \neq 1$, $t \neq 1$, $s \neq t$ and $s+t+1 \neq 0$, then $\boldsymbol{\mathfrak{L}}^s_{s,t} = \mathbb{R} \cdot \mathfrak{e}^A_{s,t}$.*
(2) *If $s = 1$ or $t = 1$ or $s = t$ or $s+t+1 = 0$, then $\boldsymbol{\mathfrak{L}}^s_{s,t} = \mathbb{R} \cdot \mathfrak{g}^A_{s,t} + \mathbb{R} \cdot \mathfrak{e}^A_{s,t}$.*

We now have completed Step 1 and 2 of Remark 1.23 for $\boldsymbol{\mathfrak{P}}^s_4$. We need an elementary lemma to proceed Step 3 and 4.

Lemma 4.14. *Let*

$$g(x,t) := 108\sqrt{3}t^3 + 36(10-3x)t^2 - \sqrt{3}(x+2)^2(4x+47)t + 6(x+2)^4,$$
$$h(x,v) := (4v + x^2 - 44x + 52)^2 + 128(x-4)^3.$$

(1) *$g(x,t) > 0$ for $0 < t < 3\sqrt{3}$ and $x < 12$.*
(2) *$h(x,v) > 0$ for $v > 27$ and $x \in \mathbb{R}$.*

Proof. (1) Consider $g(x,t)$ as a cubic function on t. The greater solution of

$$\frac{\partial g(x,t)}{\partial t} = 324\sqrt{3}t^2 + 72(10-3x)t - \sqrt{3}(x+2)^2(4x+47) = 0$$

is $t_1(x) := \dfrac{\sqrt{g_1(x)} - 2(10-3x)}{18\sqrt{3}}$, where $g_1(x) := 12x^3 + 225x^2 + 372x + 964$.

Case 1: Consider the case $g_1(x) < 0$. Since the coefficient of t^3 in $g(x,t)$ is positive, we have $g(x,t) > g(x,0) = (x+2)^4 \geq 0$ for $t \geq 0$.

Case 2: Consider the case $g_1(x) \geq 0$ and $x < 12$. Note that $g(x,0) = (x+2)^4 \geq 0$, and $g(x,t) \geq \max\{g(x,t_1(x)), g(x,0)\}$ for $t \geq 0$. Thus, it is enough to show $g(x,t_1(x)) \geq 0$. Using Mathematica, we have

$$g(x,t_1(x)) = \frac{1}{81}\big(g_2(x) - g_1(x)^{3/2}\big),$$
$$\text{here} \quad g_2(x) = 378x^4 + 2331x^3 + 13986x^2 + 21636x + 32696.$$

Since $g_2(x)^2 - g_1(x)^3 = 108(12-x)^3(x+2)^4(16x^2+25x+58)$, we have $g(x,t_1) > 0$ for $x < 12$.

Case 3: Consider the case $0 < t < 3\sqrt{3}$ and $x \geq 12$. Since $t_1(x)$ is increasing for $x \geq -2$, we have $t_1(x) \geq t_1(12) = \dfrac{49\sqrt{3}}{9} > 3\sqrt{3}$ for $x \geq 12$. It is easy to see

$$g(x,\sqrt{3}) = 3(2x^4 + 4x^3 - 141x^2 - 1520x + 11456) > 0.$$

Since $g(x,\,0) > 0$, we have $g(x,t) > 0$ for $x > 12$ and $0 < t < 3\sqrt{3}$.

(2) If $x > 4$, then $(x-4)^3 > 0$ and $h(x,v) > 0$. If $x \leq 4$,

$$\begin{aligned} h(x,y) &= (x-4)^2\big((x+24)^2 + 512\big) + 8(v-27)\big(2(v-27) + (x-4)(x-40)\big) \\ &\geq 16(v-27)^2 > 0. \end{aligned}$$

□

Proof of Theorem 4.11. For each vector subspace $V \subset \boldsymbol{\mathfrak{H}}^s_4$, let $\check{V}$ be the set of all the monic polynomials in V.

Since $f(1,1,1) = 3v$, $f \in \boldsymbol{\mathfrak{P}}^s_4$ satisfies $v \geq 0$. If $v = 0$, $f \in \boldsymbol{\mathfrak{P}}^s_4$ if and only if $r \geq p^2 - 1$, by Proposition 4.12. Now we assume $v > 0$.

We use the symbol $(x,\,y,\,z)$ instead of $(p,\,r,\,v)$ as the coordinate system of $\check{\boldsymbol{\mathfrak{H}}}^s_4$. Fix a constant $v > 0$, and let H_v be the plane $z = v$ in $\check{\boldsymbol{\mathfrak{H}}}^s_4$. Let $T_v := \boldsymbol{\mathfrak{P}}^s_4 \cap H_v$, $F_v := \boldsymbol{\mathfrak{F}}^s_4 \cap H_v$, and let C_v be the curve defined by $\mathrm{disc}_4^c(1,x,x,y,v)=0$ on H_v. Note that $F_v \subset C_v$. The curve C_v is a rational curve with a parameterization

$$x = t + \frac{v(2t+1)}{(t+2)^3}, \quad y = t^2 - 1 + \frac{v(-t^3+2t^2+3t+2)}{(t+2)^3}$$

($t \in \mathbb{P}^1_{\mathbb{R}} - \{-2\}$). We can draw the graph of C_v using the above parameterization. The boundary F_v of T_v is displayed by thick lines in Fig 4.7 and 4.8.

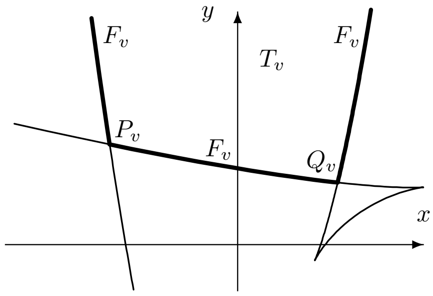

Fig.4.7. $v > 27$

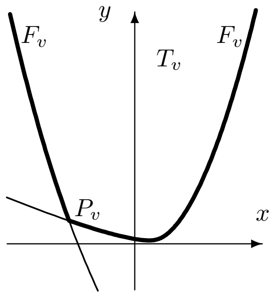

Fig.4.8. $0 < v < 27$

When $v > 0$, the curve C_v has a node at

$$P_v : (x,y) = \left(-2\sqrt{\frac{v}{3}} - 2,\ \frac{v + 2\sqrt{3v} + 9}{3}\right).$$

If $v > 27$, the following Q_v is also a node of C_v.

$$Q_v : (x,y) = \left(2\sqrt{\frac{v}{3}} - 2,\ \frac{v - 2\sqrt{3v} + 9}{3}\right).$$

Note that Q_v is an isolated zero in T_v if $0 < v < 27$. These nodes P_v, Q_v correspond to polynomials $\mathfrak{e}_k^X$.

(2) Consider the case $0 < v \le 27$. Let $t := \sqrt{v} > 0$. Substitute $y = -(t^2 + 2\sqrt{3}t + 1)$ to $\mathrm{disc}_4^c(1,x,x,y,t^2) = 0$, then we have

$$\mathrm{disc}_4^c\left(1,x,x,\left(x^2 - (t^2 + 2\sqrt{3}t + 1)\right),t^2\right) = -\sqrt{3}t\left(x + \frac{2\sqrt{3}}{3}t + 2\right)^2 g(x,t) = 0,$$

here $\ g(x,t) := 108\sqrt{3}t^3 + 36(10-3x)t^2 - \sqrt{3}(x+2)^2(4x+47)t + 6(x+2)^4.$

Since $g(x,t) > 0$ for $0 < t < 3\sqrt{3}$ and $x < 12$ by the previous lemma, we conclude that the curve C_v and the parabola $y = x^2 - (v + 2\sqrt{3v} + 1)$ intersect at only P_v. Thus, this parabola cut off extra domain which does not belong to T_v, from the domain $\mathrm{disc}_4^c(1,x,x,y,v) \ge 0$.

(3) Consider the case $v > 27$. The parabola $y = x^2/4 + 1$ passes through P_v and Q_v. Note that

$$\mathrm{disc}_4^c\left(1,x,x,\frac{x^2}{4} + 2, v\right) = -\frac{\left(3(x+2)^2 - 4v\right)^2 h(x,v)}{256},$$

here $h(x,v)$ is a polynomial of the previous lemma. Since $h(x,v) > 0$, the roots of the equation $\mathrm{disc}_4^c(1,x,x,x^2/4+2,v) = 0$ are $x = \pm 2\sqrt{\frac{v}{3}} - 2$. Thus C_v and the parabola intersect only at P_v and Q_v. Thus, this parabola cut of extra domain which does not belong to T_v, from the domain $\mathrm{disc}_4^c(1,x,x,y,v) \ge 0$. $\square$

Section 5. Quintic inequalities.

5.1. The PSD cone $\mathcal{P}_5^{s0+}$.

Since $\dim \mathcal{H}_5^{s0} = 4$, it will not be so hard to determine $\mathcal{P}_5^{s0+}$. Before the Step 1 of Remark 1.23, we introduce some elementary propositions.

Proposition 5.1. ([13] Theorem 2.1. See also the proof of Theorem 6.1.) *Let $f(x, y, z)$ be symmetric homogeneous polynomial with $3 \le \deg f \le 5$. Then $f(x, y, z) \ge 0$ for all $x \ge 0$, $y \ge 0$, $z \ge 0$, if and only if $f(s, 1, 1) \ge 0$ and $f(0, s, 1) \ge 0$ for all $s \ge 0$.*

For a generalization of the above proposition, see [23] Theorem 4.5 and [26] I Theorem 2.2.

