Extremal Cubic Inequalities of Three Variables

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Abstract. Let $\boldsymbol{\mathcal{H}}_{3,d}$ be the vector space of homogeneous three variable polynomials of degree d, and $\boldsymbol{\mathcal{P}}^{+}_{3,d}$ be the set of all elements $f \in \boldsymbol{\mathcal{H}}_{3,d}$ such that $f(x,y,z) \geq 0$ for all $x \geq 0$, $y \geq 0$, $z \geq 0$. In this article we determine all extremal elements of $\boldsymbol{\mathcal{P}}^{+}_{3,3}$. We prove that if $f \in \boldsymbol{\mathcal{P}}^{+}_{3,3}$ is an irreducible extremal element, then the zero locus $V_{\mathbb{C}}(f)$ in $\mathbb{P}^2_{\mathbb{C}}$ is a rational curve whose unique singularity is an acnode in the interior of $\mathbb{P}^2_{+}$ or a cusp on an edge of $\mathbb{P}^2_{+}$. We also prove that if $f \in \boldsymbol{\mathcal{P}}^{+}_{3,3}$ is an extremal element, then $f(x^2, y^2, z^2)$ is an extremal element of $\boldsymbol{\mathcal{P}}_{3,6}$, where $\boldsymbol{\mathcal{P}}_{3,d}$ is the set of all the elements $f \in \boldsymbol{\mathcal{H}}_{3,d}$ such that $f(x,y,z) \geq 0$ for all x, y, $z \in \mathbb{R}$.

§1. Introduction.

In this article, we determine all the extremal elements of the convex cone $\boldsymbol{\mathcal{P}}^{+}_{3,3}$, where basic symbols are defined as the following:

$$\mathbb{P}^n_{+} := \left\{(x_0 : \cdots : x_n) \in \mathbb{P}^n_{\mathbb{R}} \;\middle|\; x_0 \geq 0, \ldots, x_n \geq 0\right\},$$

$$\boldsymbol{\mathcal{H}}_{n,d} := \mathbb{R}[x_1, \ldots, x_n]_{(d)} = \left\{f \in \mathbb{R}[x_1, \ldots, x_n] \;\middle|\; f \text{ is homogeneous and } \deg f = d\right\} \cup \{0\},$$

$$\boldsymbol{\mathcal{P}}_{n,d} := \left\{f \in \boldsymbol{\mathcal{H}}_{n,d} \;\middle|\; f(x_1, \ldots, x_n) \geq 0 \text{ for all } (x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}_{\mathbb{R}}\right\},$$

$$\boldsymbol{\mathcal{P}}^{+}_{n,d} := \left\{f \in \boldsymbol{\mathcal{H}}_{n,d} \;\middle|\; f(x_1, \ldots, x_n) \geq 0 \text{ for all } (x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}_{+}\right\},$$

$$\Sigma_{n,d} := \left\{f \in \boldsymbol{\mathcal{P}}_{n,d} \;\middle|\; f \text{ is a sum of squares of some elements from } \boldsymbol{\mathcal{H}}_{n,d/2}\right\}.$$

Let $\boldsymbol{\mathcal{P}}$ be a closed convex cone, and $\boldsymbol{\mathcal{H}}$ be the vector space spanned by $\boldsymbol{\mathcal{P}}$. An element $0 \neq f \in \boldsymbol{\mathcal{P}}$ is said to be *extremal* if g, $h \in \boldsymbol{\mathcal{P}}$ and $f = g + h$, implies that g and h are divisible by f. Let

$$\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}) := \left\{f \in \boldsymbol{\mathcal{P}} \;\middle|\; f \text{ is an extremal element of } \boldsymbol{\mathcal{P}}\right\}.$$

An element $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}})$ is said to be *exposed* if there exists a hyperplane $H \subset \boldsymbol{\mathcal{H}}$ such that $H \cap \boldsymbol{\mathcal{P}} = \mathbb{R}_{+} \cdot f$, where $\mathbb{R}_{+} := \left\{a \in \mathbb{R} \;\middle|\; a \geq 0\right\}$. For $f \in \boldsymbol{\mathcal{H}}_{n,d}$ and $K = \mathbb{R}$ or $\mathbb{C}$, we denote

$$V_K(f) := \left\{(x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}_K \;\middle|\; f(x_1, \ldots, x_n) = 0\right\}.$$

A set $V_{\mathbb{R}}(f) \cap \mathbb{P}^{n-1}_{+}$ is denoted by $V_{+}(f)$. We will see that the extremal ray $\mathbb{R}_{+} \cdot f$ of $\boldsymbol{\mathcal{P}}^{+}_{3,3}$ is determined by $V_{+}(f)$ with additional information of infinitely near points.

Before we state our main result, i.e., Theorem 1.5, we present some elements of $\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^{+}_{3,3})$. For example, $\mathfrak{f}_0(x,\, y,\, z) = x^2y + y^2z + z^2x - 3xyz \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^{+}_{3,3})$ (see Proposition 3.26 or [5,

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Corollary 3.3]). The Schur's inequality type polynomial $\mathfrak{f}_1(x,\, y,\, z) = x^3 + y^3 + z^3 + 3xyz - x^2y - y^2z - z^2x - xy^2 - yz^2 - zx^2$ and xyz are also elements of $\mathcal{E}(\mathcal{P}^+_{3,3})$ (see Corollary 3.27). If $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ is a symmetric polynomial, then $f = \alpha\mathfrak{f}_1$ or $f = \alpha xyz$ ($\exists\alpha > 0$. See also [5, Theorem 3.8]). On the other hand, AM-GM inequality type polynomial

$$x^3 + y^3 + z^3 - 3xyz = \mathfrak{f}_1(x, y, z) + \mathfrak{f}_0(x, y, z) + \mathfrak{f}_0(y, x, z)$$

is not extremal in $\mathcal{P}^+_{3,3}$.

Theorem 1.1, 1.2, 1.4 below describe three new families of elements of $\mathcal{E}(\mathcal{P}^+_{3,3})$.

Theorem 1.1. *Assume that $p \geq 0$, $q \geq 0$, $r \geq 0$, $pq - p + 1 > 0$, $qr - q + 1 > 0$ and $rp - r + 1 > 0$. Then:*

(1) There exists an irreducible polynomial $\mathfrak{f}_{pqr} \in \mathcal{E}(\mathcal{P}^+_{3,3})$ which satisfies

$$\mathfrak{f}_{pqr}(1, 1, 1) = \mathfrak{f}_{pqr}(0, p, 1) = \mathfrak{f}_{pqr}(1, 0, q) = \mathfrak{f}_{pqr}(r, 1, 0) = 0.$$

(2) If $p > 0$, $q > 0$, $r > 0$, then every $f \in \mathcal{P}^+_{3,3}$ satisfying $f(1, 1, 1) = f(0, p, 1) = f(1, 0, q) = f(r, 1, 0) = 0$ is of the form $f = \alpha\mathfrak{f}_{pqr}$ for some $\alpha \in \mathbb{R}$.

