A New Proof of Shapiro Inequality

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Abstract We present a new proof of Shapiro cyclic inequality. Especially, we treat the case $n = 23$ precisely.

§1. Introduction.

Let $n \geq 3$ be an integer, x_1, x_2,..., x_n be positive real numbers, and let

$$E_n(x_1, \ldots, x_n) := \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}},$$

here we regard $x_{i+n} = x_i$ for $i \in \mathbb{Z}$. In this article, we present a new proof of the following theorem:

Theorem 1.1. (1) If n is an odd integer with $3 \leq n \leq 23$, then

$$E_n(x_1, \ldots, x_n) \geq n/2. \qquad (P_n)$$

Moreover, $E_n(x_1,\ldots, x_n) = n/2$ holds only if $x_1 = x_2 = \cdots = x_n$.

(2) If n is an even integer with $4 \leq n \leq 12$, then (P_n) holds. Moreover, the equality holds only if $(x_1,\ldots, x_n) = (a, b, a, b,\ldots, a, b)$ $(\exists a > 0, \exists b > 0)$.

(3) If n is an even integer with $n \geq 14$ or an odd integer with $n \geq 25$, then there exists $x_1 > 0,\ldots, x_n > 0$ such that $E_n(x_1,\ldots, x_n) < n/2$.

(3) was proved by [4] in 1979. It is said that (1) was proved by [6] in 1989. (2) was proved by [2] in 2002. Note that [2] treat (1) to be an open problem. The author also thinks we should give a more agreeable proof of (1). In this article, we give more precise proof of (1) than [6].

§2. Basic Facts.

Throughout this article, we use the following notations:

$$\partial_i E_n(\mathbf{x}) := \frac{\partial}{\partial x_i} E_n(\mathbf{x}) = \frac{1}{x_{i+1} + x_{i+2}} - \frac{x_{i-2}}{(x_{i-1} + x_i)^2} - \frac{x_{i-1}}{(x_i + x_{i+1})^2}$$

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$$\overline{K_n} := \left\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\middle|\; x_1\ge 0,\dots,\ x_n\ge 0\right\}$$
$$K_n^\circ := \left\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\middle|\; x_1> 0,\dots,\ x_n> 0\right\}$$
$$K_n^\bullet := \left\{ (x_1,\dots,x_n)\in\overline{K_n} \;\middle|\; \begin{array}{l}(x_1,\dots,x_n)\notin K_n^\circ,\\ (x_i,x_{i+1})\ne(0,0) \text{ for any } i\in\mathbb{Z}.\end{array}\right\}$$
$$K_n = K_n^\circ \cup K_n^\bullet$$

It is easy to see that there exists $\mathbf{a}\in K_n^\bullet$ such that

$$\inf_{\mathbf{x}\in K_n^\circ} E_n(\mathbf{x}) = E_n(\mathbf{a}).$$

Thus, we consider $E_n(\mathbf{x})$ to be a continious function on $K_n^\bullet$.

Proposition 2.1.([3]) (1) If (P_n) is false, then (P_{n+2}) is also false.

(2) If (P_n) is false for an odd integer $n\ge 3$, then (P_{n+1}) is also false.

Proof. Assume that there exists positive real numbers $a_1,\dots,\ a_n$ such that $E_n(a_1,\dots,\ a_n) < n/2$.

(1) Since, $E_{n+2}(a_1,\dots,a_n,a_1,a_2) = 1 + E_n(a_1,\dots,a_n) < \dfrac{n+2}{2}$, (P_{n+2}) is false.

(2) Note that

$$\begin{aligned}&E_{n+1}(a_1,\dots,a_{r-1},a_r,a_r,a_{r+1},\dots,a_n) - E_n(a_1,\dots,a_n) - \frac{1}{2}\\ &= \frac{a_{r-1}}{a_r+a_r} + \frac{a_r}{a_r+a_{r+1}} - \frac{a_{r-1}}{a_r+a_{r+1}} - \frac{1}{2}\\ &= \frac{(a_r-a_{r-1})(a_r-a_{r+1})}{2a_r(a_r+a_{r+1})}\end{aligned}$$

for $1\le r\le n$. Thus, it is sufficient to show that there exists r such that $(a_r-a_{r-1})(a_r-a_{r+1})\le 0$.

Assume that $(a_r-a_{r-1})(a_r-a_{r+1})>0$ for all $1\le r\le n$. Since n is odd,

$$\prod_{r=1}^{n}(a_r-a_{r+1})^2 = \prod_{r=1}^{n}(a_{r-1}-a_r)(a_r-a_{r+1}) < 0.$$

This is a contradiction. ∎

Proposition 2.2.([4]) (1) E_{14}(42, 2, 42, 4, 41, 5, 39, 4, 38, 2, 38, 0, 40, 0) < 7. Thus (P_{14}) is false.

(2) E_{25}(34, 5, 35, 13, 30, 17, 24, 18, 18, 17, 13, 16, 9, 16, 5, 16, 2, 18, 0, 21, 0, 25, 0, 29, 0) $< 25/2$. Thus (P_{25}) is false.

Thus, Theorem 1.1 (3) is proved by Proposition 2.1 and 2.2. It is essential to show (P_{12}) and (P_{23}) for a proof of Theorem 1.1 (2) and (3).

Definition 2.3. We say that $\mathbf{x}=(x_1,\dots,\ x_n)\in K_n$ and $\mathbf{y}=(y_1,\dots,\ y_n)\in K_n$ belong to the *same component* if "$x_i=0 \iff y_i=0$" for all $i=1,\dots,\ n$.

Let $\mathbf{x}=(x_1,\dots,\ x_n)\in K_n^\bullet$. If $x_{i-1}=0$, $x_i\ne 0$, $x_{i+1}\ne 0,\dots,\ x_j\ne 0$, and $x_{j+1}=0$ for $i<j\in\mathbb{Z}$, then we call $(x_i,\dots,\ x_j)$ to be a *segment* of $\mathbf{a}$, and we define $j-i+1$ to be the *length* of this segment. A segment of length l is called *l-semgent.*

For a segment $\mathbf{s} := (x_i,\ldots, x_j)$ of $\mathbf{x}$, we denote

$$S(\mathbf{s}) := \sum_{k=i}^{j-1} \frac{x_k}{x_{k+1}+x_{k+2}}, \quad Head(\mathbf{s}) := x_i, \quad Tail(\mathbf{s}) := x_j.$$

Here we define $S(\mathbf{s}) = 0$, if the length of $\mathbf{s}$ is 1.

Let $\mathbf{s}_1,\ldots, \mathbf{s}_r$ be all the segments of $\mathbf{x}$ in this order. Let l_k be the length of $\mathbf{s}_k$. Then $(l_1,\ldots, l_r)$ is called the *index* of $\mathbf{x}$. Note that

$$E_n(\mathbf{a}) = \sum_{k=1}^{r} S(\mathbf{s}_k) + \sum_{k=1}^{r} \frac{Tail(\mathbf{s}_{k-1})}{Head(\mathbf{s}_k)}.$$

Here we regard $\mathbf{s}_{k+r} = \mathbf{s}_k$ for $k \in \mathbb{Z}$.

Theorem 2.4. Assume that $\min_{\mathbf{x}\in K_n^\bullet} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} = (a_1,\ldots, a_n) \in K_n^\bullet$. Let $\mathbf{s}_1,\ldots, \mathbf{s}_r$ be all the segments of $\mathbf{a}$ in this order, and let l_k be the length of $\mathbf{s}_k$. Then the followings hold.

(1) $\dfrac{Tail(\mathbf{s}_1)}{Head(\mathbf{s}_2)} = \dfrac{Tail(\mathbf{s}_2)}{Head(\mathbf{s}_3)} = \cdots = \dfrac{Tail(\mathbf{s}_{r-1})}{Head(\mathbf{s}_r)} = \dfrac{Tail(\mathbf{s}_r)}{Head(\mathbf{s}_1)}$.

(2) Assume that $\mathbf{a} = (\mathbf{s}_1, 0, \mathbf{s}_2, 0,\ldots, \mathbf{s}_r, 0)$, and let σ be a permutation of $\{1, 2,\ldots, r\}$. Then there exist real numbers $t_1 > 0$, $t_2 > 0,\ldots, t_r > 0$ such that

$$\mathbf{b} := \big(t_1\mathbf{s}_{\sigma(\mathbf{1})}, 0, t_2\mathbf{s}_{\sigma(\mathbf{2})}, 0, \ldots, t_r\mathbf{s}_{\sigma(\mathbf{r})}, 0\big)$$

satisfies $E_n(\mathbf{b}) = E_n(\mathbf{a})$.