Proposition 5.2. *Let $f(a, b, c) := S_5 + pT_{4,1} + qT_{3,2} + rUS_2 - (1 + 2p + 2q + r)US_{1,1} \in \breve{\mathcal{H}}_5^{s0}$. Then, $f(0, s, 1) \ge 0$ for all $s \ge 0$ if and only if one of (1) and (2) holds.*

(1) *$p \ge -3$ and $p + q + 1 \ge 0$.*
(2) *$p < -3$ and $4q \ge (p + 1)^2 + 4$.*

Proof. Since $f(0, 1, 1) = 2(p + q + 1)$, $p + q + 1 \ge 0$ is a necessary. Note that

$$\begin{aligned} f(0, x, 1) &= (x^5 + 1) + p(x^4 + 1) + q(x^3 + x^2) \\ &= (1 + x)(1 + (p - 1)x + (1 - p + q)x^2 + (p - 1)x^3 + x^4). \end{aligned}$$

Thus, let $\alpha := p - 1$, $\beta := 1 - p + q$, and

$$\begin{aligned} g(x) &:= (1 + (p - 1)x + (1 - p + q)x^2 + (p - 1)x^3 + x^4) \\ &= x^4 + \alpha x^3 + \beta x^2 + \alpha x + 1 \\ &= x^2 \left(\left(x + \frac{1}{x} \right)^2 + \alpha \left(x + \frac{1}{x} \right)^2 + (\beta - 2) \right). \end{aligned}$$

Again, let $y := x + 1/x$ and $h(y) := y^2 + py + (q - 2)$. Then

$$\begin{aligned} f(x, 1, 0) \ge 0 \ (\forall x \ge 0) &\iff g(x) \ge 0 \ (\forall x \ge 0) \\ &\iff h(y) := y^2 + \alpha y + (\beta - 2) \ge 0 \ (\forall y \ge 2). \end{aligned}$$

Note that $h(2) = f(1, 1, 0) = p + q + 1$ and $h(y) = (y + \alpha/2)^2 + (4\beta - 8 - \alpha^2)/4$.

If $\alpha \ge -4$, i.e. if $p \ge -3$, then $h(2) = 2\alpha + \beta + 2 = p + q + 1 = f(1, 1, 0)$ is the minimum of h. This is the condition (1).

If $\alpha < -4$, i.e. if $p < -3$, then $h(-\alpha/2) = (4\beta - 8 - \alpha^2)/4 = (4q - (p + 1)^2 - 4)/4$ is the minimum of h. This is the condition (2). □

Proposition 5.3. *For $f(a, b, c) := S_5 + pT_{4,1} + qT_{3,2} + rUS_2 - (1 + 2p + 2q + r)US_{1,1} \in \breve{\mathcal{H}}_5^{s0}$, let*

$$\tilde{d_5}(p, q, r) := 4(p+1)(p-2)(2p-1) - 9(q(2p-1) + r(p+1)) - \left((2p-1)^2 - 3(2q+r+2)\right)^{3/2}.$$

Then $f(s, 1, 1) \ge 0$ for all $s \ge 0$ if and only if one of (1), (2), (3) or (4) holds.

(1) *$p \ge -1$ and $4p + 2q + r + 3 \ge 0$.*
(2) *$p < -1$ and $(2p - 1)^2 < 3(2q + r + 2)$.*
(3) *$4p + 2q + r + 3 < 0$ and $\tilde{d_5}(p, q, r) \ge 0$.*
(4) *$p < -1$, $(2p - 1)^2 \ge 3(2q + r + 2)$ and $\tilde{d_5}(p, q, r) \ge 0$.*

Proof. Since $f(0,1,1) = 2(p+q+1)$, it is necessary that $p+q+1 \geq 0$. Note that

$$f(x,1,1) = (x-1)^2\big(x^3 + 2(p+1)x^2 + (4p+2q+r+3)x + 2(p+q+1)\big).$$

Thus, let $g(x) := x^3 + 2(p+1)x^2 + (4p+2q+r+3)x + 2(p+q+1)$.
Then $g'(x) = 3x^2 + 4(p+1)x + (4p+2q+r+3)$. We consider the roots of $g'(x) = 0$.

Case 1: We treat the case that the quadric equation $g'(x) = 0$ has no positive roots.

In this case, $g(x) > 0$ on $(0, \infty)$ if and only if $g(0) > 0$. The axis of the parabola $y = g'(x)$ is $x = -\dfrac{2(p+1)}{3}$, and

$$g'\left(-\frac{2(p+1)}{3}\right) = \frac{3(2q+r+2) - (2p-1)^2}{3}.$$

$g'(x) = 0$ has no positive roots if and only if "$p+1 \geq 0$ and $g'(0) \geq 0$" or "$p+1 < 0$ and $g'(-2(p+1)/3) > 0$". These correspond with the conditions (1) and (2).

Case 2: We treat the case $g'(0) \geq 0$ and the quadric equation $g'(x) = 0$ has a positive root x_0.

In this case, $g(x) \geq 0$ on $[0, \infty)$ if and only if $g(x_0) \geq 0$ and $g(0) \geq 0$. $g'(x) = 0$ has a positive root x_0 if and only if $p+1 < 0$ and $g'(-2(p+1)/3) \leq 0$. Since

$$x_0 = \frac{\sqrt{(2p-1)^2 - 3(2q+r+2)} - 2(p+1)}{3}, \tag{5.4}$$

and $27g(x_0) = 2\tilde{d_5}(p,q,r)$, we have the condition (4).

Case 3: We treat the case $g'(0) < 0$.

In this case, x_0 defined by (5.4) is a positive, and $g'(x_0) = 0$. $g(x) \geq 0$ on $[0, \infty)$ if and only if $g(x_0) \geq 0$ and $g(0) \geq 0$. This corresponds with the condition (3). □

In (3) and (4), $\tilde{d_5} \geq 0$ is equivalent to $d_5 \geq 0$. Thus, Theorem 0.4 is proved. But we want to obtain extremal inequalities and to determine which are discriminants. We choose $s_0 = S_5 - US_{1,1}$, $s_1 = T_{4,1} - 2US_{1,1}$, $s_2 = T_{3,2} - US_{1,1}$, $s_3 = US_2 - US_{1,1}$ as a base of $\mathcal{H}_5^{s0}$. For Step 1 of Remark 1.23, we execute Remark 1.21(2). Then we know that $X_5^{s0+} := X(\mathbb{P}_+^2, \mathcal{H}_5^{s0})$ is defined by

$$F_{50}(x_0,x_1,x_2,x_3) := x_1^2 - x_0x_2 + x_1x_2 - x_2^2 - 2x_0x_3 - 5x_1x_3 + 2x_2x_3 = 0,$$
$$F_{51}(x_0,x_1,x_2,x_3) := x_1^2 + x_2^2 - 2x_0x_2 - 2x_1x_3 \leq 0.$$

Note that $F_{51}(s_0,s_1,s_2,s_3) = -(a-b)^2(b-c)^2(c-a)^2(a+b+c)^4 \leq 0$. Using Proposition 2.12, 2.13, 2.14, and the above, we obtain $\Delta^2(X_5^{s0+}) = \big\{\operatorname{Reg}(X_5^{s0+})\big\}$, $\Delta^1(X_5^{s0+}) = \big\{\Phi_5^{s0}(L_{F+}^b), \Phi_5^{s0}(L_{F+}^0)\big\}$, and $\Delta^0(X_5^{s0+}) = \big\{\Phi_5^{s0}(0:0:1), \Phi_5^{s0}(0:1:1)\big\}$. Next execute Step 2 of Remark 1.23. Since $\Phi_5^{s0}(0:0:1) = (1:0:0)$, $\mathcal{F}(\Phi_5^{s0}(0:0:1))$ is the set of all the polynomials at infinity in $\mathcal{P}_5^{s0}$. This can be determined using Proposition 1.33.

Proposition 5.5. *$\mathcal{P}_5^{s0+}$ and $\mathcal{P}_5^{s+}$ have no main components.*

Proof. We prove $\mathcal{F}_5^{s+} \subset \mathcal{P}_5^{s+}$ is not a face component. Assume $0 \neq \exists f \in \operatorname{Int}(\mathcal{F}_5^{s+})$. We may assume $f \in \mathcal{L}_{s,t}^{s+}$ for $0 < \exists s < \exists t < 1$, and $f \notin \mathcal{L}_{0,u}^{s+}$ for any $u \geq 0$, and $f \notin \mathcal{L}_{v,1}^{s+}$ for any $v \geq 0$. For $g \in \mathcal{H}_5^s$, we denote

$$M(g) := \max\big\{g(a) \bigm| a \in \partial A_F\big\}, \quad m(g) := \min\big\{g(a) \bigm| a \in \partial A_F\big\},$$

here $\partial A_F := \big\{(u,1,1) \bigm| 0 \leq u \leq 1\big\} \cup \big\{(1,1,u) \bigm| u \geq 1\big\} \cup \big\{(0,u,1) \bigm| 0 \leq u \leq 1\big\}$. Since $f \notin \mathcal{L}_{0,u}^{s+}$ and $f \notin \mathcal{L}_{v,1}^{s+}$, we have $m(g) > 0$. Since $M(S_5) < +\infty$, there exists $0 < \varepsilon \ll 1$ such that $m(f-\varepsilon S_5)>0$. By Proposition 5.1, $f-\varepsilon S_5 \in \boldsymbol{\mathcal{P}}_5^{s0+}$. But $f(s,t,1)-\varepsilon S_5(s,t,1)<0$. Thus $\boldsymbol{\mathcal{F}}_5^{s+}$ is not a face component.