For the explicit expression of $\mathfrak{f}_{pqr}$ see Definition 3.16.

Theorem 1.2. *For $p,\, q \in \mathbb{R}$, let*

$$\begin{aligned}\mathfrak{g}_{pq}(x, y, z) := {}& z^3 + q^2x^2z + p^2y^2z - 2qxz^2 - 2pyz^2 - (p^2 + q^2 - 4p - 4q + 3)xyz \\ & + (1 - p + q)(1 - p - q)x^2y + (1 + p - q)(1 - p - q)xy^2.\end{aligned}$$

If $p \geq 0$, $q \geq 0$ and $p + q \leq 1$, then $\mathfrak{g}_{pq} \in \mathcal{E}(\mathcal{P}^+_{3,3})$ and

$$\mathfrak{g}_{pq}(1, 1, 1) = \mathfrak{g}_{pq}(0, 1, p) = \mathfrak{g}_{pq}(1, 0, q) = \mathfrak{g}_{pq}(1, 0, 0) = \mathfrak{g}_{pq}(0, 1, 0) = 0.$$

Conversely, if $p > 0$, $q > 0$, $p + q < 1$ and $f \in \mathcal{P}^+_{3,3}$ satisfies $f(1, 1, 1) = f(1, 0, p) = f(0, 1, q) = f(1, 0, 0) = f(0, 1, 0) = 0$, then there exists $\alpha \in \mathbb{R}$ such that $f = \alpha\mathfrak{g}_{pq}$.

Remark 1.3. $\mathfrak{f}_{ppp}$ was discovered in [1, Theorem 3.1]. A special type of $\mathfrak{g}_{pq}$ was discovered in [15]. Let

$$\begin{aligned}M_t(x, y, z) := {}& (1 - 2t^2)(x^4y^2 + x^2y^4) + t^4(x^4z^2 + y^4z^2) \\ & - (3 - 8t^2 + 2t^4)x^2y^2z^2 - 2t^2(x^2 + y^2)z^4 + z^6\end{aligned}$$

be the polynomial of (1.8) or (6.17) in [15]. Then $M_t(x,\, y,\, z) = \mathfrak{g}_{t^2,t^2}(x^2,\, y^2,\, z^2)$.

Note that $V_{\mathbb{C}}(\mathfrak{f}_{pqr})$ and $V_{\mathbb{C}}(\mathfrak{g}_{pq})$ have a node at $(1{:}\,1{:}\,1) \in \mathbb{P}^2_{\mathbb{C}}$. There are extremal elements $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ such that $V_{\mathbb{C}}(f)$ has a cusp.

Theorem 1.4. *Let*

$$\begin{aligned}\mathfrak{h}_{pq}(x, y, z) := {}& 2x^3 + 3(p - q)x^2y - 6pqxy^2 + q^2(3p + q)y^3 + 3(q - p)x^2z - 6pqxz^2 \\ & + (p^3 + 36p^2q - 6pq^2 - 2q^3)y^2z + (-2p^3 - 6p^2q + 3pq^2 + q^3)yz^2 \\ & + p^2(p + 3q)z^3 + 12pqxyz.\end{aligned}$$

Assume that $p \geq 0$, $q \geq 0$ and $(p,\, q) \neq (0,\, 0)$. Then $\mathfrak{h}_{pq} \in \mathcal{E}(\mathcal{P}^+_{3,3})$ and $\mathfrak{h}_{pq}(p,\, 0,\, 1) = \mathfrak{h}_{pq}(q,\, 1,\, 0) = 0$. Moreover, $V_{\mathbb{C}}(f)$ has a cusp at $(0{:}\,1{:}\,1)$. Conversely, for $f \in \mathcal{P}^+_{3,3}$, if $V_{\mathbb{C}}(f)$ has a cusp at $(0{:}\,1{:}\,1)$ and $f(p, 0, 1) = f(q, 1, 0) = 0$, then there exists $\alpha \in \mathbb{R}$ such that $f = \alpha\mathfrak{h}_{pq}$.

Our main theorem is the characterization of the elements from $\mathcal{E}(\mathcal{P}^+_{3,3})$:

Theorem 1.5. *Let $f(x, y, z) \in \mathcal{E}(\mathcal{P}^+_{3,3})$. Then, $f(x, y, z)$ is a positive multiple of one of the following polynomials:*

(1) $\mathfrak{f}_{pqr}(x,\ y/a,\ z/b)$ *where* $a>0$, $b>0$, $p\geq 0$, $q\geq 0$, $r\geq 0$, $pq-p+1>0$, $qr-q+1>0$ *and* $rp-r+1>0$.
(2) $\mathfrak{f}_{prq}(x,\ z/b,\ y/a)$ *where* $a>0$, $b>0$, $p\geq 0$, $q\geq 0$, $r\geq 0$, $pq-q+1>0$, $qr-r+1>0$, $rp-p+1>0$ *and* $pqr=0$.
(3) $\mathfrak{g}_{pq}(x,\ y/a,\ z/b)$ *or* $\mathfrak{g}_{pq}(y/a,\ z/b,\ x)$ *or* $\mathfrak{g}_{pq}(z/b,\ x,\ y/a)$ *where* $a>0$, $b>0$, $p\geq 0$, $q\geq 0$ *and* $p+q<1$.
(4) $\mathfrak{h}_{pq}(x,\ y/a,\ z)$ *or* $\mathfrak{h}_{pq}(y,\ z/a,\ x)$ *or* $\mathfrak{h}_{pq}(z,\ x/a,\ y)$ *where* $a>0$, $p\geq 0$, $q\geq 0$ *and* $(p,\ q)\neq(0,\ 0)$.
(5) $x(ax+by+cz)^2$ *or* $y(ax+by+cz)^2$ *or* $z(ax+by+cz)^2$ *where* $a,\ b,\ c\in\mathbb{R}$, $(a,\ b,\ c)\neq(0,\ 0,\ 0)$ *and* $\dim\big(V_{\mathbb{R}}(ax+by+cz)\cap\mathbb{P}^2_+\big)=1$.
(6) *The monomial* xyz.
Conversely, all polynomials in (1)-(6) belongs to $\mathcal{E}(\mathcal{P}^+_{3,3})$.