Proof. (1) Since $E_n(a_{1+k}, a_{2+k}, \ldots, a_{n+k}) = E_n(a_1, a_2, \ldots, a_n)$, we may assume $\mathbf{a} = (\mathbf{s}_1, 0, \mathbf{s}_2, 0,\ldots, \mathbf{s}_r, 0)$. Let $x_i := Head(\mathbf{s}_i)$, $y_i := Tail(\mathbf{s}_i)$. Define $t_1,\ldots, t_r$ by $t_1 := 1$ and

$$t_j := \frac{y_1y_2\cdots y_{j-1}}{x_2x_3\cdots x_j} \cdot \left(\frac{x_1x_2\cdots x_r}{y_1y_2\cdots y_r}\right)^{\frac{j-1}{r}}$$

for $j = 2, 3,\ldots, r$. It is easy to see that

$$\frac{t_{j-1}y_{j-1}}{t_jx_j} = \sqrt[r]{\frac{y_1\cdots y_r}{x_1\cdots x_r}} = \frac{t_ry_r}{t_1x_1}.$$

Take $t_1 > 0,\ldots, t_r > 0$, and let

$$\mathbf{c} = (t_1\mathbf{s}_1, 0, t_2\mathbf{s}_2, 0, \ldots, t_r\mathbf{s}_r, 0).$$

Note that $S(t_i\mathbf{s}_i) = S(\mathbf{s}_i)$. By AM-GM inequality,

$$\begin{aligned} E_n(\mathbf{a}) &= \sum_{i=1}^{r} S(\mathbf{s}_i) + \sum_{i=1}^{r} \frac{y_{i-1}}{x_i} \\ &\geq \sum_{i=1}^{r} S(\mathbf{s}_i) + r \cdot \sqrt[r]{\frac{y_1\cdots y_r}{x_1\cdots x_r}} = \sum_{i=1}^{r} S(t_i\mathbf{s}_i) + \sum_{i=1}^{r} \frac{t_{i-1}y_{i-1}}{t_ix_i} = E_n(\mathbf{c}). \end{aligned}$$

Since $E_n(\mathbf{a})$ is the minimum, we have $E_n(\mathbf{a}) = E_n(\mathbf{c})$. By the equality condition of AM-GM inequality, we have $t_1 = t_2 = \cdots = t_r = 1$. Thus

$$\frac{y_{j-1}}{x_j} = \sqrt[r]{\frac{y_1\cdots y_r}{x_1\cdots x_r}},$$

and we have (1).

(2) By the same argument as (1), we conclude that there exists positive integers $t'_1,\ldots,$ t'_r such that

$$\mathbf{b} := (t'_1\mathbf{s}_{\sigma(1)},\, 0,\, t'_2\mathbf{s}_{\sigma(2)},\, 0, \ldots,\, t'_r\mathbf{s}_{\sigma(r)},\, 0)$$

satisfies

$$E_n(\mathbf{b}) = \sum_{i=1}^{r} S(\mathbf{s}_i) + r \cdot \sqrt[r]{\frac{y_1\cdots y_r}{x_1\cdots x_r}}.$$

Thus $E_n(\mathbf{b}) = E_n(\mathbf{a})$. □

Remark 2.5. By the above theorem, we may assume that the index $(l_1,\ldots, l_r)$ of $\mathbf{a}$ satisfies $l_1 \geq l_2 \geq \cdots \geq l_r$, if $\min_{\mathbf{x}\in K_n^\bullet} E_n(\mathbf{x}) = E_n(\mathbf{a})$. Thus, we always write the index of such $\mathbf{a}$ in descending order.

Definition 2.6. Assume that $\mathbf{a} \in K_n^\bullet$ satisfies the condition of the above theorem. Then we define $U(\mathbf{a})$ to be

$$U(\mathbf{a}) := \frac{Tail(\mathbf{s}_1)}{Head(\mathbf{s}_2)} = \frac{Tail(\mathbf{s}_2)}{Head(\mathbf{s}_3)} = \cdots = \frac{Tail(\mathbf{s}_{r-1})}{Head(\mathbf{s}_r)} = \frac{Tail(\mathbf{s}_r)}{Head(\mathbf{s}_1)}.$$

Note that $E_n(\mathbf{a}) = rU(\mathbf{a}) + \sum_{k=1}^{r} S(\mathbf{s}_k)$, for $\mathbf{a} = (\mathbf{s}_1,\, 0,\, \mathbf{s}_2,\, 0,\ldots,\, \mathbf{s}_r,\, 0)$.

§3. Bushell Theorem.

We survey and improve the results of [1]. In this section, we denote

$$\begin{aligned}
A_i(\mathbf{x}) &:= \frac{x_i}{x_{i+1}+x_{i+2}}\\
B(\mathbf{x}) &:= \big(x_2+x_3,\, x_3+x_4, \ldots,\, x_n+x_1,\, x_1+x_2\big)\\
R(\mathbf{x}) &:= \left(\frac{1}{x_n}, \frac{1}{x_{n-1}}, \frac{1}{x_{n-2}}, \ldots, \frac{1}{x_1}\right)\\
T(\mathbf{x}) &= \left(\frac{x_n}{(x_1+x_2)^2}, \ldots, \frac{x_{n+1-i}}{(x_{n+2-i}+x_{n+3-i})^2}, \ldots, \frac{x_1}{(x_2+x_3)^2}\right)
\end{aligned}$$

for $\mathbf{x} = (x_1,\ldots, x_n)$. We also denote the i-th element of $B(\mathbf{x})$ by $B(\mathbf{x})_i = x_{i+1}+x_{i+2}$. $R(\mathbf{x})_i$ and $T(\mathbf{x})_i$ are also defined similarly. The symbol $T(\mathbf{x})$ are used throughout this article.

Lemma 3.1.([1] Lemma 3.2, 4.2) The above functions satisfy the followings.

(1) $\partial_i E_n(\mathbf{x}) = (R(B(\mathbf{x}))_{n+1-i} - (B(T(\mathbf{x})))_{n+1-i}$.

(2) $(T^2(\mathbf{x}))_i = \dfrac{x_i}{\big(1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})\big)^2}$.

(3) $E_n(T(\mathbf{x})) - E_n(\mathbf{x}) = \displaystyle\sum_{i=1}^{n} \frac{x_i\big(\partial_i E_n(\mathbf{x})\big)^2}{(B(T(\mathbf{x})))_{n+1-i}}$.

(4) $E_n(\mathbf{x}) + E_n(\mathbf{y})$

$$= E_n(\mathbf{x}+\mathbf{y}) + E_n(T(\mathbf{x})+T(\mathbf{y})) - \sum_{i=1}^{n} \frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}\big(\partial_i E_n(\mathbf{x})+\partial_i E_n(\mathbf{y})\big)}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}\cdot(B(T(\mathbf{x})+T(\mathbf{y})))_{n+1-i}}.$$

Proof. (1) $\partial_i E_n(\mathbf{x}) = \dfrac{1}{x_{i+1}+x_{i+2}} - \left(\dfrac{x_{i-2}}{(x_{i-1}+x_i)^2} + \dfrac{x_{i-1}}{(x_i+x_{i+1})^2}\right) = (R(B(\mathbf{x}))_{n+1-i} - (B(T(\mathbf{x})))_{n+1-i}$.

(2) $(T(\mathbf{x}))_i = \dfrac{x_{n+1-i}}{(B(\mathbf{x}))^2_{n+1-i}}$. Combine this with (1), we obtain

$$(T^2(\mathbf{x}))_i = \frac{(T(\mathbf{x}))_{n+1-i}}{(B(T(\mathbf{x})))^2_{n+1-i}} = \frac{x_i/(B(\mathbf{x}))^2_i}{\big((R(B(\mathbf{x})))_{n+1-i} - \partial_i E_n(\mathbf{x})\big)^2}. \tag{3.1.1}$$

Since $(B(\mathbf{x}))_i \cdot (R(B(\mathbf{x})))_{n+1-i} = 1$, we obtain (2).

(3) By the similar calculation as above, we obtain

$$\begin{aligned} E_n(T(\mathbf{x})) - E_n(\mathbf{x}) &= \sum_{i=1}^n \frac{(T(\mathbf{x}))_i}{(B(T(\mathbf{x})))_i} - \sum_{i=1}^n \frac{x_i}{(B(\mathbf{x}))_i} \\ &= \sum_{i=1}^n \left(\frac{(T(\mathbf{x}))_{n+1-i}}{(B(T(\mathbf{x})))_{n+1-i}} - \frac{x_i}{(B(\mathbf{x}))_i}\right) \\ &= \sum_{i=1}^n \left(\frac{x_i}{(B(\mathbf{x}))_i\big(1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})\big)} - \frac{x_i}{(B(\mathbf{x}))_i}\right) \\ &= \sum_{i=1}^n \frac{x_i \partial_i E_n(\mathbf{x})}{1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})}. \end{aligned}$$

Since,

$$\begin{aligned} &\sum_{i=1}^n x_i\partial_i E_n(\mathbf{x}) = \sum_{i=1}^n \frac{x_i}{x_{i+1}+x_{i+2}} - \sum_{i=1}^n \frac{x_{i-2}x_i}{(x_{i-1}+x_i)^2} - \sum_{i=1}^n \frac{x_{i-1}x_i}{(x_i+x_{i+1})^2} \\ &= \sum_{i=1}^n \frac{x_{i-1}(x_i+x_{i+1})}{(x_i+x_{i+1})^2} - \sum_{i=1}^n \frac{x_{i-1}x_{i+1}}{(x_i+x_{i+1})^2} - \sum_{i=1}^n \frac{x_{i-1}x_i}{(x_i+x_{i+1})^2} = 0, \end{aligned}$$

we obtain

$$\begin{aligned} E_n(T(\mathbf{x})) - E_n(\mathbf{x}) &= \sum_{i=1}^n x_i\partial_i E_n(\mathbf{x})\left(\frac{1}{1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})} - 1\right) \\ &= \sum_{i=1}^n \frac{x_i\big(\partial_i E_n(\mathbf{x})\big)^2}{(B(T(\mathbf{x})))_{n+1-i}}. \end{aligned}$$