We prove $\boldsymbol{\mathcal{F}}_5^{s0+} \subset \boldsymbol{\mathcal{P}}_5^{s0+}$ is not a face component. Otherwise, we can take $0 \neq f$, $g \in \mathrm{Int}(\boldsymbol{\mathcal{F}}_5^{s0+})$ such that:

(a) There exists $0<s_1<t_1<1$ and $0<s_2<t_2<1$ such that $(s_1,\ t_1) \neq (s_2,\ t_2)$, $f \in \boldsymbol{\mathcal{L}}_{s_1,t_1}^{s0+}$, $g \notin \boldsymbol{\mathcal{L}}_{s_1,t_1}^{s0+}$, $f \notin \boldsymbol{\mathcal{L}}_{s_2,t_2}^{s0+}$, and $g \in \boldsymbol{\mathcal{L}}_{s_2,t_2}^{s0+}$.

(b) $f,\ g \notin \boldsymbol{\mathcal{L}}_{0,u}^{s0+}$ for any $u \geq 0$.

(c) $f,\ g \notin \boldsymbol{\mathcal{L}}_{v,1}^{s0+}$ for any $v \geq 0$, $v \neq 1$.

Then, at least one of $m(f/g)>0$ or $m(g/f)>0$ holds. We may assume $m(f/g)>0$. Take $0<\varepsilon<m(f/g)$. Then $f-mg \in \boldsymbol{\mathcal{P}}_5^{s0+}$. But $f(s_1,t_1,1)-\varepsilon g(s_1,t_1,1)<0$. Thus $\boldsymbol{\mathcal{F}}_5^{s0+}$ is not a face component. ∎

It is complicated to write the conditions in homogeneous form. So, for $f \in \boldsymbol{\mathcal{H}}_5^{s0}$ which does not lie on infinity, we normalize it in monic form. Let $p := p_1/p_0$, $q := p_2/p_0$ and $r := p_3/p_0$, that is

$$f = S_5 + pT_{4,1} + qT_{3,2} + rUS_2 - (1+2p+2q+r)US_{1,1} \in \boldsymbol{\mathcal{H}}_5^{s0}.$$

For each vector subspace $V \subset \boldsymbol{\mathcal{H}}_5^{s0}$, let $\breve{V}$ be the set of all the monic polynomials in V.

Theorem 5.6. *We use the same symbols as above.*

(1) *Let* $P = \Phi_5^{s0}(0:1:1) = (1:1:1:0)$, *and* $O = \Phi_5^{s0}(0:0:1) = (1:0:0:0)$. *Then the dual of* P *is described as* $\breve{\boldsymbol{\mathcal{F}}}(P) = \boldsymbol{\mathcal{A}}_1' \cup \boldsymbol{\mathcal{A}}_2'$, *where*

$$\boldsymbol{\mathcal{A}}_1' := \left\{\ (p,q,r) \in \breve{\boldsymbol{\mathcal{H}}}_5^{s0} \ \middle|\ p+q+1=0,\ -3 \leq p \leq -1,\ r \geq p^2 \right\}$$
$$\boldsymbol{\mathcal{A}}_2' := \left\{\ (p,q,r) \in \breve{\boldsymbol{\mathcal{H}}}_5^{s0} \ \middle|\ p+q+1=0,\ -1 \leq p,\ 2p+r+1 \geq 0 \right\}.$$

(2) *Let* $C_1 = \Phi_5^{s0}(L_{F+}^b) = \left\{\Phi_5^{s0}(s:1:1) \ \middle|\ s>0\right\}$. *Then*

$$\breve{\boldsymbol{\mathcal{F}}}(C_1) = \left\{\ (p,q,r) \in \breve{\boldsymbol{\mathcal{H}}}_5^{s0} \ \middle|\ 4q=(p+1)^2+4,\ p \leq -3,\ d_5(p,q,r) \geq 0 \right\}.$$

(3) *Let* $C_2 = \Phi_5^{s0}(L_{F+}^0) = \left\{\Phi_5^{s0}(0:s:1) \ \middle|\ s>0\right\}$. *Then* $\breve{\boldsymbol{\mathcal{F}}}(C_2) = \boldsymbol{\mathcal{B}}_1' \cup \boldsymbol{\mathcal{B}}_2' \cup \boldsymbol{\mathcal{B}}_3'$, *where*

$$\boldsymbol{\mathcal{B}}_1' := \left\{\ (p,q,r) \in \breve{\boldsymbol{\mathcal{H}}}_5^{s0} \ \middle|\ d_5(p,q,r)=0,\ p<-3,\ 4q \geq (p+1)^2+4 \right\},$$
$$\boldsymbol{\mathcal{B}}_2' := \left\{\ (p,q,r) \in \breve{\boldsymbol{\mathcal{H}}}_5^{s0} \ \middle|\ \begin{array}{l} d_5(p,q,r)=0,\ -3 \leq p \leq -1, \\ p+q+1 \geq 0,\ (q,r) \neq (-p-1,\ -2p-1) \end{array} \right\},$$
$$\boldsymbol{\mathcal{B}}_3' := \left\{\ (p,q,r) \in \breve{\boldsymbol{\mathcal{H}}}_5^{s0} \ \middle|\ d_5(p,q,r)=0,\ -1 \leq p,\ p+q+1 \geq 0,\ 2p+r+1 \leq 0 \right\}.$$

Especially, $d_5(p,q,r) = \mathrm{disc}(C_1)$, $4q-(p+1)^2-4 = \mathrm{disc}(C_2)$. $p+q+1 = \mathrm{disc}(P)$ *and* $p_0 = \mathrm{disc}(O)$.

Proof. Fix $p \in \mathbb{R}$, and observe the plane section $V_p = \left\{(q,r) \in \mathbb{R}^2 \ \middle|\ (1,p,q,r) \in \boldsymbol{\mathcal{H}}_5^{s0+}\right\}$. Let C_p be the curve defined by $d_5(p,q,r)=0$ on the $(q,\ r)$-plane V_p. C_p is a rational cubic curve with the parameterization

$$q = \frac{1}{27(2t+1)^3} \cdot \Big(-3^6t^3 + (p+1)\Big(4(8p^2-65p+116)t^3$$
$$+6(8p^2-38p-19)t^2 + 3(8p^2-11p-73)t + (4p^2+8p-23)\Big)\Big),$$
$$r = \frac{1}{27(2t+1)^3} \cdot \Big(-8(2p-1)^3t^3 - 3(2p-1)^2(8p+23)t^2$$
$$-6(2p-1)(p+4)(4p+7)t - (p+4)^2(8p+5)\Big).$$

C_p has a cusp at $P_p = (q, r) = \left(\frac{4p^3 + 12p^2 - 15p - 23}{27}, -\frac{(2p-1)^3}{27} \right)$. The cusp P_p lies on a line $3(2q + r + 2) = (2p - 1)^2$. The vertical line $p + q + 1 = 0$ tangents to C_p at $(q, r) = (-p-1, -2p-1)$, and intersects with C_p at $(q, r) = (-p-1, p^2)$. The graph of C_p in the case $p > -1$ is as in Figure 5.1.

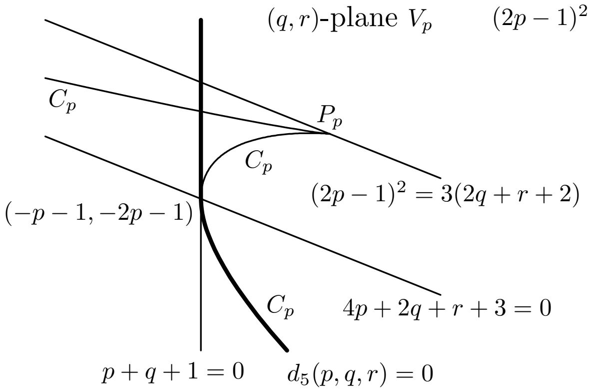

Fig 5.1. Case: $p > -1$

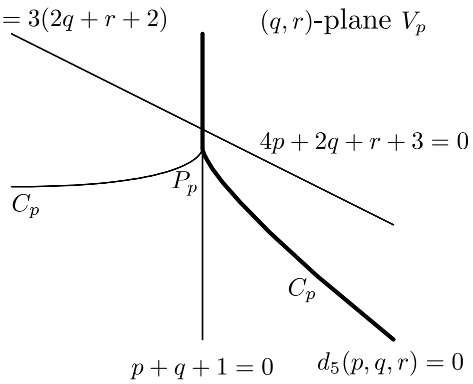

Fig 5.2. Case: $p = -1$

$\partial \mathbf{P}_5^{s0+} \cap V_p$ is shown by the thick curve. The graph of C_p in the case $p = -1$ is as in Figure 5.2. In this case, the line $4p + 2q + r + 3 = 0$ coincides with $3(2q + r + 2) = (2p - 1)^2$, and these lines intersect to $p + q + 1 = 0$ at the cusp P_p.

The graph of C_p in the case $p < -1$ is as in Figure 5.3. In this case $(-p-1, -2p-1) \notin V_p$, by the condition $4(p + 1)(p - 2)(2p - 1) - 9(q(2p - 1) + r(p + 1)) \geq 0$.