This theorem will be proved in §3.6.

Hilbert proved that $\mathcal{P}_{n,d}=\Sigma_{n,d}$ if and only if $n\leq 2$ or $d=2$ or $(n,\ d)=(3,\ 4)$. Moreover, every element of $\mathcal{E}(\mathcal{P}_{3,4})$ is a square of a quadratic polynomial [11]. We shall give an alternative proof of this fact at Theorem 4.1.

The first part of the following theorem is proved in [3, Remark 8], and the second part follows from [15, Theorem 7.2],

Theorem 1.6. *If* $f\in\mathcal{P}_{3,6}$ *is an exposed extremal element which is not the square of a cubic polynomial, then* $V_{\mathbb{C}}(f)$ *is an irreducible rational curve which has 10 acnodes* $P_1,\ldots,\ P_{10}$*, and* $V_{\mathbb{R}}(f)=\{P_1,\ldots,\ P_{10}\}$*. On the other hand, if* $f\in\mathcal{P}_{3,6}$ *and* $V_{\mathbb{C}}(f)$ *is an irreducible curve which has 10 acnodes in* $\mathbb{P}^2_{\mathbb{R}}$*, then* $f\in\mathcal{E}(\mathcal{P}_{3,6})$.

In spite of this general theorem, only a few concrete elements of $\mathcal{E}(\mathcal{P}_{3,6})$ ware known (see [14]). But as a corollary of Theorem 1.5, we obtain the following:

Theorem 1.7. *If* $f(x,y,z)\in\mathcal{E}(\mathcal{P}^+_{3,3})$*, then* $f(x^2,y^2,z^2)\in\mathcal{E}(\mathcal{P}_{3,6})$.

The convex cone $\mathcal{P}_{3,6}$ is studied in [5, 6, 7, 8, 14, 15], and the convex cone $\mathcal{P}^e_{3,6}:=\mathcal{P}_{3,6}\cap\mathbb{R}[x^2,y^2,z^2]$ is studied in [9]. Since $\mathcal{P}^+_{3,3}\cong\mathcal{P}^e_{3,6}$ by the correspondence $f(x,y,z)\to f(x^2,y^2,z^2)$, our results characterize $\mathcal{E}(\mathcal{P}^e_{3,6})$, and prove the following (see Corollary 3.25):

Corollary 1.8. $\mathcal{E}(\mathcal{P}^e_{3,6})\subset\mathcal{E}(\mathcal{P}_{3,6})$.

Here, we sketch our idea of proof of Theorem 1.5. To classify $f\in\mathcal{E}(\mathcal{P}^+_{3,3})$, we observe the complex cubic curve $C_{\mathbb{C}}:=V_{\mathbb{C}}(f)\subset\mathbb{P}^2_{\mathbb{C}}$ and the real cubic curve $C_{\mathbb{R}}:=V_{\mathbb{R}}(f)\subset\mathbb{P}^2_{\mathbb{R}}$.

If $C_{\mathbb{C}}$ is reducible, classification is easy (Proposition 3.1—3.4).

Consider the case $C_{\mathbb{C}}$ is irreducible. Then $C_{\mathbb{C}}$ is a rational curve with a singular point P_0 (Lemma 3.5). If P_0 is a node, it lies inside of $\mathbb{P}^2_+$ (Lemma 3.6—3.11). After a suitable projective transformation, we may assume $P_0=(1\colon 1\colon 1)$. If P_0 is a cusp, it lies on an edge of a triangle $\partial\mathbb{P}^2_+$ and it is not a vertex (Lemma 3.10, 3.6). In this case, we may assume $P_0=(0\colon 1\colon 1)$. In the both cases, $C_{\mathbb{R}}-\{P_0\}$ contacts to $\partial\mathbb{P}^2_+$ at some points $P_1,\ldots,\ P_r$.

We shall study a general theory of infinitely near zeros in §2. As a monic polynomial in one variable is determined by its roots when we count the multiplicity exactly, an extremal ray $\mathbb{R}_+\cdot f$ is essentially determined by $V_+(f)$ when we consider infinitely near zeros. By Theorem 2.11, $f\in\mathcal{E}(\mathcal{P}^+_{3,3})$ satisfies

$$\mathbb{R}_+\cdot f=\mathcal{P}_0\cap\mathcal{P}_1\cap\cdots\cap\mathcal{P}_r$$

where $\mathcal{P}_i$ is a local cone or an infinitesimal local cone of $\mathcal{P}_{3,3}^{+}$ at P_i (see Definition 2.9). $\mathcal{P}_i$ has informationis on infinitely near points of P_i. This condition also gives a lower bound of r. On the other hand, at most two P_i can exist on an edge of the triangle $\partial\mathbb{P}^2_+$. If two points P_i, P_j exist on an edge, one of them must be a vertex. Thus $r \leq 4$. Our classification essentially depends on this result.

In §3.3, we study the case that $P_0 = (0\colon 1\colon 1)$ is a cusp. In this case, we can easily prove $f = c\mathfrak{h}_{pq}$ using Theorem 1.5(see Theorem 3.13).

In §3.4, we study the case that $P_0 = (1\colon 1\colon 1)$ is a node. In this case, there are several types of configurations of $P_1,\ldots, P_r$. After studying each case using Theorem 1.1 and 1.2, we conclude that f can be represented by $\mathfrak{f}_{pqr}$ or $\mathfrak{g}_{pq}$.

The idea of infinitely near zeros is also useful to reform Theorem 1.6. If $f \in \mathcal{E}(\mathcal{P}_{3,6}) - \Sigma_{3,6}$ is not always exposed, $V_{\mathbb{R}}(f)$ consists of just 10 points including infinitely near points. Theorem 2.11 states that extremal elements of PSD cones $\mathcal{P}_{n,2d}$ or $\mathcal{P}_{n,d}^{+}$ are determined by their equality conditions including infinitely near points. In [15], properties of $f \in \mathcal{E}(\mathcal{P}_{3,6}) - \Sigma_{3,6}$ with $\#V_{\mathbb{R}}(f) \leq 9$ are studied using the idea of 'divisor' instead of 'infinitely near points'. When we treat $V_{\mathbb{R}}(f)$ as a divisor, a notion of multiplicity is included in it, and works well for $(n,\, 2d) = (3,\, 6)$. But it will not determine $f \in \mathcal{E}(\mathcal{P}_{n,2d})$ for large d. Notion of 'infinitely near points' gives more complete condition.