(4) Let $a := x_i$, $b := x_{i+1}+x_{i+2} = (B(\mathbf{x}))_i$, $c := y_i$, $d := (B(\mathbf{y}))_i$.

$$\begin{aligned} &\frac{x_i+y_i}{(B(\mathbf{x}+\mathbf{y}))_i} + \frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}} \\ &= \frac{a+c}{b+d} + \frac{a/b^2+c/d^2}{1/b+1/d} = \frac{a}{b}+\frac{c}{d} = A_i(\mathbf{x}) + A_i(\mathbf{y}) \end{aligned} \tag{3.1.2}$$

By (1), we have

$$\begin{aligned} &\frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}}{(B(T(\mathbf{x})+T(\mathbf{y})))_{n+1-i}} - \frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}} \\ &= \frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}\big(\partial_i E_n(\mathbf{x})+\partial_i E_n(\mathbf{y})\big)}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}\cdot(B(T(\mathbf{x})+T(\mathbf{y})))_{n+1-i}}. \end{aligned} \tag{3.1.3}$$

Take $\sum_{i=1}^{n}$ of (3.1.2) and (3.1.3), we obtain (4). $\square$

Theorem 3.2.([1] Theorem 3.3) (1) $E_n(T(\mathbf{x})) \geq E_n(\mathbf{x})$ holds for $\mathbf{x} \in K_n$. Moreover, if $E_n(T(\mathbf{x})) = E_n(\mathbf{x})$, then $T^2(\mathbf{x}) = \mathbf{x}$ holds.

(2) If $\min_{\mathbf{x} \in K_n^\bullet} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n$, then the following holds.

$$T^2(\mathbf{a}) = \mathbf{a}, \quad E_n(T(\mathbf{a})) = E_n(\mathbf{a}).$$

Proof. (1) $E_n(T(\mathbf{x})) \geq E_n(\mathbf{x})$ follows from Lemma 3.1 (3). Assume that $E_n(T(\mathbf{x})) = E_n(\mathbf{x})$. Then $x_i\big(\partial_i E_n(\mathbf{x})\big)^2 = 0$ ($\forall i = 1, \ldots, n$), by Lemma 3.1 (3). Thus $x_i = 0$ or $\partial_i E_n(\mathbf{x}) = 0$. By Lemma 3.1 (2), we obtain $(T^2(\mathbf{x}))_i = x_i$.

(2) If E_n is minimum at $\mathbf{a}$, then $a_i = 0$ or $\partial_i E_n(\mathbf{a}) = 0$. By Lemma 3.1 (2), we have $(T^2(\mathbf{a}))_i = a_i$. We also have $E_n(T(\mathbf{a})) = E_n(\mathbf{a})$ by Lemma 3.1 (3). $\square$

Lemma 3.3.([1] Lemma 4.3) Let a, b, c, d, e be positive real numbers, and p, q be real numbers. Assume that

$$p\frac{1+\lambda a}{(1+\lambda c)^2} + q\frac{1+\lambda b}{(1+\lambda d)^2} = \frac{1}{1+\lambda e} \tag{3.3.1}$$

for all real numbers $\lambda \geq 0$. Then the followings hold.

(1) If $p = 0$, then $q = 1$ and $b = d = e$.
(2) If $q = 0$, then $p = 1$ and $a = c = e$.
(3) If $p \neq 0$ and $q \neq 0$, then $c = d = e$.

Proof. (1) Substitute $\lambda = 0$, $p = 0$ for (3.3.1), we have $q = 1$. In this case, (3.3.1) is equivalent to

$$(1+\lambda b)(1+\lambda e) = (1+\lambda d)^2.$$

As an equality of a polynomial in λ, we have $b = d = e$.

(2) can be proved similarly as (1).

(3) Let

$$\begin{aligned} g(\lambda) := {} & p(1+\lambda a)(1+\lambda d)^2(1+\lambda e) \\ & + q(1+\lambda b)(1+\lambda c)^2(1+\lambda e) - (1+\lambda c)^2(1+\lambda d)^2. \end{aligned} \tag{3.3.2}$$

$g(\lambda) = 0$ as a polynomial in λ. Thus

$$0 = g\left(-\frac{1}{e}\right) = -\left(1-\frac{c}{e}\right)^2\left(1-\frac{d}{e}\right)^2,$$

and we have $c = e$ or $d = e$.

Assume that $d \neq e$. Then $c = e$. From (3.3.2), we obtain

$$p(1+\lambda a)(1+\lambda d)^2 + q(1+\lambda b)(1+\lambda e)^2 - (1+\lambda e)(1+\lambda d)^2 = 0. \tag{3.3.3}$$

Substitute $\lambda = -1/e$ for (3.3.3), we obtain $p(1-a/e)(1-d/e)^2 = 0$. Thus $a = e$. Then

$$p(1+\lambda d)^2 + q(1+\lambda b)(1+\lambda e) - (1+\lambda d)^2 = 0. \tag{3.3.4}$$

Substitute $\lambda = -1/e$ for (3.3.4), we have $d = e$. A contradiction. Thus $d = e$.

Similarly, we have $c = e$. $\square$

Theorem 3.4. (1) Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a}) = E_n(\mathbf{b})$ at $\mathbf{a}$, $\mathbf{b} \in K_n^\bullet$ and that $\mathbf{a}$ and $\mathbf{b}$ belong to the same component. Then, there exists a real number $\mu > 0$ such that $\mathbf{a} = \mu\mathbf{b}$.

(2) Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^\circ$. Then $E_n(\mathbf{a}) = n/2$. Moreover $\mathbf{a} = (a, a, a, \ldots, a)$ $(\exists a > 0)$, or $\mathbf{a} = (a, b, a, b, \ldots, a, b)$ $(\exists a > 0, b > 0)$.

Proof. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a}) = E_n(\mathbf{b})$ for $\mathbf{a}$, $\mathbf{b} \in K_n$, and that $\mathbf{a}$ and $\mathbf{b}$ belong to the same component. Let $\lambda > 0$ be any real number.

If $a_i \neq 0$, then $\partial_i E_n(\mathbf{a}) = \partial_i E_n(\lambda\mathbf{b}) = 0$. If $a_i = 0$, then $b_i = 0$ and $(T(\mathbf{a}))_{n+1-i} = 0$, $(T(\lambda\mathbf{b}))_{n+1-i} = 0$. Thus we have

$$(T(\mathbf{a}) + T(\lambda\mathbf{b}))_{n+1-i} \cdot \big(\partial_i E_n(\mathbf{a}) + \partial_i E_n(\lambda\mathbf{b})\big) = 0$$

$(\forall i \in \mathbb{Z})$. We use the Lemma 3.1 (4) with $\mathbf{x} = T(\mathbf{a})$, $\mathbf{y} = \lambda\mathbf{b}$. Since the numerators of the fractions in $\sum$ in Lemma 3.1 (4) are zero, we have

$$E_n(\mathbf{a}) + E_n(\lambda\mathbf{b}) = E_n(\mathbf{a} + \lambda\mathbf{b}) + E_n(T(\mathbf{a}) + T(\lambda\mathbf{b})).$$

Since $E_n(\lambda\mathbf{b}) = E_n(\mathbf{b}) = E_n(\mathbf{a})$ is minimum, we have

$$E_n(\mathbf{a} + \lambda\mathbf{b}) = E_n(T(\mathbf{a}) + T(\lambda\mathbf{b})) = E_n(\mathbf{a}).$$

Since $E_n(\mathbf{x})$ is minimum at $\mathbf{x} = \mathbf{a} + \lambda\mathbf{b}$ for any $\lambda > 0$, we have

$$0 = \partial_i E_n(\mathbf{a} + \lambda\mathbf{b}) = \frac{1}{(B(\mathbf{a} + \lambda\mathbf{b}))_i} - \frac{a_{i-2} + \lambda b_{i-2}}{(B(\mathbf{a} + \lambda\mathbf{b}))_{i-2}^2} - \frac{a_{i-1} + \lambda b_{i-1}}{(B(\mathbf{a} + \lambda\mathbf{b}))_{i-1}^2} \tag{3.4.1}$$

when $a_i \neq 0$. Let

$$a := \frac{b_{i-2}}{a_{i-2}}, \quad b := \frac{b_{i-1}}{a_{i-1}}, \quad c := \frac{(B(\mathbf{b}))_{i-2}}{(B(\mathbf{a}))_{i-2}}, \quad d := \frac{(B(\mathbf{b}))_{i-1}}{(B(\mathbf{a}))_{i-1}},$$

$$e := \frac{(B(\mathbf{b}))_i}{(B(\mathbf{a}))_i}, \quad p := \frac{a_{i-2}(B(\mathbf{a}))_i}{(B(\mathbf{a}))_{i-2}^2}, \quad q := \frac{a_{i-1}(B(\mathbf{a}))_i}{(B(\mathbf{a}))_{i-1}^2}.$$

Then, (3.4.1) become (3.3.1). It is easy to see that the cases (1) and (2) of Lemma 3.3 do not occur. Lemma 3.3 (3) implies

$$\frac{(B(\mathbf{b}))_{i-2}}{(B(\mathbf{a}))_{i-2}} = \frac{(B(\mathbf{b}))_{i-1}}{(B(\mathbf{a}))_{i-1}} = \frac{(B(\mathbf{b}))_i}{(B(\mathbf{a}))_i} =: \frac{1}{\mu} > 0.$$

Thus

$$a_{i+1} + a_{i+2} = B(\mathbf{u}) = \mu B(\mathbf{v}) = \mu(b_{i+1} + b_{i+2}) \tag{3.4.2}$$

$(\forall i \in \mathbb{Z})$. If n is odd, then $a_i = \mu b_i$ $(\forall i \in \mathbb{Z})$ from (3.4.2). Thus $\mathbf{a} = \mu\mathbf{b}$.