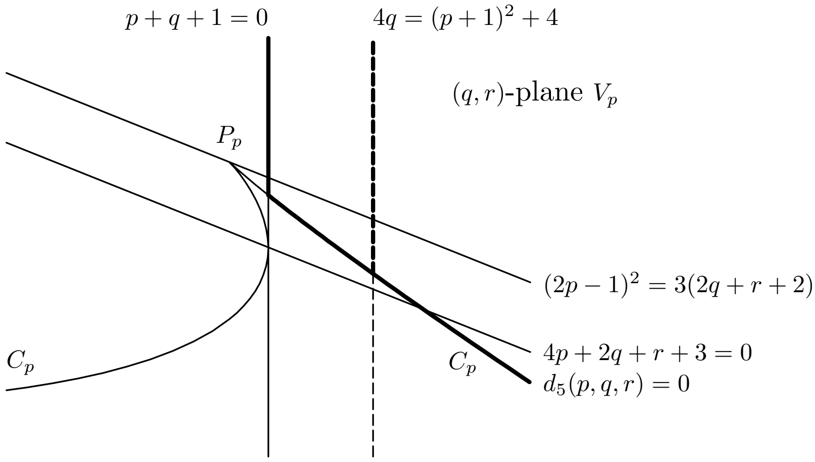

Fig.5.3. Case: $p < -1$

By the above observation, we complete the proof. □

Corollary 5.7. *Let*

$$f_p^A(a, b, c) := s_0 + ps_1 - (p + 1)s_2 + p^2 s_3,$$
$$\ell(t) := 2 - t^2 + t\sqrt{(t - 1)(t + 2)},$$
$$s_m(t) := (1/2)(\ell(t) - \sqrt{\ell(t)^2 - 4},$$

$$
\begin{aligned}
f_t^B(a,b,c) &:= s_0 + (1-2\ell(t))s_1 + (t^3+2t^2-2-2(t^2-1)\ell(t))s_2 \\
&\qquad - ((t+1)^2(2t+3) - 4(t+1)^2\ell(t))s_3, \\
g_t(a,b,c) &:= s_1 + (t^2-1)s_2 - 2(s+1)^2 s_3.
\end{aligned}
$$

(1) *For all $t \geq 0$, g_t is an extrelal element of $\mathcal{P}_5^{s0+}$, and $g_t \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,0}^{s0+}$.*

(2) *Let $t \geq 2$, and put $s := s_m(t)$. Then $0 < s \leq 1$, and $f_t^B \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,s}^{s0+}$. f_t^B is an extrelal element of $\mathcal{P}_5^{s0+}$.*

(3) *Let $0 \leq t \leq 2$, and put $p := -t-1$. Then $f_p^A \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,1}^{s0+}$, and f_p^A is an extremal element of $\mathcal{P}_5^{s0+}$.*

(4) *All the extremal elements of $\mathcal{P}_5^{s0+}$ are positive multiples of f_p^A $(-3 \leq p \leq -1)$, f_t^B $(t \geq 2)$, g_t $(t \geq 0)$, s_2 and s_3.*

Proof. (1) $f \in \mathcal{L}_{0,0}^{s0+}$ implies the cofficient of s_0 in f is equal to zero. Since,

$$
\begin{aligned}
g_t(s,1,1) &= 2(s-1)^2(s-t)^2, \\
g_t(0,s,1) &= s(s+1)((s-1)^2+t^2),
\end{aligned}
$$

we have $g_t \in \mathcal{L}_{t,1}^{s0+}$ by Proposition 5.1.

(2) It is easy exercise to veryfy that $s_m(t)$ varies (0, 1] when $t \geq 2$. Since

$$
\begin{aligned}
f_t^B(s,1,1) &= (s-t)^2(s-1)^2(s+2(t-\sqrt{(t-1)(t+2)})^2), \\
f_t^B(0,s,1) &= (s+1)(s^2-(2-t^2-t\sqrt{(t-1)(t+2)})s+1)^2,
\end{aligned}
$$

we have $f_t^B \in \mathcal{L}_{t,1}^{s0+} \cap \mathcal{L}_{0,s}^{s0+}$.

(3) follow from

$$
\begin{aligned}
f_p^A(t,1,1) &= t(t-1)^2(t+p+1)^2, \\
f_p(0,t,1) &= (t+1)(t-1)^2(t^2+(p+1)t+1).
\end{aligned}
$$

(4) All the extremal elements of $\mathcal{L}_{0,0}^{s0+}$ are positive multiples of g_t $(0 \geq 0)$ and $g_\infty := s_2$. $\mathcal{L}_{0,s}^{s0+} \cap \mathcal{L}_{0,0}^{s0+} = \mathbb{R}_+ \cdot s_3$. Thus we obtain (4). □

5.2. Edge discriminant of $\mathcal{P}_5^{c+}$.

We cannot yet obtain the main discriminant of $\mathcal{P}_5^{c0+}$ and $\mathcal{P}_5^{c+}$. It will be extraordinary long polynomial. But we determined the edge discriminant of $\mathcal{P}_5^{c+}$.

We choose $s_0 = S_5 - US_2$, $s_1 = S_{4,1} - US_2$, $s_2 = S_{3,2} - US_2$, $s_3 = S_{2,3} - US_2$, $s_4 = S_{1,4} - US_2$, $s_5 = US_2 - US_{1,1}$ as a base of $\mathcal{H}_5^{c0}$, and we choose $s_0, \ldots, s_5$ and $s_6 = US_{1,1}$ as a base of $\mathcal{H}_5^{c}$. Let

Lemma 5.8. *Let $\mathcal{L}_s^{c0+}$ be the local cone of $\mathcal{P}_5^{c0+}$ at the point $(0:s:1) \in \mathbb{P}_+^2$. Note that $\dim \mathcal{L}_s^{c0+} = 6 - 2 \leq 4$ by Proposition 1.27(1). The following $F_{1,s}$, $F_{2,s}$, $F_{3,s}$, $F_{4,s}$ are linearly independent elements in $\mathcal{L}_{0,s}^{c0+}$.*

$$
\begin{aligned}
\mathcal{H}_{0,s}^{c0} &:= \{ f \in \mathcal{H}_5^{c0} \mid f(0,s,1) = 0 \}, \\
F_{1,s}(a,b,c) &:= 3s^4 s_0 - (4s^5-1)s_1 + (s^8 - 4s^3)s_4, \\
F_{2,s}(a,b,c) &:= 2s_1 - 3ss_2 + s^3 s_4, \\
F_{3,s}(a,b,c) &:= s_1 - 3s^2 s_3 + 2s^3 s_4, \\
F_{4,s}(a,b,c) &:= s_5.
\end{aligned}
$$

Proof. It is easy to check $F_{i,s}(0,s,1)=0$, $F_{i,s}=(1,1,1)=0$ for $1\le i\le 4$, and $F_{j,s}(0,0,1)=0$ for $2\le j\le 4$. Since $\dim \mathcal{H}^{c0}_{0,s}=\dim \mathcal{H}^{c0}_{5}-2=4$, $F_{1,s},\ldots, F_{1,s}$ is a base of $\mathcal{H}^{c0}_{0,s}$. We need long analytic argument to prove $F_{1,s}$, $F_{2,s}\in \mathcal{L}^{c0+}_{0,s}$. Thus, I will publish this proof in an other article. $F_{3,s}\ge 0$ follows from $F_{3,s}(a,b,c)=s^3F_{2,s}(b,a,c,1/s)\ge 0$. $F_{4,s}\ge 0$ follows from $S_2\ge S_{1,1}$. ∎

Theorem 5.9. *For* $\breve{\mathcal{H}}^c_5\ni f=S_5+xS_{4,1}+yS_{1,4}+zS_{3,2}+wS_{2,3}+uUS_2+vUS_{1,1}\in\mathcal{H}^c_5$, *the edge discriminant of* $\mathcal{P}^{c+}_5$ *is the following* disc^{c+}_5.

$$
\begin{aligned}
&\mathrm{disc}^{c+}_5(x,y,z,w)\\
&:= -27x^4y^4-4x^3y^2w^3-4x^2y^3z^3+18x^3y^3zw+x^2y^2z^2w^2\\
&\quad +144x^4y^2w+144x^2y^4z-6x^3y^2z^2-6x^2y^3w^2+16x^3w^4+16y^3z^4\\
&\quad -80x^3yzw^2-80xy^3z^2w+18x^2yz^3w+18xy^2zw^3-4x^2z^2w^3-4y^2z^3w^2\\
&\quad -36x^3y^3-192x^4yz-192xy^4w-128x^4w^2-128y^4z^2\\
&\quad +24x^2yw^3+24xy^2z^3-27x^2z^4-27y^2w^4-746x^2y^2zw\\
&\quad +144x^3z^2w+144y^3zw^2-72xzw^4-72yz^4w+356xyz^2w^2+16z^3w^3\\
&\quad +256x^5+256y^5+160x^3yw+160xy^3z+1020x^2yz^2+1020xy^2w^2\\
&\quad +560x^2zw^2+560y^2z^2w-630xz^3w-630yzw^3+108z^5+108w^5\\
&\quad -50x^2y^2-1600x^3z-1600y^3w-900xw^3-900yz^3-2050xyzw+825z^2w^2\\
&\quad +2000x^2w+2000y^2z+2250xz^2+2250yw^2-2500xy-3750zw+3125.
\end{aligned}
$$

Proof. By Proposition 2.8, the edge discriminants of $\mathcal{P}^{c+}_5$ and $\mathcal{P}^{c0+}_5$ agrees. Thus we shall calculate that of $\mathcal{P}^{c0+}_5$. Eliminate α_2, α_3 and s from

$$x=\frac{-(4s^5-1)}{3s^4}+2\alpha_2+\alpha_3,\quad y=\frac{s^8-4s^3}{3s^4}+s^3\alpha_2+2s^3\alpha_3,\quad z=-3s\alpha_2,\quad w=-3s^2\alpha_3,$$

we obtain $\mathrm{disc}^{c+}_5=0$. ∎

If f is symmetric, then

$$\mathrm{disc}^{c+}_5(p,p,q,q)=(p+q+1)(5-3p+q)^3(4q-(p+1)^2-4)^2.$$

Recall that $4q-(p+1)^2-4$ is a discriminant of $\mathcal{P}^{s0+}_5$. Since $\mathrm{disc}^{c+}_5(p,p,q,q)$ is a multiple of $(4q-(p+1)^2-4)^2$, we know that $\mathcal{E}^{s+}_5\subset \mathrm{Sing}(\mathcal{E}^{c+}_5)$. The following is an example of Lemma 5.8.