We also mention that the $\mathbb{R}$-scheme structure of $\mathbb{P}^2_{\mathbb{R}}$ is not unique. We can blow up $\mathbb{P}^2_{\mathbb{R}}$ at conjugate imaginal points. So, we must be careful to treat 'divisors' on $\mathbb{P}^2_{\mathbb{R}}$. There also infinitely many real projective surfaces X such that $\mathbb{P}^2_+ \subset X$. This is inconvenient to treat blowing ups. We often need to get rid of information outside of $\mathbb{P}^2_+$, and there shoud be a unique structure sheaf $\mathcal{R}_{\mathbb{P}^2_+}$. So, in this article, we use the notion of semialgebraic variety.

§2. Infinitely near zeros.

One of the key idea to prove Theorem 1.5 is to introduce the notion of infinitely near zeros. Max Noether introduced the notion of infinitely near points on algebraic surfaces. The notion of infinitely near zeros of inequalities is similar one. We need blowing ups to treat infinitely near zeros. So, we should generalize some notions. The notion of semialgebraic varieties is introduced in [1], and the precise properties of semialgebraic variety are explained in [2, §5]. But we don't need deep understanding for semialgebraic varieties in this article. We present here minimum definitions.

Definition 2.1.(Semialgebraic variety) A locally ringed space $(A,\, \mathcal{R}_A)$ is called *semialgebraic variety*, if there exists a real algebraic variety $(X,\, \mathcal{R}_X)$ in the sence of [4] which satisfies the following:

(1) The ring $\mathcal{R}_X(X)$ is an integral domain such that Krull dim $\mathcal{R}_X(X) = \dim X$. We denote $\mathrm{Rat}(X) := Q(\mathcal{R}_X(X))$ (the field of fractions).

(2) There exists an injective morphism $\iota\colon (A,\, \mathcal{R}_A) \longrightarrow (X,\, \mathcal{R}_X)$ as locally ringed spaces, and ι induces a homeomorphism $A \to \iota(A)$ with respect to the Euclidean topology. Moreover, $\iota(A)$ is a semialgebraic subset of X such that $\mathrm{Zar}_X(\iota(A)) = X$, i.e. the Zariski closure of $\iota(A)$ in X agrees with X.

(3) The induced map $\iota_P^*\colon \mathcal{R}_{X,\iota(P)} \longrightarrow \mathcal{R}_{A,P}$ is an isomorphism for every $P \in A$, and

$$\mathcal{R}_A(U) = \bigcap_{P\in U} \iota_P^*\big(\mathcal{R}_{X,\iota(P)}\big) \subset \iota^*\,\mathrm{Rat}(X)$$

for any non-empty Euclidean open subset $U \subset A$.

The above definition is based on the fact that $\mathfrak{R}_X$ is genereted by its global section (see [2, Corollary 5.13]). On a senialgebraic variety A, we use the Euclidean topology and the Zariski topology induced from X. Many terminologies for semialgebraic varieties can be defined by the similar way as complex algebraic varieties.

Definition 2.2.(Signed linear system) Let $(A, \mathfrak{R}_A)$ be a semialgebraic variety, and $\mathfrak{C}^0_A$ be the sheaf of germs of real contineous functions on A.

(1) Let $\mathfrak{I}$ be an invertible $\mathfrak{R}_A$-sheaf. $\mathfrak{I}$ is called a *signed invertible sheaf* on A if
  (i) there exists $\mathfrak{C}^0_A$-invertible sheaf $\mathfrak{J}$ such that $\mathfrak{I} \otimes_{\mathfrak{R}_A} \mathfrak{C}^0_A = \mathfrak{J} \otimes_{\mathfrak{C}^0_A} \mathfrak{J}$, and
  (ii) there exists $e \in \mathfrak{J}(A)$ such that $e^2 \in \mathfrak{I}(A)$ and $\mathfrak{I}(A) = \mathfrak{R}_A(A) \cdot e^2$.
  Then, for $f \in H^0(A, \mathfrak{I})$, there exists $g \in H^0(A, \mathfrak{R}_A)$ such that $f = ge^2$. We define $\mathrm{sign}(f(P)) \in \{0, \pm 1\}$ by $\mathrm{sign}(f(P)) = \mathrm{sign}(g(P))$ for $P \in A$.

(2) Let $\mathfrak{I}$ be a signed invertible $\mathfrak{R}_A$-sheaf. A finite dimensional vector subspace $\mathfrak{H} \subset H^0(A, \mathfrak{I})$ is called a *signed linear system* on A. For $f \in \mathfrak{H}$, we say f is *PSD* on A if $f(P) \geq 0$ for all $P \in A$.

Example 2.3. Let $A = \mathbb{P}^{n-1}_+ \subset \mathbb{P}^{n-1}_{\mathbb{R}} = X$ and $e := \sqrt{x_1}^d$. Then $\mathfrak{R}_A(d) = \mathfrak{R}_A \cdot e^2$. Thus $\mathfrak{R}_A(d)$ is a signed invertible $\mathfrak{R}_A$-sheaf, and $\mathfrak{H}_{n,d} \subset H^0(A, \mathfrak{R}_A(d))$ is a signed lenear system on $\mathbb{P}^n_+$,

Similarly, $\mathfrak{H}_{n,2d}$ is a signed linear systen on $\mathbb{P}^{n-1}_{\mathbb{R}}$.

Proposition 2.4. *Let $(A, \mathfrak{R}_A)$ and $(B, \mathfrak{R}_B)$ be semialgebraic varieties and $\varphi\colon B \to A$ be a morphism. If $\mathfrak{H}$ is is a signed linear system on A, then $\varphi^*\mathfrak{H} := \big\{\varphi^*(f) := f \circ \varphi \;\big|\; f \in \mathfrak{H}\big\}$ is a signed linear system on B.*

Proof. Let $\mathfrak{I}$, $\mathfrak{J}$ and e be same as in Definition 2.2. Then $\mathfrak{I}_B := \varphi^*\mathfrak{I} \otimes_{\varphi^*\mathfrak{R}_A} \mathfrak{R}_B$, $\mathfrak{J}_B := \varphi^*\mathfrak{J} \otimes_{\varphi^*\mathfrak{C}^0_A} \mathfrak{C}^0_B$ and $e_B := \varphi^*(e)$ satisfy conditions so that $\varphi^*\mathfrak{H}$ is a signed linear system on B. □

Definition 2.5.(PSD cone) Let $(A, \mathfrak{R}_A)$ be a semialgebraic variety, and $\mathfrak{H}$ be a signed linear sytem on A. The cone

$$\mathfrak{P} = \mathfrak{P}(A, \mathfrak{H}) := \big\{f \in \mathfrak{H} \;\big|\; f \text{ is PSD on } A\big\}$$

is called the *PSD cone* on A in $\mathfrak{H}$.