We treat the case n is even. Let $\mathbf{w} = (1, -1, 1, -1, \ldots, -1) \in \mathbb{R}^n$. By elementary linear algebra, we conclude that the solutions of the system of equations (3.4.2) is of the form

$$\mathbf{a} - \mu\mathbf{b} = \nu\mathbf{w} \quad (\exists\nu \in \mathbb{R}).$$

If $\mathbf{a} \in K_n^\bullet$, then $\mathbf{a}$ and $\mathbf{b}$ have zeros at the same place. Thus, ν must be zero. Thus we obtain (1).

We shall prove (2). Apply above argument to $\mathbf{b} = (a_2, a_3, \ldots, a_n, a_1)$. If n is odd, then $\mathbf{a} = \mu\mathbf{b}$. Thus $\mu = 1$, and $a_1 = a_2 = \cdots = a_n$. In this case, $E_n(\mathbf{a}) = n/2$.

If n is even, $\mathbf{a} - \mu\mathbf{b} = \nu\mathbf{w}$. Thus $\mathbf{a} = (a_1, a_2, a_1, a_2, \ldots, a_1, a_2)$. Then $E_n(\mathbf{a}) = n/2$. □

Corollary 3.5. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^\bullet$. Let $\mathbf{s}$ and $\mathbf{t}$ be segments of $\mathbf{a}$ with the same length l. Then, there exists a real number $c > 0$ such that $\mathbf{s} = c\mathbf{t}$.

Proof. We construct a vector $\mathbf{b}$ as in the proof of Theorem 2.4 (2), where σ is the transposition of $\mathbf{s}$ and $\mathbf{t}$. Then $E_n(\mathbf{a}) = E_n(\mathbf{b})$. By Theorem 3.4, $\mathbf{a} = \mu\mathbf{b}$ $(\exists\mu > 0)$. Thus $\mathbf{s} = c\mathbf{t}$ $(\exists c > 0)$. ∎

Corollary 3.6. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^\bullet$. Let $\mathbf{s} = (a_1,\dots, a_l)$ be a l-segment of $\mathbf{a}$ with $l \geq 2$. Let $U := U(\mathbf{a})$. Then there exists a real number $\mu > 0$ such that

$$\begin{aligned}&\left(\frac{U^2}{a_l}, \frac{a_{l-1}}{a_l^2}, \frac{a_{l-2}}{(a_{l-1}+a_l)^2}, \frac{a_{l-3}}{(a_{l-2}+a_{l-1})^2}, \dots, \frac{a_2}{(a_3+a_4)^2}, \frac{a_1}{(a_2+a_3)^2}\right)\\ &= \mu(a_1, a_2, a_3, a_4, \dots, a_{l-1}, a_l). \end{aligned} \tag{3.6.1}$$

Proof. We may assume that $\mathbf{a} = (\mathbf{s}, 0,\dots)$. Rotate the elements of $T(\mathbf{a})$ so that the segment corresponding to $\mathbf{s}$ comes to be the same place with $\mathbf{s}$, and we denote this vector by $\mathbf{b}$. Then the top segment of $\mathbf{b}$ is

$$\left(\frac{a_l}{a_{l+2}^2}, \frac{a_{l-1}}{a_l^2}, \frac{a_{l-2}}{(a_{l-1}+a_l)^2}, \frac{a_{l-3}}{(a_{l-2}+a_{l-1})^2}, \dots, \frac{a_2}{(a_3+a_4)^2}, \frac{a_1}{(a_2+a_3)^2}\right).$$

By Theorem 3.2 (2), $E_n(\mathbf{b}) = E_n(T(\mathbf{a})) = E_n(\mathbf{a})$. By Theorem 3.4, $\mathbf{b} = \mu\mathbf{a}$ $(\exists\mu > 0)$. Since $U = a_l/a_{l+2}$, $a_l/a_{l+2}^2 = U^2/a_l$. Thus, we have (3.6.1). ∎

§4. Bushell-McLead Theorem.

The aim of this section is to explain Theorem 4.3, according to [2]. In This section, we denote

$$\begin{aligned} K_n^\triangle &:= \left\{(x_1,\dots,x_n) \in K_n^\bullet \;\middle|\; x_{n-1} = 1,\, x_n = 0\right\} \\ y_i &:= \frac{x_i}{x_{i+1}+x_{i+2}} = A_i(\mathbf{x}). \end{aligned}$$

Note that $y_n = 0$, $y_{n-1} = x_{n-1}/x_1$, and $y_{n-2} = x_{n-2}$ for $\mathbf{x} = (x_1,\dots, x_n) \in K_n^\triangle$. The map $\Phi\colon K_n^\triangle \to \Phi(K_n^\triangle)$ defined by $\Phi(x_1,\dots, x_n) = (y_1,\dots, y_n)$ is bijective. The inverse map Φ^{-1} is obtained as the solution of the system of equations $y_i(x_{i+1} + x_{i+2}) - x_i = 0$ $(i = 1,\dots, n-2)$. Let

$$P_k(z_1, z_2, \dots, z_k) := \begin{vmatrix} z_1 & z_1 & & & & & & \\ -1 & z_2 & z_2 & & & & & \\ & -1 & z_3 & z_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & z_{k-2} & z_{k-2} & \\ & & & & & -1 & z_{k-1} & z_{k-1} \\ & & & & & & -1 & z_k \end{vmatrix}.$$

Inductively, we can prove that $x_i = P_{n-i-1}(y_i, y_{i+1},\dots, y_{n-2})$. By the properties of determinant, we can prove the following lemma.

Lemma 4.1.([2] Lemma 3.1) The followings hold. Here we put $P_0 := 1$ and $P_{-1} = 1$.

(1) $P_k(z_1,\dots, z_k) = z_k P_{k-1}(z_1,\dots, z_{k-1}) + z_{k-1}P_{k-2}(z_1,\dots, z_{k-2})$.

(2) For $1 \le j < k$,

$$P_k(z_1, \dots, z_k) = P_j(z_1, \dots, z_j)P_{k-j}(z_{j+1}, \dots, z_k) + z_j P_{j-1}(z_1, \dots, z_{j-1})P_{k-j-1}(z_{j+2}, \dots, z_k).$$

Lemma 4.2.([2] Lemma 3.2) Let $\mathbf{x} = (x_1, \dots, x_n) \in K_n^{\triangle}$, and $(y_1, \dots, y_n) = \Phi(x_1, \dots, x_n)$. Assume that $x_i \partial_i E_n(\mathbf{x}) = 0$ for all $i = 1, 2, \dots, n$. Then the followings hold.
(1) $y_i = y_1^2 P_{i-1}(y_1, \dots, y_{i-1})P_{n-i-1}(y_i, \dots, y_{n-2})$
(2) $y_1 - y_i = y_1^2 y_{i-1} P_{i-2}(y_1, \dots, y_{i-2})P_{n-i-2}(y_{i+1}, \dots, y_{n-2})$

Proof. Put $p_i := P_i(y_1, \dots, y_i)$. Then (1), (2) can be written as (1) $y_i = y_1^2 p_{i-1} x_i$, and (2) $y_1 - y_i = y_1^2 y_{i-1} p_{i-2} x_{i+1}$.

(1) As a formal rational function

$$x_i \partial_i E_n(\mathbf{x}) = \frac{x_i}{x_{i+1} + x_{i+2}} - \frac{x_{i-2}x_i}{(x_{i-1} + x_i)^2} - \frac{x_{i-1}x_i}{(x_i + x_{i+1})^2} = y_i - \frac{y_{i-2}^2 x_i}{x_{i-2}} - \frac{y_{i-1}^2 x_i}{x_{i-1}}.$$

So, the condition $x_i \partial_i E_n(\mathbf{x}) = 0$ can be represented as

$$\frac{y_i}{x_i} = \frac{y_{i-2}^2}{x_{i-2}} + \frac{y_{i-1}^2}{x_{i-1}} \tag{4.2.1}$$

as an equation in the field $\mathbb{R}(x_1, \dots, x_{n-2})$. Here, we regard $x_0 = x_n = 0$, $x_{-1} = x_{n-1} = 1$, $y_0 = y_n = 0$, and $y_{-1} = y_{n-1} = 1/x_1$. It is enough to show

$$\frac{y_i}{x_i} = y_1^2 p_{i-1} \tag{4.2.2}$$

in $\mathbb{R}(x_1, \dots, x_{n-2})$.

Consider the case $i = 1$. Then, $p_0 = 1$. (4.2.1) can be written as $y_1/x_1 = 1/x_1^2$. Multiply $x_1^2 y_1$, then we have (4.2.2).

Consider the case $i = 2$. By (4.2.1) and $x_1 y_1 = 1$, $y_1 = P_1(y_1) = p_1$, we have

$$\frac{y_2}{x_2} = \frac{y_1^2}{x_1} = y_1^3 = y_1^2 p_1.$$

Thus we obtain (4.2.2).