Example 5.10. For any a, b, $c\in\mathbb{R}_+$, the following inequalities hold:

$$S_5+\left(\frac{5\sqrt[5]{4}}{4}-1\right)US_2\ge \frac{5\sqrt[5]{4}}{4}S_{4,1},\quad 2S_{4,1}+S_{1,4}\ge 3S_{3,2}.$$

Proof. The first is $F_{1,\sqrt[5]{4}}(a,b,c)\ge 0$. The last is $F_{2,1}(a,b,c)\ge 0$. ∎

Section 6. Sextic inequalities.

6.1. Convex analysis.

Choi, Lam and Rezenick studied some sextic inequalities in [9]. Cîrtoaje presented a nice theorem in [11] Theorem 2.4. But, its proof has error, and its statement is not correct. Example 6.10(5) gives a counterexample for Theorem 2.4 of [11]. Some corrected versions are published in [13] and [14]. Here, we provide a corrected and extended version.

Theorem 6.1. *Let $f(x,\, y,\, z)$ be a homogeneous symmetric polynomial with $6 \leq \deg f \leq 8$. Let $p := x+y+z$, $q := xy+yz+zx$, $r := xyz$, and denote*

$$f(x,y,z) = g_0(p,q)r^2 + g_1(p,q)\,r + g_2(p,q) \quad (g_0,\, g_1,\, g_2 \in \mathbb{R}[p,q]).$$

We also fix the following symbols.

$$\begin{aligned}
&D(p,q) := g_1(p,q)^2 - 4g_0(p,q)g_2(p,q),\\
&h_1(s) := 2sg_0(s+2, 2s+1) + g_1(s+2, 2s+1),\\
&h_2(t) := 2t^2 g_0(2t+1, t^2+2t) + g_1(2t+1, t^2+2t),\\
&I_1 := \big\{\ s \in \mathbb{R} \ \big|\ -2 \leq s \leq 1,\ g_0(s+2,2s+1) > 0, \text{ and } D(s+2,\, 2s+1) > 0\big\},\\
&I_2 := \big\{\ s \in I_1 \ \big|\ -1/2 \leq s \leq 1\big\}.
\end{aligned}$$

(I) *Assume $\deg f = 6$ or 8. Then, $f(x,\, y,\, z) \geq 0$ for all $x,\, y,\, z \in \mathbb{R}$, if and only if the following condition* (1) *holds, and for every $s \in I_1$ (depending on s), one of* (2) *or* (3) *holds.*

(1) *$f(0,0,1) \geq 0$ and $f(x,1,1) \geq 0$ for all $x \in \mathbb{R}$.*

(2) *$h_1(s) \geq 0$.*

(3) *$h_2((1+2s)/(4-s)) \leq 0$.*

(II) *Assume $6 \leq \deg f \leq 8$. Then, $f(x,\, y,\, z) \geq 0$ for all $x,\, y,\, z \in \mathbb{R}_+$, if and only if the following condition* (4) *holds, and for every $s \in I_2$ (depending on s), one of* (5) *or* (3) *holds.*

(4) *$f(0,x,1) \geq 0$ and $f(x,1,1) \geq 0$ for all $x \in \mathbb{R}_+$.*

(5) *$h_1(s) \geq 0$ or $g_1(s+2, 2s+1) \geq 0$,*

Proof. Step 0: For $p,\, q \in \mathbb{R}$ with $p^2 - 3q \geq 0$, let

$$\begin{aligned}
&X_{p,q} := \big\{(x,y,z) \in \mathbb{R}^3 \ \big|\ x \leq y \leq z,\ x+y+z = p,\ xy+yz+zx = q\big\},\\
&\alpha_1(p,q) := \frac{p - 2\sqrt{p^2-3q}}{3}, \quad \alpha_2(p,q) := \frac{p + 2\sqrt{p^2-3q}}{3},\\
&\beta_1(p,q) := \frac{p - \sqrt{p^2-3q}}{3}, \quad \beta_2(p,q) := \frac{p + \sqrt{p^2-3q}}{3},\\
&r_1(p,q) := \alpha_1(p,q)\beta_2(p,q)^2, \quad r_2(p,q) := \beta_1(p,q)^2\alpha_2(p,q).
\end{aligned}$$

Note that $X_{p,q} \neq \emptyset$. For $r := xyz$,

$$27r^2 - 2(9pq - 2p^3)r + (4q^3 - p^2q^2) = -(x-y)^2(y-z)^2(z-x)^2 \leq 0.$$

The solution of the above inequality on r is

$$r_1(p,q) = \frac{9pq - 2p^3 - 4(p^2-3q)^{3/2}}{27} \leq r \leq \frac{9pq - 2p^3 + 4(p^2-3q)^{3/2}}{27} = r_2(p,q).$$

Moreover, if $r = r_1(p,q)$ or $r = r_2(p,q)$, then $-(x-y)^2(y-z)^2(z-x)^2 = 0$. Thus we have:

(0-a) $\displaystyle\min_{(x,y,z)\in X_{p,q}} xyz = r_1(p,q)$. The equality holds if and only if $x = \alpha_1(p,q)$ and $y = z = \beta_2(p,q)$.

(0-b) $\max_{(x,y,z)\in X_{p,q}} xyz = r_2(p,q)$. The equality holds if and only if $x = y = \beta_1(p,q)$ and $y = z = \alpha_2(p,q)$.

We shall prove sufficiency of (I) and (II).

Assume that (1) holds in the case (I) or (4) holds in the case (II), and assume that $f(a, b, c) < 0$ for certain a, b, c. In the case (II), we assume $a \geq 0$, $b \geq 0$, $c \geq 0$. It is enough to show that (2), (3) and (5) cannot hold.

Let $P := a+b+c$, $Q := ab+bc+ca$. We may assume $P > 0$, since $f(-a, -b, -c) = f(a, b, c)$ in case (I). Let $\alpha_1 := \alpha_1(P,Q)$, $\alpha_2 := \alpha_2(P,Q)$, $\beta_1 := \beta_1(P,Q)$, $\beta_2 := \beta_2(P,Q)$, $r_1 := r_1(P,Q)$ and $r_2 := r_2(P,Q)$. Since $P^2 - 3Q = \big((a-b)^2 + (b-c)^2 + (c-a)^2\big)/2 \geq 0$, we have $\alpha_2 \geq 0$ and $\beta_2 \geq 0$. If $\alpha_2 = (P + 2\sqrt{P^2 - 3Q})/3 = 0$, then $P = Q = 0$ and $a = b = c = 0$. This contradicts to $f(a, b, c) < 0$. Thus $\alpha_2 > 0$. Similarly, we have $\beta_2 > 0$. In the case (II), since $Q \geq 0$, we have $\beta_1 \geq 0$. Let

$$g(r) := g_0(P,Q)r^2 + g_1(P,Q)\,r + g_2(P,Q).$$

Note that $g(abc) = f(a, b, c) < 0$. Let $S := \alpha_1/\beta_2$ and $T := \beta_1/\alpha_2$. Since $ST+2S-4T+1 = 0$, we have $T = (1+2S)/(4-S)$ and $S = (4T-1)/(T+2)$.

Since $\alpha_1 + 2\beta_2 = P \geq 0$, $\beta_2 > 0$, and $\alpha_1 \leq \beta_2$, we have $-2 \leq S \leq 1$. In the case (II), since $\beta_1 \geq 0$, we have $T \geq 0$. Thus $-1/2 \leq S \leq 1$. Note that the following relations:

$$P = \alpha_1 + 2\beta_2 = (S+2)\beta_2 = \alpha_2 + 2\beta_1 = (1+2T)\alpha_2,$$
$$Q = 2\alpha_1\beta_2 + \beta_2^2 = (2S+1)\beta_2^2 = \beta_1^2 + 2\alpha_2\beta_1 = (T^2+2T)\alpha_2^2,$$
$$g_0(P,Q) = \beta_2^{d-6} g_0(S+2, 2S+1) = \alpha_2^{d-6} g_0(2T+1, T^2+2T), \tag{6.2}$$
$$g_1(P,Q) = \beta_2^{d-3} g_1(S+2, 2S+1) = \alpha_2^{d-3} g_1(2T+1, T^2+2T), \tag{6.3}$$
$$g_2(P,Q) = \beta_2^{d} g_2(S+2, 2S+1) = \alpha_2^{d} g_2(2T+1, T^2+2T),$$
$$D(P,Q) = \beta_2^{d} D(S+2, 2S+1) = \alpha_2^{d} D(2T+1, T^2+2T), \tag{6.4}$$
$$r_1 = \beta_2^{d-3} S, \quad r_2 = \alpha_2^{d-3} T^2,$$
$$h_1(S) = \beta_2^{3-d}\big(2r_1 g_0(P,Q) + g_1(P,Q)\big), \quad h_2(T) = \alpha_2^{3-d}\big(2r_2 g_0(P,Q) + g_1(P,Q)\big). \tag{6.5}$$

Step 1: We shall show $g_0(S+2,\, 2S+1) > 0$.

Assume that $g_0(S+2,\, 2S+1) \leq 0$. Then $g_0(P,\, Q) \leq 0$ by (6.2), and $g(r)$ is a concave function or a linear function. Let $r_1' := r_1$ in the case (I), and $r_1' := \max\{r_1,\, 0\}$ in the case (II). Since $g(abc) < 0$, we have $g(r_1') < 0$ or $g(r_2) < 0$.