We can represent $\mathfrak{P}^+_{n,d} = \mathfrak{P}(\mathbb{P}^{n-1}_+, \mathfrak{H}_{n,d})$, and $\mathfrak{P}_{n,2d} = \mathfrak{P}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathfrak{H}_{n,2d})$.

Definition 2.6. Let A be a non-singular semialgebraic variety, $\mathfrak{H}$ be a signed linear system on A, and $\mathfrak{P} = \mathfrak{P}(A, \mathfrak{H})$ be a PSD cone.

(1) Take $P \in A$ and $f \in \mathfrak{H}$. Assume that f can be represented as $f = ge^2$ as Definition 2.2. If we take a suitable open subset $U \subset A$ and an analytic coordinate system $(x_1,\ldots, x_n)$ on U whose origin is P, we can regard $g|_U \in \hat{\mathfrak{R}}_{A,P} := \mathbb{R}[[x_1,\ldots, x_n]]$. Let $\mathfrak{m}$ be the maximal ideal of $\hat{\mathfrak{R}}_{A,P}$ corresponding to the point P. The *multiplicity* of f at P is defined as
$$\mathrm{mult}_P\, f := \sup\big\{d \geq 0 \;\big|\; g|_U \in \mathfrak{m}^d\big\}.$$

(2) For $x \in A$, we put
$$\mathrm{mult}_x\, \mathfrak{P} := \min_{h \in \mathfrak{P}-\{0\}} \mathrm{mult}_x\, h.$$

Assume that $\dim \mathcal{P} \geq 2$, where $\dim \mathcal{P}$ implies the dimension of $\mathcal{P}$ as a semialgebraic variety which agree with the dimension as a convex cone or as a manifold. For $0 \neq f \in \mathcal{P}$, we put

$$\mathcal{Z}_f(\mathcal{P}) := \{P \in A \mid \operatorname{mult}_P f > \operatorname{mult}_P \mathcal{P}\}.$$

(3) Take a point $P \in A$ and put

$$\mathcal{P}_P := \{g \in \mathcal{P} \mid \operatorname{mult}_P g > \operatorname{mult}_P \mathcal{P}\}.$$

If $\mathcal{P}_P \neq \{0\}$, then $\mathcal{P}_P$ is called the *local cone* of $\mathcal{P}$ at P.

When we consider a vector subspace $\mathcal{H}_P := \{g \in \mathcal{H} \mid \operatorname{mult}_P g > \operatorname{mult}_P \mathcal{P}\}$, then $\mathcal{P}_P = \mathcal{P}(A, \mathcal{H}_P)$. Thus $\mathcal{P}_P$ is a semialgebraic closed convex cone.

Proposition 2.7. *Let A, $\mathcal{P}$ be same as in Definition 2.6 where $\dim \mathcal{P} \geq 2$.*
(1) *There exists $h_0 \in \mathcal{P}$ such that $\operatorname{mult}_x h_0 = \operatorname{mult}_x \mathcal{P}$ for all $x \in A$.*
(2) *Assume that $0 \neq f \in \mathcal{P}_a$ and $f = g + h$, where $g, h \in \mathcal{P}$ and $a \in A$. Then $g, h \in \mathcal{P}_a$.*

Proof. (1) For $a \in A$, there exists $h_a \in \mathcal{P}$ such that $\operatorname{mult}_a h_a = \operatorname{mult}_a \mathcal{P}$. Note that $\operatorname{mult}_x g$ and $\operatorname{mult}_x \mathcal{P}$ are upper semicontinuous functions on $x \in A$ with respect to Zariski topology (see [10, II Exercise 5.8]). For any point $a \in A$, there exists an open neighborhood $a \in U \subset A$ and $h \in \mathcal{P}$ such that $\operatorname{mult}_x h = \operatorname{mult}_x \mathcal{P}$ for all $x \in U$. Since $\operatorname{mult}_x h_a \geq \operatorname{mult}_x \mathcal{P}$, h_1/h is holomorphic on U and is upper semicontinuous. Thus $\operatorname{mult}_x h_a - \operatorname{mult}_x \mathcal{P} \in \mathbb{Z}$ is also a Zariski upper semicontinuous functions on $x \in A$, and whose minimum value is equal to 0. Thus $U_a := \{x \in A \mid \operatorname{mult}_x h_a = \operatorname{mult}_x \mathcal{P}\}$. is a Zariski open subset of A. Since A is quasi-compact with respect to Zariski topology, we can choose $a_1, \ldots, a_r \in A$ such that $U_{a_1} \cup \cdots \cup U_{a_r} = A$. Let $h_0 = h_{a_1} + \cdots + h_{a_r}$. Then

$$\operatorname{mult}_x h_0 = \inf_{1 \leq i \leq r} \operatorname{mult}_x h_{a_i} = \operatorname{mult}_x \mathcal{P}$$

for all $x \in A$.

(2) Take a general $h_0 \in \operatorname{Int}(\mathcal{P})$ such that $\operatorname{mult}_x h_0 = \operatorname{mult}_x \mathcal{P}$ for all $x \in A$, where the word 'general' is used in the sense [13, §7.9 **a**]. Then $f_1 := f/h_0$, $g_1 := g/h_0$ and $h_1 := h/h_0$ are holomorphic functions on A (i.e. $f_1, g_1, h_1 \in H^0(A, \mathcal{R}_A)$, for f_1, g_1, h_1 have no poles on A by (1). Moreover, f_1, g_1 and h_1 are PSD on A. Since $0 = f_1(a) = g_1(a) + h_1(a)$, $g_1(a) \geq 0$ and $h_1(a) \geq 0$, we have $g_1(a) = h_1(a) = 0$. Thus $\operatorname{mult}_a g > \operatorname{mult}_a \mathcal{P}$ and $\operatorname{mult}_a h > \operatorname{mult}_a \mathcal{P}$. This implies $g, h \in \mathcal{P}_a$. □

Proposition 2.8. *Let A, $\mathcal{P}$ be same as in Definition 2.6 with $\dim \mathcal{P} \geq 2$. Moreover, we assume that A is compact with respect to the Euclidean topology. Take $f \in \mathcal{E}(\mathcal{P})$. Then,*
(1) *$\mathcal{Z}_f(\mathcal{P}) \neq \emptyset$.*
(2) *If $a \in \mathcal{Z}_f(\mathcal{P})$, then $\mathcal{E}(\mathcal{P}_a) = \mathcal{P}_a \cap \mathcal{E}(\mathcal{P})$.*

Proof. (1) Assume that $\mathcal{Z}_f(\mathcal{P}) = \emptyset$. Then, $f \in \mathcal{P}$ satisfies $\operatorname{mult}_x f = \operatorname{mult}_x \mathcal{P}$ for all $x \in A$. Moreover, a general $h_0 \in \operatorname{Int}(\mathcal{P}) - \mathbb{R}_+ \cdot f$ satisfy $\operatorname{mult}_x h_0 = \operatorname{mult}_x \mathcal{P}$ for all $x \in A$. We can regard $g := f/h_0$ as a holomorphic function on A. Since A is compact, $\varepsilon := \inf_{x \in A} g(x) > 0$. Then $h := f - \varepsilon h_0 \in \mathcal{P} - \mathbb{R}_+ \cdot f$. Thus $f = h + \varepsilon h_0$ is not extremal in $\mathcal{P}$.