Consider the case $i \ge 3$. We shall prove (4.2.2) by induction on i. By induction assumption, $y_j/x_j = y_1^2 p_{j-1}$ for $1 \le j < i$. By Lemma 4.1 (1), $p_{i-1} = y_{i-1}p_{i-2} + y_{i-2}p_{i-3}$. Thus

$$\frac{y_i}{x_i} = \frac{y_{i-2}^2}{x_{i-2}} + \frac{y_{i-1}^2}{x_{i-1}} = y_1^2(y_{i-2}p_{i-3} + y_{i-1}p_{i-2}) = y_1^2 p_{i-1}.$$

(2) Apply Lemma 4.1 (5) with $k = n - 2$, $j = i - 1$, then we obtain $x_1 = p_{i-1}x_i + y_{i-1}p_{i-2}x_{i+1}$. Since $x_1 = 1/y_1$, after multiplying y_1^2 to the both hand sides, we obtain $y_1 = y_1^2 p_{i-1} x_i + y_1^2 y_{i-1} p_{i-2} x_{i+1}$. By (1),

$$y_1 - y_i = y_1 - y_1^2 p_{i-1} x_i = y_1^2 y_{i-1} p_{i-2} x_{i+1}.$$

Thus we obtain (2). ∎

Theorem 4.3.([2] Proposition 3.3) If $\min_{\mathbf{x} \in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^{\bullet}$, then $U(\mathbf{a}) \ge 1/2$.

Proof. We may assume $\mathbf{a} = (x_1, \ldots, x_n) \in K_n^{\triangle}$. By Lemma 4.2 (1), (2), we have $0 \le x_i/(x_{i+1} + x_{i+2}) = y_i \le y_1 = 1/x_1 = U(\mathbf{a})$ $(i = 1, \ldots, n)$. Assume that $U(\mathbf{a}) < 1/2$. Then $x_1 > 2$, and $2x_i \le x_{i+1} + x_{i+2}$. Take $\sum$, we obtain

$$2\sum_{i=1}^{n} x_i < \sum_{i=1}^{n}(x_{i+1} + x_{i+2}) = 2\sum_{i=1}^{n} x_i.$$

A contradiction. □

§5. Short segments.

The following Theorem is an extenstion of [2] Lemma 4.1, [5] §4, §5 and [6] §5.

Theorem 5.1. Assume that $\min_{\mathbf{x} \in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^{\bullet}$. Then $\mathbf{a}$ does not contain segments of length 2, 3, 4, 5, 7, or 9.

Proof. Let $\mathbf{s} = (a_1, \ldots, a_l)$ be a l-segment of $\mathbf{a}$ $(l \ge 2)$. Put $U := U(\mathbf{a})$, $V := \dfrac{a_{l-1} + a_l}{a_l} > 1$. Note that $a_{l+1} = 0$, $a_{l+2} = a_l/U$ by Theorem 2.4 (1). By Theorem 4.3, $U \ge 1/2$.

Since $a_{l+2} + a_{l+3} \ge a_{l+2} = a_l/U$, we have

$$0 \le \partial_{l+1} E_n(\mathbf{a}) = \frac{1}{a_{l+2} + a_{l+3}} - \frac{a_{l-1}}{a_l^2} - \frac{a_l}{a_{l+2}^2} \le \frac{1}{a_l}\big(U - (V-1) - U^2\big).$$

Thus, we have $V \le 1 + U - U^2$. Since $1 < V \le 1 + U - U^2$, we have $U < 1$ and $1 < V \le \dfrac{5}{4} - \left(U - \dfrac{1}{2}\right)^2 \le \dfrac{5}{4}$. Thus (U, V) is included in the set

$$D := \left\{(u, v) \in \mathbb{R}^2 \;\middle|\; 1/2 \le u < 1,\ 1 < v \le 1 + u - u^2\right\}.$$

By (3.6.1), $\dfrac{a_1 a_l}{U^2} = \dfrac{1}{\mu} = \dfrac{a_2 a_l^2}{a_{l-1}}$. Thus we have

$$a_2 = \frac{a_1 a_{l-1}}{a_l U^2} = \frac{V-1}{U^2} a_1.$$

Since $\partial_{i-2} E_n(\mathbf{a}) = 0$ $(i = 3, 4, \ldots, l+2)$, we have

$$a_i = \frac{1}{\dfrac{a_{i-4}}{(a_{i-3} + a_{i-2})^2} + \dfrac{a_{i-3}}{(a_{i-2} + a_{i-1})^2}} - a_{i-1}.$$

Here $a_{-1} = a_{n-1} = U a_1$ and $a_0 = a_n = 0$. Inductively, we obtain

$$a_3 = \frac{1}{a_{n-1}/a_1^2} - a_2 = \frac{U - V + 1}{U^2} a_1 \qquad (\text{if } l \ge 3)$$

$$a_4 = \frac{V - U}{U^2} a_1 \qquad (\text{if } l \ge 4)$$

$$a_5 = \frac{1 + UV - V^2}{U^2 V} a_1 \qquad (\text{if } l \ge 5).$$

Thus, we define a series of rational functions by

$$f_1(u,v) := 1, \quad f_2(u,v) := \frac{v-1}{u^2}, \quad f_3(u,v) := \frac{u-v+1}{u^2}, \quad f_4(u,v) := \frac{v-u}{u^2}$$

$$f_i(u,v) := \frac{1}{\dfrac{f_{i-4}(u,v)}{(f_{i-3}(u,v)+f_{i-2}(u,v))^2} + \dfrac{f_{i-3}(u,v)}{(f_{i-2}(u,v)+f_{i-1}(u,v))^2}} - f_{i-1}(u,v)$$

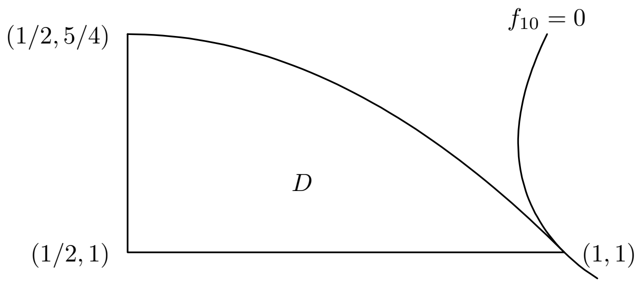
$(i \geq 5)$. Then, $a_i = f_i(U, V)a_1$ for $1 \leq i \leq l+2$. Especially, $f_{l+1}(U, V) = a_{l+1}/a_1 = 0$.

Since $u - v + 1 > 0$, $v - u > 0$, $1 + uv - v^2 > 0$ on D, we obtain $f_i(u, v) > 0$ on D for $i = 3, 4, 5$. Thus $a_{l+1} \neq 0$ for $l = 2, 3, 4$. Therefore, $\mathbf{a}$ does not contain segments of length 2, 3, or 4.

Similarly, $f_i(u, v) > 0$ on D for $i = 6, 8, 10$. We need numerical analysis to prove this. If you have 'Mathematica', execute the following.

```
<< Graphics`ImplicitPlot`;
fi[i_, u_, v_] := (a = 1; b = (v-1)/u^2;
   c = (1+u-v)/u^2; d = (v-u)/u^2;
   Do[(e=1/(a/(b+c)^2 + b/(c+d)^2) - d; a=b; b=c; c=d; d = e),
   {k, 5, i, 1}]; e)
G1[i_]:=(Plot3D[fi[i, u, v], {u, 1/2, 1}, {v, 1, 1 + u - u^2}])
G2[i_]:=(ImplicitPlot[(u^2 - u + v - 1) fi[i, u, v] == 0,
   {u, 1/2, 1}, {v, 1, 5/4}])
```

For example, you can observe the graph of $f_{10}(u, v)$ by `G1[10]`. You can also draw the graph of $f_{10}(u, v) = 0$ by `G2[10]`.



$f_{10}(u, 1 + u - u^2)$ have a zero of the order 2 at $u = 1$. Thus, as the above figure, the graph of $f_{10}(u, v) = 0$ tangents to the parabola $v = 1+u-u^2$ at $(1, 1)$, but have no common point with D. Thus we know that $f_{10}(u, v) > 0$ on D.

We know also $f_8(u, v) > 0$ on D similarly.

It is possible to prove $f_6(u, v) > 0$ on D directly. $f_6(u, v)$ can be written as $f_6(u,v) = \dfrac{f_{6,1}(u,v)f_{6,2}(u,v)}{u^2 v f_{6,3}(u,v)}$, here

$$f_{6,1}(u,v) := 1 - v + v^3 - uv^2$$
$$f_{6,2}(u,v) := (1 + v - v^2) + uv$$
$$f_{6,3}(u,v) := -1 + v + v^3 - v^3 + uv^2.$$

It is easy too see that $f_{6,1}(u, v) > 0$, $f_{6,2}(u, v) > 0$, $f_{6,3}(u, v) > 0$ on D. Thus $f_6(u, v) > 0$ on D. Since $f_6(u, v) > 0$, $f_8(u, v) > 0$ and $f_{10}(u, v) > 0$ on D, we conclude that $\mathbf{a}$ does not contain segments of length 5, 7, or 9. □

Corollary 5.2. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^\bullet$.

(1) If $n = 12$, then the index of $\mathbf{a}$ must be (11).