If $g(r_2) < 0$, then there exists $(x,\, y,\, z) \in X_{P,Q}$ such that $xyz = r_2$ and $f(x,\, y,\, z) < 0$. Then $(x,\, y,\, z) = (\beta_1,\, \beta_1,\, \alpha_2)$ by Step 0. Thus

$$0 > g(r_2) = f(\beta_1, \beta_1, \alpha_2) = \beta_1^d f(\alpha_2/\beta_1, 1, 1) \geq 0.$$

If $g(r_1) < 0$, then we can derive similar contradiction.

If $g(0) < 0$ in the case (II), then there exists $(x,\, y,\, z) \in X_{P,Q}$ such that $xyz = 0$ and $f(x,\, y,\, z) < 0$. Since $0 \leq x \leq y \leq z$, we have $x = 0$. Then $0 > g(0) = f(0,\, 1,\, y/z) \geq 0$. Thus we have $g_0(S+2,\, 2S+1) > 0$.

Since $g_0(S+2,\, 2S+1) > 0$, we have $g_0(P,\, Q) > 0$ by (6.2). Thus, $g(r)$ is a convex function. Since $g(abc) < 0$, we have $D(P,Q) > 0$. This implies $D(S+2, 2S+1) > 0$ by (6.4). Thus $S \in I_1$. In the case (II), since $-1/2 \leq S \leq 1$, we have $S \in I_2$.

Step 2. We shall show $h_2((1+2S)/(4-S)) = h_2(T) > 0$, i.e. (3) cannot hold.

Assume that $h_2(T) \leq 0$. Then $-g_1(P,Q)/2g_0(P,Q) \geq r_2$. That is, the axis of quadratic function $g(r)$ exists in the right side of r_2. Thus

$$0 > f(a,b,c) = g(abc) \geq g(r_2) = f(\beta_1, \beta_1, \alpha_2) = \beta_1^d f(\alpha_2/\beta_1, 1, 1) \geq 0.$$

A contradiction. Thus, (3) cannot hold.

Step 3. We shall show that $h_1(S) < 0$, i.e. (2) cannot hold.

Assume that $h_1(S) \geq 0$. Then $-g_1(P,Q)/2g_0(P,Q) \leq r_1$. The axis of $g(r)$ exists in the left side of r_1. Thus

$$0 > f(a,b,c) = g(abc) \geq g(r_1) = f(\alpha_1, \beta_2, \beta_2) = \beta_2^d f(\alpha_1/\beta_2, 1, 1) \geq 0.$$

A contradiction. Thus, (2) cannot hold.

Step 4. We shall show that if $S \in I_2$ then $g_1(S+2,\, 2S+1) < 0$. i.e. (5) cannot hold.

Assume that $g_1(S+2,\, 2S+1) \geq 0$. Then $g_1(P,\, Q) \geq 0$ by (6.3).

If $S \geq 0$, then $r_1 = \beta_2^{d-3} S \geq 0$ and by (6.5). This derives a contradiction:

$$0 > \beta_2^{d-3} h_1(S) = 2r_1 g_0(P,Q) + g_1(P,Q) \geq 0.$$

If $S < 0$, then $r_1 < 0$, and $0 > \beta_2^{d-3} h_1(s) \geq g_1(P,\, Q)$. Thus, the axis of $g(r)$ exists in $r \leq 0$. Thus

$$0 > f(a,b,c) = g(abc) \geq g(0) \geq 0.$$

A contradiction. Thus, (5) cannot hold.

Here we complete the proof of the sufficiency part.

Step 5. We shall prove the necessity part of (I) and (II).

For $x,\, y,\, z \in \mathbb{R}$, let $p := x + y + z$, $q := xy + xz + zx$, and

$$A_1(x,y,z) := \frac{p - 2\sqrt{p^2 - 3q}}{3}, \quad B_2(x,y,z) := \frac{p + \sqrt{p^2 - 3q}}{3}.$$

Assume that $f(x,\, y,\, z) \geq 0$ for all $x,\, y,\, z \in \mathbb{R}$ in the case (I), or for all $x,\, y,\, z \in \mathbb{R}_+$ in the case (II). Then (1) holds in the case (I), and (4) holds in the case (II).

Let $s \in I_1$. There exists $a,\, b,\, c \in \mathbb{R}$ such that $A_1(a,b,c)/B_2(a,b,c) = s$. If $s \in I_2$, we can choose $a \geq 0$, $b \geq 0$ and $c \geq 0$. Let $P := a + b + c$, $Q := ab + bc + ca$. Using this P, Q, we define $X_{P,Q}$, α_1, α_2, β_1, β_2, r_1, r_1', r_2, S and T same as the proof of sufficiency part. Note that $\alpha_1 = A_1(a,b,c)$, $\beta_2 = B_2(a,b,c)$, and $s = S$.

Since $S = s \in I_1$, we have $D(S+2,\, 2S+1) > 0$ and $g_0(S+2,\, 2S+1) > 0$. Thus $D(P,\, Q) > 0$ and $g_0(P,\, Q) > 0$, by (6.2) and (6.4). This implies that the quadric equation $g(r) = 0$ has two real roots $r = r_3,\, r_4$, here $r_3 < r_4$. Since $g(r)$ is convex and $g(r) \geq 0$ for all $r_1' \leq r \leq r_2$, then we have $r_3 \geq r_2$ or $r_4 \leq r_1'$. This implies that the axis of $g(r)$ satisfies $-g_1(P,Q)/2g_0(P,Q) \geq r_2$ or $-g_1(P,Q)/2g_0(P,Q) \leq r_1'$.

Consider the case (I). If $-g_1(P,Q)/2g_0(P,Q) \geq r_2$, then by (6.5), we have

$$h_2((1+2s)/(4-s)) = h_2(T) = \alpha_2^{3-d}\big(2r_2 g_0(P,Q) + g_1(P,Q)\big) \leq 0.$$

Thus (3) holds. If $-g_1(P,Q)/2g_0(P,Q) \leq r_1$, then

$$h_1(s) = h_1(S) = \beta_2^{3-d}\big(2r_1 g_0(P,Q) + g_1(P,Q)\big) \geq 0.$$

Thus (2) holds.

Consider the case (II). In this case, $S = s \in I_2 \subset I_1$. As the above arguments, if $-g_1(P,Q)/2g_0(P,Q) \geq r_2$ then (3) holds. Consider the case $-g_1(P,Q)/2g_0(P,Q) \leq r_1'$. If $r_1 \geq 0$, i.e. if $r_1' = r_1$, as the above arguments, we have $h_1(s) \geq 0$. If $r_1 < 0$, i.e. if $r_1' = 0$, then $-g_1(P,\, Q) \leq 0$ by (6.3). This implies $g_1(s+2,\, 2s+1) \geq 0$. Thus, (5) holds. □

Note. Even if $\deg f \geq 9$, if f can be written as

$$f(x,y,z) = g_0(p,q)r^2 + g_1(p,q)\,r + g_2(p,q),$$

then the above theorem holds.

Using this theorem, we immediately obtain the following:

Example 6.6. For any $a,\ b,\ c \in \mathbb{R}$, the following hold:
(1) $6S_6 + 5T_{4,2} \geq 8T_{5,1}$.
(2) $8S_6 + 10S_{3,3} \geq 9T_{5,1}$.
(3) $T_{4,2} + 18U^2 \geq 4UT_{2,1}$.
(4) $\dfrac{3+\sqrt{3}}{18}S_6 + \dfrac{15-\sqrt{3}}{6}U^2 \geq US_3$.
Equality holds if $a=b=c$ or $(a:b:c) = (\sqrt{3}-1:1:1)$ or so on.
(5) $2(1-\alpha)S_6 + \alpha UT_{2,1} \geq T_{5,1}$, here $\alpha \sim 0.4235227783$ is a root of

$$257\alpha^6 - 1512\alpha^5 + 3598\alpha^4 - 4386\alpha^3 + 2865\alpha^2 - 950\alpha + 125 = 0.$$

Equality holds if $a=b=c$ or $(a:b:c) = (\beta:1:1)$, here $\beta \sim 0.5712944281$ is a root of

$$\beta^6 + 4\beta^5 + 7\beta^4 + 6\beta^3 - 2\beta - 1 = 0.$$

(6) $(2-\gamma)S_6 + 3\gamma U^2 \geq T_{5,1}$, here $\gamma \sim 0.762794619$ the unique real root of

$$54\gamma^5 - 378\gamma^4 + 1017\gamma^3 - 1330\gamma^2 + 900\gamma - 250 = 0.$$

Equality holds if $a=b=c$ or $(a:b:c) = (\delta:1:1)$, here $\delta \sim 0.5701772717$ is the unique real root of

$$\delta^5 + 3\delta^4 + 7\delta^3 + \delta^2 - 2 = 0.$$

(7) $(2-\varepsilon)S_6 + 3\varepsilon U^2 \geq UT_{2,1}$, here $\varepsilon \sim 1.8010662235$ is a root of

$$108\varepsilon^3 - 414\varepsilon^2 + 538\varepsilon - 257 = 0.$$

Equality holds if $a=b=c$ or $(a:b:c) = (\zeta:1:1)$, here $\zeta \sim 0.5666113232$ is a root of $3\zeta^3 + \zeta^2 + 2\zeta - 2 = 0$.

6.2. Edge discriminant of $\mathcal{P}_6^{c+}$.

For $\mathcal{P}_6^{c+}$, we could only determine the edge discriminant.