(2) $\mathcal{E}(\mathcal{P}_a) \supset \mathcal{P}_a \cap \mathcal{E}(\mathcal{P})$ is trivial. We shall show $\mathcal{E}(\mathcal{P}_a) \subset \mathcal{E}(\mathcal{P})$. Take $g \in \mathcal{E}(\mathcal{P}_a)$. Assume that $g = h_1 + h_2$ for $h_1, h_2 \in \mathcal{P} - \mathbb{R}_+ \cdot g$. Then $h_1, h_2 \in \mathcal{P}_a$ by Proposition 2.7. So, $g \notin \mathcal{E}(\mathcal{P}_a)$. A contradiction. Thus $g \in \mathcal{E}(\mathcal{P})$, and $\mathcal{E}(\mathcal{P}_a) \subset \mathcal{E}(\mathcal{P})$. □

Note that when $\mathfrak{Q} \subset \mathfrak{P}$ is a sub PSD cone, $\mathfrak{E}(\mathfrak{Q}) = \mathfrak{Q} \cap \mathfrak{E}(\mathfrak{P})$ does not hold in general. We have many counter examples.

The following definition is an anlogue of a resolution of the base locus of a linear system by a sequence of blowing ups. Words and symbols are based on algebraic geometry.

Definition 2.9.(infinitesimal local cone) Let A be a non-singular semialgebraic variety which is compact with respect to the Euclidean topology. Let $\mathfrak{P} = \mathfrak{P}(A, \mathfrak{H})$ be a PSD cone with $\dim \mathfrak{P} \geq 2$. Fix $f \in \mathfrak{E}(\mathfrak{P})$. Then $\mathfrak{Z}_f(\mathfrak{P}) \neq \emptyset$.

(1) Take $a \in \mathfrak{Z}_f(\mathfrak{P})$. Assume that $\dim \mathfrak{P}_a \geq 2$. Put $A_0 := A$, $a_0 := a$, $f_0 := f$ and $\mathfrak{L}_0 := \mathfrak{P}_a$. Then $f_0 \in \mathfrak{E}(\mathfrak{L}_0)$ by Proposition 2.8.

Inductively, we shall define A_i, a_i, $\mathfrak{L}_i$ for $i \geq 0$, and $\psi_i \colon A_i \to A_{i-1}$ for $i \geq 1$. Now fix $i \in \mathbb{N}$, and assume that A_j, a_j and $\mathfrak{L}_j$ are defined for all $0 \leq j < i$. Put $\overline{\psi}_j := \psi_1 \circ \cdots \circ \psi_j$: $A_j \longrightarrow A$ whenever $\psi_1,\ldots, \psi_j$ will be defined. In this process, we assume that $\psi_j(a_j) = a_{j-1}$, $f_j := \overline{\psi}_j^*(f) = f \circ \overline{\psi}_j \in \mathfrak{E}(\mathfrak{L}_j)$ and $a_j \in \mathfrak{Z}_{f_j}(\mathfrak{L}_j)$ for $0 \leq j < i$. Consider

$$\mathfrak{L}_i' := \left\{ g \in \mathfrak{L}_{i-1} \;\middle|\; \operatorname{mult}_{a_{i-1}} g > \operatorname{mult}_{a_{i-1}} \mathfrak{L}_{i-1} \right\}.$$

We divide into two cases.

(1-i) The case $0 \neq \mathfrak{L}_i' \subsetneqq \mathfrak{L}_{i-1}$ and $f_{i-1} \in \mathfrak{L}_i'$.

Then, let $A_i := A_{i-1}$, $\psi_i \colon A_i \longrightarrow A_{i-1}$ be the identity map, $a_i := a_{i-1}$ and we put $\mathfrak{L}_i := \mathfrak{L}_i'$. Note that $a_i \in \mathfrak{Z}_{f_i}(\mathfrak{L}_i)$. Now we repeat the process increasing i.

(1-ii) The case $\mathfrak{L}_i' = 0$ or $\mathfrak{L}_i' = \mathfrak{L}_{i-1}$ or $f_{i-1} \notin \mathfrak{L}_i'$.

Then, let $\psi_i \colon A_i \longrightarrow A_{i-1}$ be the blowing up of A_{i-1} at the point $a_{i-1} \in A_{i-1}$. $\mathfrak{H}_i := \overline{\psi}_i^* \mathfrak{H}$ is a signed linear system on A_i and $\overline{\psi}_i^* \mathfrak{P} := \left\{ g \circ \overline{\psi}_i \;\middle|\; g \in \mathfrak{P} \right\}$ satisfies $\overline{\psi}_i^* \mathfrak{P} = \mathfrak{P}(A_i, \mathfrak{H}_i)$. Note that $\dim \psi_i^* \mathfrak{L}_{i-1} = \dim \mathfrak{L}_{i-1} \geq 2$.

(1-ii-a) Consider the case that we can find $a_i \in \mathfrak{Z}_{f_i}(\psi_i^* \mathfrak{L}_{i-1})$ such that $\psi_i(a_i) = a_{i-1}$ where $f_i := \overline{\psi}_i^* f$. Let $\mathfrak{L}_i$ be the local cone of $\psi_i^* \mathfrak{L}_{i-1}$ at the point a_i. Note that $f_i \in \mathfrak{E}(\mathfrak{L}_i)$, because $\mathfrak{E}(\mathfrak{L}_i) = \mathfrak{L}_i \cap \mathfrak{E}(\psi_i^* \mathfrak{L}_{i-1})$. Then, we repeat the process increasing i.

(1-ii-b) Termination of the process.