(2) If $n = 23$, then the index of $\mathbf{a}$ must be one of the following 17 indexes: (22), (20, 1), (18, 1, 1), (16, 1, 1, 1), (15, 6), (14, 1, 1, 1, 1), (13, 8), (13, 6, 1), (12, 1, 1, 1, 1, 1), (11, 10), (11, 8, 1), (11, 6, 1, 1), (10, 1, 1, 1, 1, 1, 1), (8, 6, 6), (8, 1, 1 ,1, 1, 1, 1, 1), (6, 6, 6, 1), (6, 1, 1, 1, 1, 1, 1, 1, 1).

Definition 5.3. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^\bullet$, and that $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ is a l-segment of $\mathbf{a}$ with $l \geq 2$. Then, we define

$$V_l(\mathbf{a}) := 1 + \frac{s_{l-1}}{s_l},$$
$$R_l(\mathbf{a}) := \frac{s_1}{s_l} = \frac{Head(\mathbf{s})}{Tail(\mathbf{s})}.$$

If there are no segment of length l in $\mathbf{a}$, we define $R_l(\mathbf{a}) := 1$. Moreover we define $R_1(\mathbf{a}) := 1$. By Corollary 3.5, $V_l(\mathbf{a})$ and $R_l(\mathbf{a})$ do not depend the choice of $\mathbf{s}$.

Theorem 5.4. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^\bullet$.

(1) If $\mathbf{a}$ contains segment of length 6, then the following holds.
$$1/2 \leq U(\mathbf{a}) < 0.63894, \quad R_6(\mathbf{a}) < 1/2$$

(2) If $\mathbf{a}$ contains a segment of length 8, then the following holds.
$$1/2 \leq U(\mathbf{a}) < 0.73254, \quad R_8(\mathbf{a}) < 0.65994$$

(3) If $\mathbf{a}$ contains a segment of length 10, then the following holds.
$$0.63893 < U(\mathbf{a}) < 0.78332, \quad R_{10}(\mathbf{a}) < 0.90213$$

(4) If $\mathbf{a}$ contains a segment of length 11, then the following holds.
$$0.94197 < U(\mathbf{a}) < 1$$

(5) If $\mathbf{a}$ contains a segment of length 12, then the following holds.
$$0.73253 < U(\mathbf{a}) < 0.81295, \quad R_{12}(\mathbf{a}) < 1.20768$$

(6) If $\mathbf{a}$ contains a segment of length 13, then the following holds.
$$0.90868 < U(\mathbf{a}) < 1$$

(7) If $\mathbf{a}$ contains a segment of length 14, then the following holds.
$$0.78331 < U(\mathbf{a}) < 0.83098, \quad R_{14}(\mathbf{a}) < 1.61530$$

(8) If $\mathbf{a}$ contains a segment of length 15, then the following holds.
$$1/2 \leq U(\mathbf{a}) < 0.63894 \quad \text{or} \quad 0.88942 < U(\mathbf{a}) < 0.94198$$

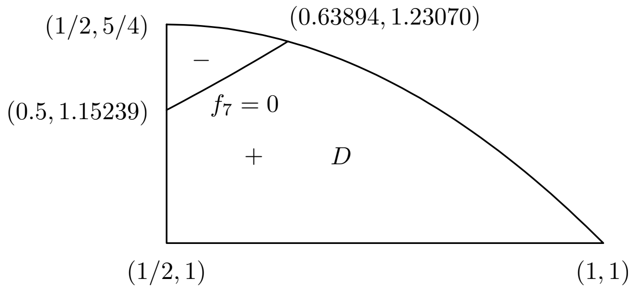
(9) If $\mathbf{a}$ contains a segment of length 16, then the following holds.
$$0.81294 < U(\mathbf{a}) < 0.84220, \quad R_{16}(\mathbf{a}) < 2.20409$$

Proof. We use the same notation with the proof of Theorem 5.1. Moreover put $U := U(\mathbf{a})$, $V := V_l(\mathbf{a})$, and

$$D'_i := \{(u, v) \in D \mid f_i(u, v) > 0\},$$
$$D_i := D'_2 \cap D'_3 \cap D'_4 \cap \cdots \cap D'_i.$$

Note that $D'_2 = D'_3 = D'_4 = D'_5 = D'_6 = D'_8 = D'_{10} = D$.

(1) Consider the case $l = 6$. The graph Γ_7 of $f_7(u,v) = 0$ on D is as following.



This curve Γ_7 is the hyper elliptic curve defined by

$$(2v - 2v^2 - v^3 + v^4) + u(-1 + 2v + v^2 - 2v^3) + u^2v^2 = 0.$$

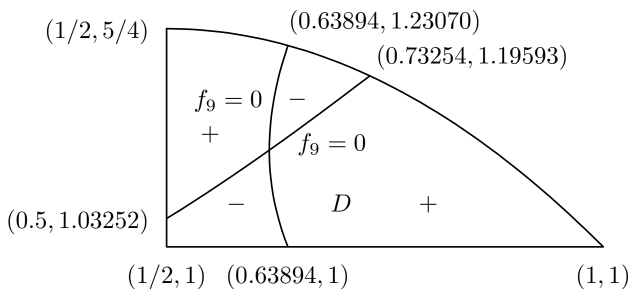
Thus, we put

$$f_{7,1}(v) := \frac{(v^2-1)(2v-1) + \sqrt{(v-1)(v^3+v^2+3v-1)}}{2v^2}.$$

We obtain the intersection of Γ_7 and the parabola $v = 1 + u - u^2$ on D by solving $f_7(u, 1 + u - u^2) = 0$. This root is $u \sim 0.6389355101$ (rounded up). If $\mathbf{a}$ has a 6-segment, then $f_7(U, V) = 0$. Thus $1/2 \leq U < 0.6389355101$. Since $f_6(f_{7,1}(v), v)$ is monotonically increasing on $1.15239 < v < 1.23070$, we have

$$R_6(\mathbf{a}) \leq 1/f_6(f_{7,1}(1.23070), 1.23070) < 0.42657 < 1/2$$

(2) Consider the case $l = 8$. The graph Γ_9 of $f_9(u,v) = 0$ on D is as following.



We can calculate the root of $f_9(u, 1 + u - u^2) = 0$ with $1/2 \leq u < 1$ by

```
FindRoot[fi[9, u, 1+u-u^2] == 0, {u, 0.7}]
```

and we have $u \sim 0.7325361425$ (rounded up). Thus $1/2 \leq U < 0.7325361425$. Execute

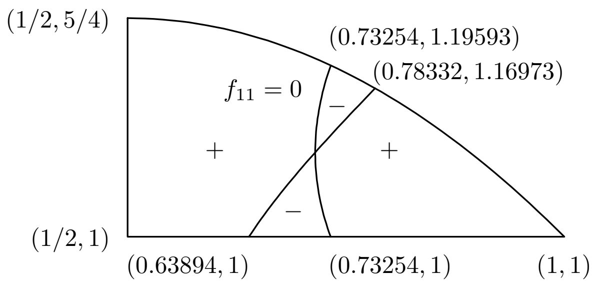
```
Plot3D[1/fi[8, u, v], {u, 1/2, 0.7325361425}, {v, 1, 1 + u - u^2}]
Maximize[{1/fi[8, 0.7325361425, v], 1<v <= 5/4}, v] // N
```

and we conclude that

$$\frac{1}{f_8(u,v)} < \frac{1}{f_8(0.73254, 1.10735)} < 0.65994$$

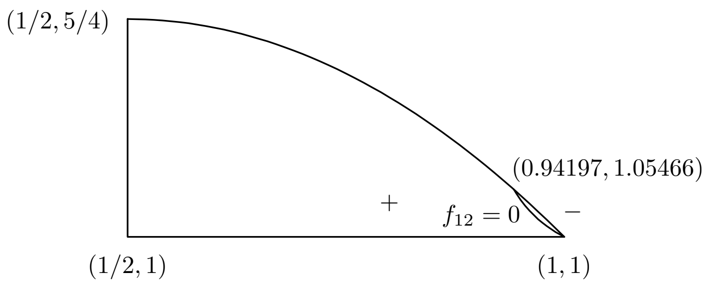
on $\Gamma_9 \cap D$. Thus $R_8(\mathbf{a}) < 0.65994$.

(3) Consider the case $l = 10$. The graph Γ_{11} of $f_{11}(u,v) = 0$ on D is as following.



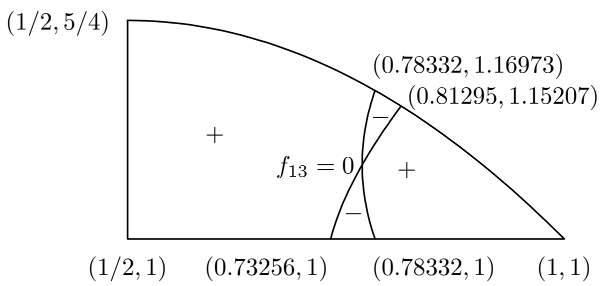
Thus, $0.6389355100 < U < 0.7833151924$. Since $1/f_{10} < 1/f_{10}(0.78332,\ 1.09863) < 0.90213$ on $\Gamma_{11} \cap D$, we have $R_{10}(\mathbf{a}) < 0.90213$.

(4) Consider the case $l = 11$. The graph of $f_{12}(u,v) = 0$ on D is a curve connecting (1, 1) and (0.94197, 1.05466) as following.