Lemma 6.7. *Let $\mathcal{L}_s^{c0+}$ be the local cone of $\mathcal{P}_6^{c0+}$ at the point $(0:s:1) \in \mathbb{P}_+^2$. Note that $\dim \mathcal{L}_s^{c0+} = 9 - 2 \leq 7$ by Proposition 1.27(1). The following $G_{i,s}$ are linearly independent elements in $\mathcal{L}_{0,s}^{c0+}$.*

$$\begin{aligned}
G_{1,s} &:= s^4 S_6 - (2s^6-1)S_{4,2} + (s^8 - 2s^2)S_{2,4} - 3(s^8 - 2s^6 + s^4 - 2s^2 + 1)U^2,\\
G_{2,s} &:= 2S_{5,1} - 3sS_{4,2} + s^3 S_{2,4} - 3(s-1)^2(s+2)U^2,\\
G_{3,s} &:= 2s^3 S_{1,5} + S_{4,2} - 3s^2 S_{2,4} - 3(s-1)^2(2s+1)U^2,\\
G_{4,s} &:= S_{4,2} + s^2 S_{2,4} - 2sS_{3,3} - 3(s-1)^2U^2,\\
G_{5,s} &:= US_3 - US_{2,1}, \quad G_{6,s} := US_3 - US_{1,2}, \quad G_{7,s} := US_3 - 3U^2.
\end{aligned}$$

Proof. It is easy to check $G_{i,s}(0,s,1) = 0$, $G_{i,s} = (1,1,1) = 0$. A proof of $G_{2,s}(a,b,c) \in \mathcal{L}_{0,s}^{c0+}$ will be published in other article. $G_{1,s}(a,b,c) = \mathfrak{f}_{s^2}(a^2,\ b^2,\ c^2) \geq 0$ by Theorem 3.1.

$G_{3,s} \geq 0$ follows from $G_{3,s}(a,b,c) = s^3 G_{2,1/s}(b,a,c)$.

$$
\begin{aligned}
G_{4,s} &= s^2(S_{2,4} - 3U^2) - 2s(S_{3,3} - 3U^2) + (S_{4,2} - 3U^2) \\
&= (S_{2,4} - 3U^2) \\
&\quad \times \left\{ \left(s - \frac{S_{3,3} - 3U^2}{S_{2,4} - 3U^2} \right)^2 + \frac{U^2(S_6 + 6S_{3,3} + 3U^2 - 3T_{4,2} - 2US_3)}{(S_{2,4} - 3U^2)^2} \right\}.
\end{aligned}
$$

Using Theorem 6.1, we have $S_6 + 6S_{3,3} + 3U^2 - 3T_{4,2} - 2US_3 \geq 0$. Thus, $G_{4,s} \geq 0$.

$G_{5,s} \geq 0$, $G_{6,s} \geq 0$, and $G_{7,s} \geq 0$ are trivial. □

Theorem 6.8. *For* $\breve{\mathfrak{H}}_6^c \ni f = S_6 + xS_{5,1} + yS_{1,5} + zS_{4,2} + wS2,4 + uS_{3,3} + x_6US_3 + x_7US_{2,1} + x_8US_{1,2} - 3(1 + x + y + z + w + u + x_6 + x_7 + x_8)U^2$, *the edge discriminant of* $\mathfrak{P}_6^{c+}$ *is the following* disc_6^{c+}.

$$
\begin{aligned}
&\mathrm{disc}_6^{c+}(x,y,z,w,u) \\
&:= 256x^5y^5 - 27x^4y^2w^4 - 27x^2y^4z^4 - 192x^4y^4zw - 6x^3y^3z^2w^2 - 4x^2y^2z^3w^3 \\
&\quad + 144x^4y^3w^2u + 144x^3y^4z^2u + 18x^3y^2zw^3u + 18x^2y^3z^3wu - 128x^4y^4u^2 \\
&\quad - 80x^3y^3zwu^2 + x^2y^2z^2w^2u^2 - 4x^3y^2w^2u^3 - 4x^2y^3z^2u^3 + 16x^3y^3u^4 \\
&\quad - 1600x^5y^3w - 1600x^3y^5z - 36x^3y^3z^3 - 36x^3y^3w^3 + 108x^4w^5 + 108y^4z^5 \\
&\quad + 1020x^4y^2zw^2 + 1020x^2y^4z^2w + 24x^3yz^2w^3 + 24xy^3z^3w^2 + 16x^2z^3w^4 \\
&\quad + 16y^2z^4w^3 + 144x^2y^2z^4w + 160x^4y^3zu + 160x^3y^4wu - 630x^4yw^3u \\
&\quad - 630xy^4z^3u - 746x^3y^2z^2wu - 746x^2y^3zw^2u - 72x^3zw^4u - 72y^3z^4wu \\
&\quad - 80x^2yz^3w^2u - 80xy^2z^2w^3u + 560x^4y^2wu^2 + 560x^2y^4zu^2 + 356x^3yzw^2u^2 \\
&\quad + 356xy^3z^2wu^2 - 6x^2y^2z^3u^2 - 6x^2y^2w^3u^2 - 4x^2z^2w^3u^2 - 4y^2z^3w^2u^2 \\
&\quad + 24x^3y^2zu^3 + 24x^2y^3wu^3 + 16x^3w^3u^3 + 16y^3z^3u^3 + 18x^2yz^2wu^3 \\
&\quad + 18xy^2zw^2u^3 - 72x^3ywu^4 - 72xy^3zu^4 + 320x^4y^4 - 50x^4y^2z^2 - 50x^2y^4w^2 \\
&\quad + 2250xy^5z^2 + 2250x^5yw^2 + 144x^3yw^4 + 144xy^3z^4 + 9768x^3y^3zw \\
&\quad + 160x^3yz^3w + 160xy^3zw^3 - 900x^4zw^3 - 900y^4z^3w - 576x^2zw^5 - 576y^2z^5w \\
&\quad - 5428x^2y^2z^2w^2 - 128x^2z^4w^2 - 128y^2z^2w^4 - 96xyz^3w^3 + 144x^2y^2zw^4 \\
&\quad - 64z^4w^4 + 2000x^5y^2u + 2000x^2y^5u - 2050x^4yzwu - 2050xy^4zwu \\
&\quad - 682x^3y^2w^2u - 682x^2y^3z^2u - 192x^2yz^4u - 192xy^2w^4u + 3272x^2yzw^3u \\
&\quad + 3272xy^2z^3wu + 320xz^2w^4u + 320yz^4w^2u - 208x^3y^3u^2 + 825x^4w^2u^2 \\
&\quad + 825y^4z^2u^2 + 1020x^3yz^2u^2 + 1020xy^3w^2u^2 + 24x^2w^4u^2 + 24y^2z^4u^2 \\
&\quad + 144x^2z^3wu^2 + 144y^2zw^3u^2 - 1584xyz^2w^2u^2 + 16z^3w^3u^2 - 900x^4yu^3 \\
&\quad - 900xy^4u^3 - 630x^3zwu^3 - 630y^3zwu^3 - 108x^2yw^2u^3 - 108xy^2z^2u^3 \\
&\quad - 72xzw^3u^3 - 72yz^3wu^3 - 27x^2z^2u^4 - 27y^2w^2u^4 + 324xyzwu^4 + 108x^3u^5 \\
&\quad + 108y^3u^5 - 2500x^5yz - 2500xy^5w - 1700x^2y^4z - 1700x^4y^2w + 248x^2y^2z^3 \\
&\quad + 248x^2y^2w^3 + 256x^2z^5 + 256y^2w^5 + 2000x^4z^2w + 2000y^4zw^2 - 13040x^3yzw^2 \\
&\quad - 13040xy^3z^2w + 4816x^2z^2w^3 + 4816y^2z^3w^2 + 512z^5w^2 + 512z^2w^5 \\
&\quad - 640xyzw^4 - 3750x^5wu - 3750y^5zu - 12330x^3y^2zu - 12330x^2y^3wu \\
&\quad - 1600x^3z^3u - 1600y^3w^3u - 120x^3w^3u - 120y^3z^3u + 560x^3z^2w^2u
\end{aligned}
$$

$$
\begin{aligned}
&+560y^3z^2w^2u+10152x^2yz^2wu+10152xy^2zw^2u+768xw^5u+768yz^5u\\
&-2496xz^3w^2u-2496yz^2w^3u+2250x^4zu^2+2250y^4wu^2+1980x^3ywu^2\\
&+1980xy^3zu^2-4536x^2zw^2u^2-4536y^2z^2wu^2-4464xyz^3u^2-4464xyw^3u^2\\
&-576z^4wu^2-576zw^4u^2+3942x^2yzu^3+3942xy^2wu^3+2808xz^2wu^3\\
&+2808yzw^2u^3+162x^2wu^4+162y^2zu^4+108z^3u^4+108w^3u^4-486xzu^5\\
&-486ywu^5+3125x^6+410x^3y^3+3125y^6+15600x^3yz^2+15600xy^3w^2\\
&+1500y^4z^2+1500x^4w^2-192x^2w^4-192y^2z^4-10560x^2z^3w-10560y^2zw^3\\
&+8748x^2y^2zw-640xyz^4w+15264xyz^2w^2-1024z^6-4352z^3w^3-1024w^6\\
&+2250x^4yu+2250xy^4u+19800x^3zwu+19800y^3zwu+16632x^2yw^2u\\
&+16632xy^2z^2u+6912xz^4u+6912yw^4u-5760xzw^3u-5760yz^3wu\\
&+15417x^2y^2u^2-2412x^2y^2zwu^2-9720x^2z^2u^2-9720y^2w^2u^2-22896xyzwu^2\\
&+8208z^2w^2u^2-1350x^3u^3-1350y^3u^3+5832xw^2u^3+5832yz^2u^3-6318xyu^4\\
&-4860zwu^4+729u^6-22500x^4z-22500y^4w-1800xy^3z-1800x^3yw\\
&-21888xyz^3-21888xyw^3-6480x^2zw^2-6480y^2z^2w+9216z^4w+9216zw^4\\
&-31320x^2yzu-31320xy^2wu-3456xz^2wu-3456yzw^2u-27540x^2wu^2\\
&-27540y^2zu^2-8640z^3u^2-8640w^3u^2+21384xzu^3+21384ywu^3+540x^2y^2\\
&+43200x^2z^2+43200y^2w^2+31968xyzw-17280z^2w^2+27000x^3u+27000y^3u\\
&+46656yz^2u+46656xw^2u+15552xyu^2+3888zwu^2-8748u^4-32400x^2w\\
&-32400y^2z-13824z^3-13824w^3-77760xzu-77760ywu+38880xy\\
&+62208zw+34992u^2-46656
\end{aligned}
$$