Since $\dim \mathfrak{P} > \dim \mathfrak{L}_0 > \dim \mathfrak{L}_1 > \cdots$, there exists $l \in \mathbb{N}$ such that $\left\{ a \in \mathfrak{Z}_{f_{l+1}}(\psi_{l+1}^* \mathfrak{L}_l) \;\middle|\; \psi_{l+1}(a) = a_l \right\} = \emptyset$. Then, we stop to repeat the process. We say $a_1, a_2,\ldots, a_l$ is a sequence of zeros of f *infinitely near* to a. Each a_i $(1 \leq i \leq l)$ is called a zero of f infinitely near to a in $\mathfrak{P}$. The convex cone $\overline{\psi_l}(\mathfrak{L}_l)$ is called an *infinitesimal local cone* of $\mathfrak{P}$ at a_l or at a with respect to f.

Assume that $f(a) = 0$ and f has only finitely many zeros $b_1,\ldots, b_N$ infinitely near to a in $\mathfrak{P}$. Then, we define $\operatorname{length}_a f := N + 1$. If $\dim \mathfrak{P}_a = 1$ or there exists no $a_1 \in \mathfrak{Z}_{\psi_1^* f}(\psi_1^* \mathfrak{P}_a)$ such that $\psi_1(a_1) = a$, then we put $\operatorname{length}_a f := 1$. If $f(a) \neq 0$, then we put $\operatorname{length}_a f := 0$.

Lemma 2.10. *Let A, $\mathfrak{P}$ and $f \in \mathfrak{E}(\mathfrak{P})$ be as in Definition 2.9. Assume that a, $b \in \mathfrak{Z}_f(\mathfrak{P})$ with $a \neq b$. Let $\mathfrak{Q} := \mathfrak{P}_b$, and assume that $\dim \mathfrak{Q} \geq 2$. If $\mathfrak{L}$ is a local cone or an infinitesimal local cone of $\mathfrak{Q}$ at a point $a \in A$, then there exists a local cone or an infinitesimal local cone $\overline{\mathfrak{L}}$ of $\mathfrak{P}$ at a such that $\overline{\mathfrak{L}} \cap \mathfrak{Q} = \mathfrak{L}$.*

Proof. If $\mathfrak{L}$ is an infinitesimal local cone of $\mathfrak{Q}$, we take a sequences $\{A_i\}_{i=0}^l$, $\{a_i\}_{i=0}^l$, $\{\psi_i\}_{i=1}^l$ and $\{\mathfrak{L}_i\}_{i=0}^l$ as in Definition 2.9, such that $\mathfrak{L}_0 = \mathfrak{Q}_a$, $\overline{\psi_l}(\mathfrak{L}_l) = \mathfrak{L}$, $A_0 = A$ and $f_0 = f$.

In the case that $\mathfrak{L} = \mathfrak{Q}_a$ is a local cone of $\mathfrak{Q}$, we put $l = 0$ at the above sequences.

Put $\overline{\mathfrak{L}}_0 := \mathfrak{P}_a$. We need to find convex cones $\overline{\mathfrak{L}}_i$ on A_i such that $\overline{\mathfrak{L}}_i \cap \overline{\psi_i}^* \mathfrak{Q} = \mathfrak{L}_i$ and that $\overline{\mathfrak{L}}_i$ is a local cone or an infinitesimal local cone of $\psi_i^* \mathfrak{L}_{i-1}$. To construct $\overline{\mathfrak{L}}_i$, we may need to refine the sequences. In the process of the refinement, we always put $A_i^0 = A_i^1 = \cdots = A_i^{k_i} = A_i$ $(0 \leq i \leq l)$, and $\psi_i^j = \mathrm{id}\colon A_i^{j+1} \to A_i^j$ $(0 \leq j < k_i,\ 0 \leq i \leq l)$ and $\psi_{i+1}^{k_i} = \psi_{i+1}\colon A_{i+1}^0 \longrightarrow A_i^{k_i}$ $(0 \leq i < l)$.

(1) If $1 \leq i \leq l$ and $\overline{\mathfrak{L}}_{i-1}$ is already defined so that $\overline{\mathfrak{L}}_{i-1} \cap \overline{\psi_{i-1}}^* \mathfrak{Q} = \mathfrak{L}_{i-1}$, we put $\mathfrak{L}_i^0 := \left(\psi_{i-1}^* \overline{\mathfrak{L}}_{i-1}\right)_{a_i}$.

It is easy to see that $\mathfrak{L}_i^0 \cap \overline{\psi_i}^* \mathfrak{Q} \supset \mathfrak{L}_i$.

(2) If $\mathfrak{L}_i^0 \cap \overline{\psi_i}^* \mathfrak{Q} = \mathfrak{L}_i$, then we put $\overline{\mathfrak{L}}_i := \mathfrak{L}_i^0$, $k_i := 0$, and we don't need a refinement.

(3) Consider the case $\mathfrak{L}_i^0 \cap \overline{\psi_i}^* \mathfrak{Q} \supsetneqq \mathfrak{L}_i$.

This can happen only if $\mathrm{mult}_{a_i} \mathfrak{L}_i^0 < \mathrm{mult}_{a_i} \mathfrak{L}_i$. Let $A_i^1 := A_i$ and $\psi_i^1 = \mathrm{id}\colon A_i^1 \to A_i^0$. Put $\mathfrak{L}_i^1 := (\mathfrak{L}_i^0)_{a_i}$. Since $\mathrm{mult}_{a_i} \mathfrak{L}_i \leq \mathrm{mult}_{a_i} f_i$, we have $f_i \in \mathfrak{L}_i^1 \subsetneqq \mathfrak{L}_i^0$. Note that $\mathrm{mult}_{a_i} \mathfrak{L}_i^0 < \mathrm{mult}_{a_i} \mathfrak{L}_i^1 \leq \mathrm{mult}_{a_i} \mathfrak{L}_i \leq \mathrm{mult}_{a_i} f_i$. Thus, $\mathfrak{L}_i^1 \cap \overline{\psi_i}^* \mathfrak{Q} \supset \mathfrak{L}_i$.

(4) If $\mathfrak{L}_i^1 \cap \overline{\psi_i}^* \mathfrak{Q} = \mathfrak{L}_i$, then we put $\overline{\mathfrak{L}}_i := \mathfrak{L}_i^1$, $k_i := 1$, and we stop this refinement.

(5) If $\mathfrak{L}_i^1 \cap \overline{\psi_i}^* \mathfrak{Q} \supsetneqq \mathfrak{L}_i$, Put $\mathfrak{L}_i^2 := (\mathfrak{L}_i^1)_{a_i}$. Repeat this process till $\mathfrak{L}_i^{k_i} \cap \overline{\psi_i}^* \mathfrak{Q} = \mathfrak{L}_i$. Then we put $\overline{\mathfrak{L}}_i := \mathfrak{L}_i^{k_i}$.