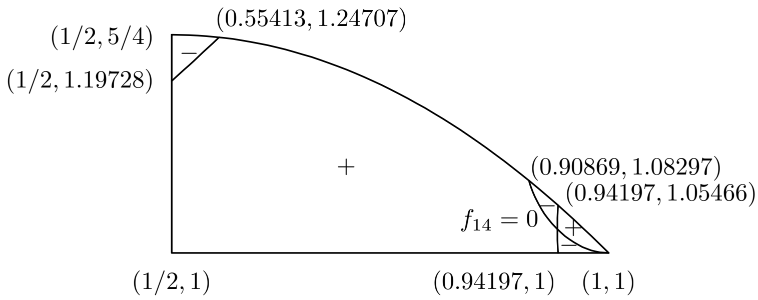
Thus, $0.9419748741 < U < 1$.

(5) Consider the case $l = 12$. The graph Γ_{13} of $f_{13}(u,v) = 0$ on D is as following.



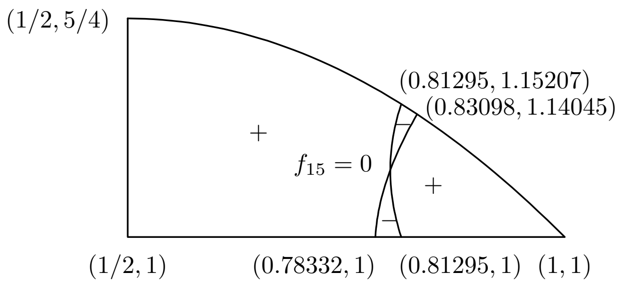
Thus, $0.7325361424 < U < 0.8129451277$. Since $1/f_{13}(u,\ v) < 1/f_{13}(0.81295,\ 1.08843) < 1.20768$ on $\Gamma_{13} \cap D$, we have $R_{12}(\mathbf{a}) < 1.20768$.

(6) Consider the case $l = 13$. The graph of $f_{14}(u,v) = 0$ on D is as following. But the curve connecting (1/2, 1.19728) and (0.55413, 1.24707) is included in $D - D'_6$ on which $a_6 < 0$. Thus, we omit this curve.



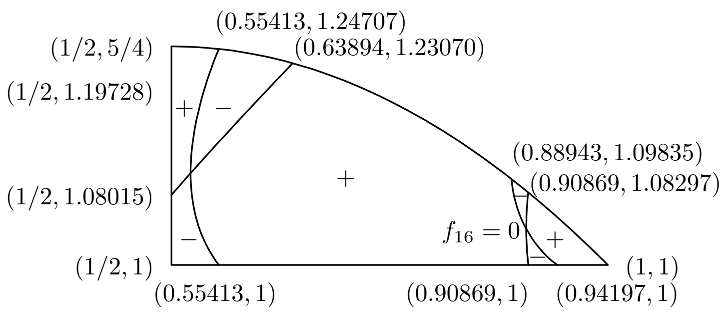
Thus we have $0.9086897811 < U < 1$.

(7) Consider the case $l = 14$. The graph Γ_{15} of $f_{15}(u,v) = 0$ on D is as following.



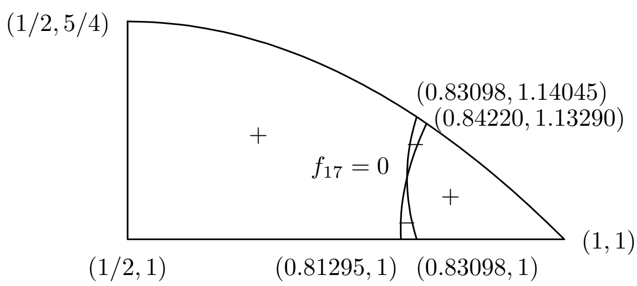
Thus, $0.7833151923 < U < 0.8309779815$. Since $1/f_{14}(u,\ v) < 1/f_{14}(0.83098,\ 1.08039) < 1.61530$, we have $R_{14}(\mathbf{a}) < 1.61530$.

(8) Consider the case $l = 15$. The graph Γ_{16} of $f_{16}(u,v) = 0$ on D is as following.



Thus, $1/2 \leq U < 0.6389355101$ or $0.8894259160 < U < 0.9419748742$.

(9) Consider the case $l = 16$. The graph Γ_{17} of $f_{17}(u,v) = 0$ on D is as following.



Thus, $0.8129451276 < U < 0.8421985095$. Since $1/f_{16}(u,\, v) < 1/f_{16}(0.84220,\, 1.07460) < 2.20409$ on $\Gamma_{17} \cap D$, we have $R_{16}(\mathbf{a}) < 2.20409$. □

§6. Proof of Theorem 1.1.

Theorem 6.1. Assume that $\min_{\mathbf{x} \in K_{23}} E_{23}(\mathbf{x}) = E_{23}(\mathbf{a})$ at $\mathbf{a} \in K_{23}^{\bullet}$. Then the index of $\mathbf{a}$ can not be any of the following values.
(1) (6, 6, 6, 1), (6, 1, 1, 1, 1, 1, 1, 1, 1).
(2) (8, 6, 6), (8, 1, 1 ,1, 1, 1, 1, 1).
(3) (10, 1, 1, 1, 1, 1, 1).
(4) (11, 10), (11, 8, 1), (11, 6, 1, 1).
(5) (13, 8), (13, 6, 1).
(6) (15, 6).
(7) (12, 1, 1, 1, 1, 1).
(8) (14, 1, 1, 1, 1).
(9) (16, 1, 1, 1).

Proof. We use the same notation with the proof of Theorem 5.1. Let $U := U(\mathbf{a})$, $R_l := R_l(\mathbf{a})$, and let m_i be the number of l_i-segments in $\mathbf{a}$ ($i = 1, \ldots, q$), and let $r := m_1 + m_2 + \cdots + m_q$ be the number of segments in $\mathbf{a}$. Then,

$$U^r R_{l_1}^{m_1} \cdots R_{l_q}^{m_q} = 1. \qquad (6.1.1)$$

(1) In these cases, $U < 1$, $R_6 < 1$ by Theorem 5.4 (1). Thus (6.1.1) can not hold.

(2) In these cases, $U < 1$, $R_6 < 1$, $R_8 < 1$ by Theorem 5.4 (1), (2). Thus (6.1.1) can not hold.

(3) In this case, $U < 1$, $R_{10} < 1$ by Theorem 5.4 (3). Thus (6.1.1) can not hold.

(4) In these cases, $0.94197 < U < 1$ by Theorem 5.4 (4). But if $\mathbf{a}$ have a segment of length 10, 8 or 6, then $0.63893 < U < 0.78332$, $1/2 \le U < 0.73254$, $1/2 \le U < 0.63894$ respectively. There exists no such U.

(5) is similar to (4).

(6) Consider the case (15, 6). $1/2 \le U < 0.63894$ and $R_6(\mathbf{a}) < 1/2$ by Theorem 5.4 (1), (8). Execute

```
Plot3D[Ri[15, u, v], {u, 1/2, 0.6389355101}, {v, 1, 1 + u - u^2}]
Maximize[{Ri[15, 0.6389355101, V], 1 <= V <= 5/4}, V] // N
```

Thus we have $1/f_{15}(u,\, v) < 1/f_{15}(0.63894,\, 1.09583) < 0.08952$ on the set $\Gamma_{16} \cap \big\{(u,v) \in D \;\big|\; 1/2 \le u \le 0.63894\big\}$. Thus $R_{15} < 0.08952$ and (6.1.1) can not hold.

(7) In this case, $1 = U^6 R_{12} < 0.81295^6 \times 1.20768 < 1$. A contradiction.

(8) In this case, $1 = U^5 R_{14} < 0.83098^5 \times 1.61530 < 1$. A contradiction.

(9) In this case, $1 = U^4 R_{16} < 0.84220^4 \times 2.20409 < 1$. A contradiction. □

The left cases are (11) when $n = 12$, and (22), (20, 1), (18, 1, 1) when $n = 23$.

Theorem 6.2. (1) Assume that $\min_{\mathbf{x} \in K_{12}} E_{12}(\mathbf{x}) = E_{12}(\mathbf{a})$ at $\mathbf{a} \in K_{12}^{\bullet}$. Then the index of $\mathbf{a}$ can not be (11). Thus, Theorem 1.1 (2) holds.

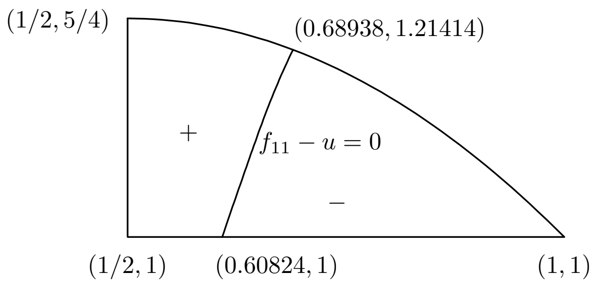
(2) Assume that $\min_{\mathbf{x}\in K_{23}} E_{23}(\mathbf{x}) = E_{23}(\mathbf{a})$ at $\mathbf{a} \in K_{23}^{\bullet}$. Then the index of $\mathbf{a}$ can not be (22).

Proof. We use the same notation with the proof of Theorem 6.1.