Proof. Compare the coefficients of $G_{1,s}+\sum_{i=2}^{7}\alpha_i G_{i,s}$ and f in Theorem 6.7, we have

$$
\begin{aligned}
&x=2\alpha_2,\quad y=2s^3\alpha_3,\quad z=\frac{1-2s^6}{s^4}-3s\alpha_2+\alpha_3+\alpha_4,\\
&w=\frac{s^8-2s^2}{s^4}+s^3\alpha_2-3s^2\alpha_3+s^2\alpha_4,\quad u=-2s\alpha_4.
\end{aligned}
$$

Eliminate α_2, α_3, α_4 and s from the above equalities, we have $\mathrm{disc}_6^+(x,y,z,w,u)=0$. ∎

Corollary 6.9. *For* $\breve{\mathfrak{H}}_6^s \ni f = S_6+pT_{5,1}+qT_{4,2}+rS_{3,3}+sUS_3+tUT_{2,1}-3(1+2p+2q+r+s+2t)U^2$, *the edge discriminant of* $\mathfrak{P}_6^{c+}$ *is the following* disc_6^{s+}.

$$
\begin{aligned}
\mathrm{disc}_6^{s+}(p,q,r) := {}& 8p^4+p^2q^2-4p^3r-42p^2q-4q^3+18pqr\\
&+9p^2+36q^2+54pr-27r^2-108q+108.
\end{aligned}
$$

Proof. Since

$$
\mathrm{disc}_6^+(p,p,q,q,r)=(2p-2q+r-2)(2p+2q+r+2)\big(\mathrm{disc}_6^{s0+}(p,q,r)\big)^2,
$$

we have the conclusion. ∎

We don't know a lot about $\mathfrak{P}_6^c$ and $\mathfrak{P}_6^{c+}$, but we can prove the following examples at once using the results in this section.

Example 6.10. For any a, b, $c \in \mathbb{R}_+$, the following hold:

(1) $2S_6 + 16S_{3,3} \geq 9T_{4,2}$.
(2) $S_9 + 6U^3 \geq US_3^2$.
(3) $(\sqrt{2}-1)T_{5,1} + 4S_{3,3} \geq 2(\sqrt{2}+1)US_3$. Equality holds if $a = b = c$ or $(a : b : c) = (1+\sqrt{2} : 1 : 1)$ or so on.
(4) $(1-\alpha)T_{5,1} + 6\alpha U^2 \geq 2US_3$, here $\alpha \sim 0.5681144549$ is a root of $23\alpha^4 - 38\alpha^3 + 35\alpha^2 - 40\alpha + 16 = 0$. Equality holds if $a = b = c$ or $(a : b : c) = (\beta : 1 : 1)$, here $\beta \sim 0.8499070444$ is a root of $\beta^4 + 4\beta^3 + \beta^2 - 2\beta - 2 = 0$.
(5) $(1-\gamma)T_{5,1} + 6\gamma U^2 \geq UT_{2,1}$, here $\gamma \sim 0.8392059669$ is a root of $23\gamma^3 - 21\gamma^2 + 5\gamma - 3 = 0$. Equality holds if $a = b = c$ or $(a : b : c) = (\delta : 1 : 1)$, here $\delta \sim 0.5651977174$ is a root of $2\delta^3 + 2\delta^2 - 1 = 0$.
(6) $(1-\varepsilon)S_6 + \varepsilon S_{3,3} \geq US_3$, here $\varepsilon \sim 0.9384024897$ is a root of $27\varepsilon^3 - 108\varepsilon^2 + 117\varepsilon - 37 = 0$. Equality holds if $a = b = c$ or $(a : b : c) = (\zeta : 1 : 1)$, here $\zeta \sim 1.8793852416$ is a root of $\zeta^3 - 3\zeta - 1 = 0$.
(7) $(1-2\eta)S_6 + \eta UT_{2,1} \geq US_3$, here $\eta \sim 0.4070962548$ is a root of
$$514\eta^5 - 1501\eta^4 + 1824\eta^3 - 1106\eta^2 + 326\eta - 37 = 0.$$
Equality holds if $a = b = c$ or $(a : b : c) = (\theta : 1 : 1)$, here $\theta \sim 0.8236644431$ is a root of $2\theta^5 + 5\theta^4 + 4\theta^3 - 4\theta - 2 = 0$.
(8) $\sqrt[3]{4}S_6 + (3 - \sqrt[3]{4})U^2 \geq 3S_{2,4}$.
(9) $S_6 + \dfrac{\sqrt{13+16\sqrt{2}}-1}{2}S_{2,4} \geq \dfrac{\sqrt{13+16\sqrt{2}}+1}{2}S_{4,2}$.

About four variables cases, $\mathcal{P}^{c0+}_{4,3} = \mathcal{P}(\mathbb{P}^3_+, \mathcal{H}^{c0}_3)$, $\mathcal{P}^{s0}_{4,4} = \mathcal{P}(\mathbb{P}^3_{\mathbb{R}}, \mathcal{H}^{s0}_4)$, and $\mathcal{P}^{s0+}_{4,4} = \mathcal{P}(\mathbb{P}^3_+, \mathcal{H}^{s0}_4)$ are determined in [4].

References

[1] T. Ando, Some homogeneous cyclic inequalities of three variables of degree three and four, Aust. J. Math. Anal. Appl. **7** (2) Art. 12 (2010)
[2] T. Ando, Cubic and quartic cyclic homogeneous inequalities of Three Variables, Math. Inequal. Appl., **16**, 127-142 (2013)
[3] T. Ando, Semialgebraic Variety, Preprint.
[4] T. Ando, Discriminants of Some Families of Cubic and Quartic Homogeneous Inequalities of Four Variables, Preprint.
[5] G. Blekherman, Nonnegative polynomials and sums of squares, J. Amer. Math. Soc. **25**(3), 617-635, (2012).
[6] G. Blekherman & P. A. Parrilo & R. Thomas, ed., Semidefinite Optimization and Convex Algebraic Geometry, Cambridge University Press 2013.
[7] J. Bochnak & M. Coste & M.F. Roy, Real Algebraic Geometry, Springer (1998).
[8] M.D. Choi & T.Y. Lam, Extremal Positive Semidefinite Forms, Math. Ann., **231**, 1-18 (1977)
[9] M.D. Choi & T.Y. Lam & B. Reznik, Even Symmetric Sextics, Math. Z., **195**, 559-580 (1987)
[10] V. Cîrtoaje, On the cyclic homogeneous polynomial inequalities of degree four, J. Inequal. Pure and Appl. Math., **10**(3) Art. 67 (2009)

[11] V. Cîrtoaje, Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree four and six in real variables, J. Nonlinear Sci. Appl., **5**, 307-320 (2012)

[12] V. Cîrtoaje, Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real variables, Aust. J. Math. Anal. Appl. **9** (1) (2012)

[13] V. Cîrtoaje & Y. Zhou, Some strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real variables, Nonlinear Ann. Appl., jnaa-00151 (2012)

[14] V. Cîrtoaje, Necessary and Sufficient Conditions for Symmetric Homogeneous Polynomial Inequalities in Nonnegative Real Variables, Math. Ineq. Appl., **16**, 413-438 (2013).

[15] V. Cîrtoaje, A Strong Method for Symmetric Homogeneous Polynomial Inequalities of Degree Six in Nonnegative Real Variables, Brit. J. Math. & Comp. Sci., **4**, 685-703 (2014)

[16] R. Hartshorne, Algebraic Geometry, Springer 1977.

[17] D. Hilbert, Uber die Darstellung definiter Formen als Summe von Formenquadraten, Math. Ann., **32**(1888), 342-350.

[18] T.Y. Lam, An introduction to real algebra, Rocky My. J. Math., **14**, 767-814 (1984)

[19] R. Muirhead, Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters, Proc. Edinb. Math. Soc., **21** 144-157 (1902)

[20] J. Nie, Discriminant and nonnegative polynimials, J. Symbolic Comput, **47**, 167-191 (2012).

[21] C. Prosecy & G. Schwarz, Inequalities defining orbit spaces, Invemt. math., **81** 539-554 (1985).

[22] B. Reznick, Forms Derived from Arithmetic-Geometric Inequality, Math. Ann., **238** 431-464 (1989)

[23] C. Riener, On the degree and half-degree principle for symmetric polynomials. J. Pure Appl. Alg. **216.4** (2012): 850-856.

[24] C. Riener & T. Theobald & L.J. Andrén & J. B. Lasserre, Exploiting symmetries in SDP-relaxations for polynomial optimization, Math. Oper. Res. **38**(1), 122-141 (2013).

[25] R. Sinn, Algebraic boundaries of semi-algebraic sets, Research in the Math. Sci., (2015) Dec. 2:3.

[26] V. Timofte, On the positivity of symmetric polynomial functions I & II, J. Math. Anal. Appl., **284** 174-190 (2003) & **304** 652-667 (2005).