(1) We may assume $\mathbf{a} = (1,\, a_2, \ldots,\, a_{11},\, 0)$. Note that $a_{11} = Ua_1 = U$. We draw the graph of $f_{11}(u,\, v) - u = 0$ on D. Execute

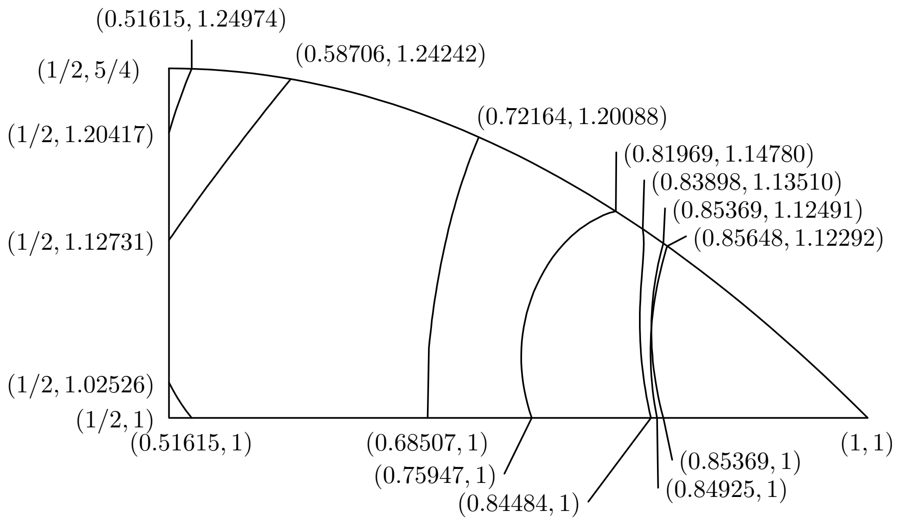
```
Plot3D[Ai[11,u,v]-u, {u, 0.5, 1}, {v, 1, 1.25}]
ImplicitPlot[(u^2-u+v-1) (Ai[11,u,v]-u)==0, {u, 0.5, 1}, {v, 1, 1.25}]
```

We obtain the following.



Thus $0.6082388995 < U < 0.6893774937$. But $0.94197 < U < 1$ by Theorem 5.4 (4). Thus the index (11) can not occur.

(2) We may assume $\mathbf{a} = (1,\, a_2, \ldots,\, a_{21},\, 0)$, here $a_{21} = U$. The graph of $f_{23}(u,\, v) = 0$ and the graph of $f_{22}(u,\, v) - u = 0$ on D are as following.



The graph $\varGamma_{23}$ of $f_{23}(u,\, v) = 0$ consists of five parts. The first is the curve connecting (1/2, 1.20417) and (0.51615, 1.24974), the second is (1/2, 1.12731) — (0.58706, 1.24242), the third is (1/2, 1.02526) — (0.51615, 1), the fourth is (0.84925, 1) — (0.85648, 1.12292), and the fifth is (0.85369, 1) — (0.85369, 1.12491). The graph $\varGamma_{22}'$ of $f_{22}(u,\, v) - u = 0$ consists of three parts. The first is (0.68507, 1) — (0.72164, 1.20088), the second is (0.75947, 1) —

(0.81969, 1.14780), and the third is (0.84484, 1) — (0.83898, 1.13510). As the above figure, $\Gamma_{23} \cap \Gamma'_{22} \cap D = \emptyset$. Thus, (U, V_{23}) can not exists if the index of $\mathbf{a}$ is (23). ∎

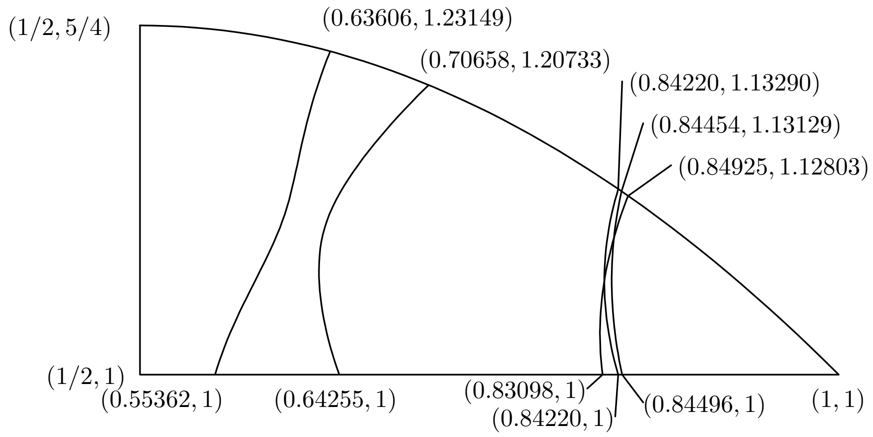
Theorem 6.3. Assume that $\min_{\mathbf{x} \in K_{23}} E_{23}(\mathbf{x}) = E_{23}(\mathbf{a})$ at $\mathbf{a} \in K^{\bullet}_{23}$. Then, the index of $\mathbf{a}$ can not be any of the following values. Thus, Theorem 1.1 (1) holds.

(1) (18, 1, 1).

(2) (20, 1).

Proof. (1) We may assume that $\mathbf{a} = (1, a_2, \ldots, a_{18}, 0, a_{20}, 0, a_{22}, 0)$. Let $U := U(\mathbf{a})$ and $V := V_{18}(\mathbf{a})$. Then, $a_{22} = U$, $a_{20} = U^2$, $a_{18} = U^3$, $f_{19}(U, V) = 0$ and $f_{18}(U, V) = U^3$.

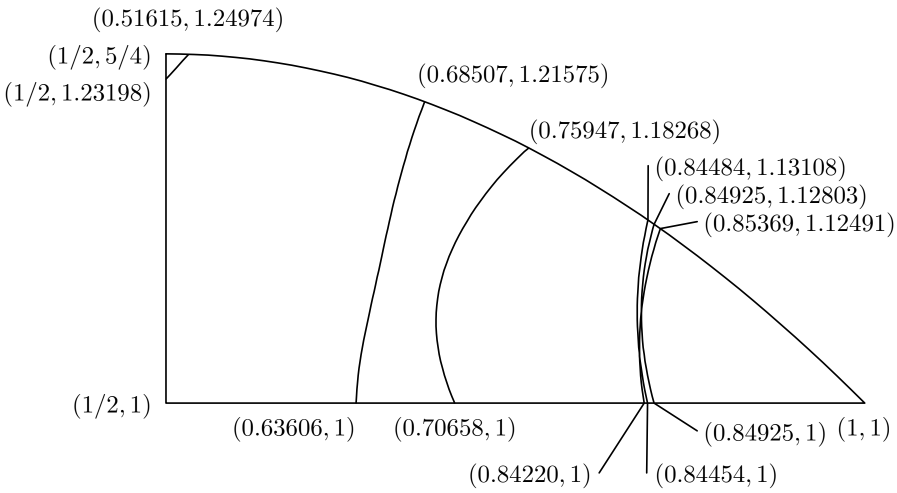
The graph of $f_{19}(u, v) = 0$ and the graph of $f_{18}(u, v) - u^3 = 0$ on D are as following.



The graph Γ_{19} of $f_{19}(u, v) = 0$ consists of two parts. The first is the curve C_1 connecting (0.83098, 1) and (0.84925, 1.12803), and the second is (0.84220, 1) — (0.84220, 1.13290). The graph Γ'_{18} of $f_{18}(u, v) - u^3 = 0$ consists of three parts. The first is (0.55362, 1) — (0.63606, 1.23149), the second is (0.64255, 1) — (0.70658, 1.20733), and the third is the curve C_2 connecting (0.84496, 1) and (0.84454, 1.13129). As the above figure, $\Gamma_{19} \cap \Gamma'_{18} \cap D = C_1 \cap C_2 \sim (0.8391429974, 1.0981287467)$. Thus $U \sim 0.8391429974$ and $V \sim 1.0981287467$. In this case $E_{23}(\mathbf{a}) > 11.511 > 23/2 = E_{23}(1, 1, \ldots, 1)$. So, $E_{23}(\mathbf{a})$ can not be minimum.

(2) We may assume $\mathbf{a} = (1, a_2, \ldots, a_{20}, 0, a_{22}, 0)$. Let $U := U(\mathbf{a})$ and $V := V_{18}(\mathbf{a})$. Then $a_{22} = U$, $a_{20} = U^2$, $f_{21}(U, V) = 0$ and $f_{20}(U, V) = U^3$.

The graph of $f_{21}(u, v) = 0$ and the graph of $f_{20}(u, v) - u^2 = 0$ on D are as following.



The graph Γ_{21} of $f_{21}(u,\, v) = 0$ consists of three parts. The first is (1/2, 1.23198) — (0.51615, 1.24974), the second is the curve C_3 connecting (0.84220, 1) and (0.85369, 1.12491), and the third is (0.84925, 1) — (0.84925, 1.12803). The graph Γ'_{20} of $f_{20}(u,\, v) - u^2 = 0$ consists of three parts. The first is (0.63606, 1) — (0.68507, 1.21575), the second is (0.70658, 1) — (0.75947, 1.18268), and the third is the curve C_4 connecting (0.84454, 1) and (0.84484, 1.13108). As the above figure, $\Gamma_{21} \cap \Gamma'_{20} \cap D = C_3 \cap C_4 \sim (0.8388196493,\, 1.0346467269)$. Thus $U \sim 0.8388196493$, and $V \sim 1.0346467269$. Then $E_{23}(\mathbf{a}) > 11.512 > 23/2 = E_{23}(1, \ldots, 1)$. Thus $E_{23}(\mathbf{a})$ can not be minimum. □

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