When $i = l$, we put $\overline{\mathfrak{L}} := \overline{\psi_l}(\overline{\mathfrak{L}}_l)$. Then $\overline{\mathfrak{L}} \cap \mathfrak{Q} = \mathfrak{L}$. ∎

Theorem 2.11. *Let A, $\mathfrak{P}$ and $f \in \mathfrak{E}(\mathfrak{P})$ be as in Definition 2.9. Assume that $\dim \mathfrak{P} \geq 2$. Then, there exists points $P_1, \ldots, P_r \in A$ (not always distinct), and local cones or infinitesimal local cones $\mathfrak{P}_1, \ldots, \mathfrak{P}_r \subset \mathfrak{P}$ with respect to f which satisfy*

(1) $\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_r = \mathbb{R}_+ \cdot f$.

(2) *$\mathfrak{P}_i$ is the local cone $\mathfrak{P}_{P_i}$ or an infinitesimal local cone of $\mathfrak{P}$ at $P_i \in A$ with respect to f for $i = 1, \ldots, r$.*

Proof. We prove by induction on $\dim \mathfrak{P}$. Take $c \in \mathfrak{Z}_f(\mathfrak{P})$, and put $\mathfrak{Q} := \mathfrak{P}_c$. Since $\dim \mathfrak{Q} < \dim \mathfrak{P}$, there exists points $P_1, \ldots, P_r \in A$, and local cones or infinitesimal local cones $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_r$ of $\mathfrak{Q}$ which satisfy $\mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_r = \mathbb{R}_+ \cdot f$, and that $\mathfrak{Q}_j$ is the local cone $\mathfrak{Q}_{P_j}$ or an infinitesimal local cone of $\mathfrak{Q}$ at P_j. Then there exists a local cone or an infinitesimal local cone $\mathfrak{P}_j$ of $\mathfrak{P}$ at P_j such that $\mathfrak{P}_j \cap \mathfrak{Q} = \mathfrak{Q}_j$ by the above lemma. Thus

$$\mathfrak{P}_c \cap \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_r = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_r = \mathbb{R}_+ \cdot f.$$

If there exists $1 \leq i \leq r$ such that $\mathfrak{P}_i \subset \mathfrak{P}_c$, we have $\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_r = \mathbb{R}_+ \cdot f$. Otherwise, put $P_{r+1} := c$ and $\mathfrak{P}_{r+1} := \mathfrak{P}_c$. Then $\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_{r+1} = \mathbb{R}_+ \cdot f$. ∎

Definition 2.12. Let A be a non-singular semialgebraic variety which is compact with respect to the Euclidean topology. Let $\mathfrak{P} = \mathfrak{P}(A, \mathfrak{H})$ be a PSD cone. Take $a \in A$ and put $\mathfrak{L}_0 := \mathfrak{P}_a$, $A_0 := A$, $a_0 = a$. Assume that $\dim \mathfrak{L}_0 \geq 2$.

Let $\{A_i\}_{i=0}^l$, $\{a_i\}_{i=0}^l$, $\{\psi_i\}_{i=1}^l$, $\{\mathfrak{L}_i\}_{i=0}^l$ be the sequence such that :

(1) $\psi_i\colon A_i \longrightarrow A_{i-1}$ is the blowing up of A_{i-1} at the point $a_{i-1} \in A_{i-1}$ or the identity map $(A_i = A_{i-1},\ \psi_i = \mathrm{id})$ $(1 \leq i \leq l)$.

(2) $\psi_i(a_i) = a_{i-1}$ $(1 \leq i \leq l)$.

(3) $\mathfrak{L}_i$ is the local cone of $\psi_i^* \mathfrak{L}_{i-1}$ at the point $a_i \in A_i$ $(1 \leq i \leq l)$.

(4) $\dim \mathfrak{L}_i \geq 2$ for $1 \leq i < l$ and $\dim \mathfrak{L}_l \geq 1$.

Put $\overline{\psi}_i := \psi_1 \circ \cdots \circ \psi_i : A_i \longrightarrow A$ $(1 \le i \le l)$. Then $\overline{\psi}_l(\boldsymbol{\mathcal{L}}_l)$ is called an *infinitesimal local cone* of $\boldsymbol{\mathcal{P}}$ at a.

Proposition 2.13. *Let A be a non-singular compact semialgebraic variety. $\boldsymbol{\mathcal{P}}$ be a PSD cone on A, and $f \in \boldsymbol{\mathcal{P}}$. Assume that $\dim \boldsymbol{\mathcal{P}} \ge 2$, and there exists local cones or infinitesimal local cones $\boldsymbol{\mathcal{P}}_1, \ldots, \boldsymbol{\mathcal{P}}_r \subset \boldsymbol{\mathcal{P}}$ such that $\boldsymbol{\mathcal{P}}_1 \cap \cdots \cap \boldsymbol{\mathcal{P}}_r = \mathbb{R}_+ \cdot f$. Then, $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}})$.*

Proof. Assume that there exists $g,\ h \in \boldsymbol{\mathcal{P}} - \mathbb{R}_+ \cdot f$ such that $f = g + h$. Fix $1 \le k \le r$.

(1) We shall prove that $g,\ h \in \boldsymbol{\mathcal{P}}_k$.

(1-i) If $\boldsymbol{\mathcal{P}}_k$ is a local cone, (1) follow from Proposition 2.7.

(1-ii) Consider the case that $\boldsymbol{\mathcal{P}}_k$ is an infinitesimal local cone of $\boldsymbol{\mathcal{P}}$ at a. Take sequences $\{A_i\}_{i=0}^l$, $\{a_i\}_{i=0}^l$, $\{\psi_i\}_{i=1}^l$, $\{\boldsymbol{\mathcal{L}}_i\}_{i=0}^l$ with $\overline{\psi}_l(\boldsymbol{\mathcal{L}}_l) = \boldsymbol{\mathcal{P}}_k$ as in the above definition. Formally, put $\overline{\psi}_0 := \mathrm{id}\colon A_0 \to A_0$. Put $f_i := \overline{\psi}_i^*(f)$, $g_i := \overline{\psi}_i^*(g)$ and $h_i := \overline{\psi}_i^*(h)$ $(0 \le i \le l)$. Then $f_i = g_i + h_i \in \boldsymbol{\mathcal{L}}_i$. We shall show that $g_i,\ h_i \in \boldsymbol{\mathcal{L}}_i$ by induction on i.

If $i = 0$, we have $g_0,\ h_0 \in \boldsymbol{\mathcal{P}}_a = \boldsymbol{\mathcal{L}}_0$ by Proposition 2.7.

Assume that $i \ge 1$ and $g_{i-1},\ h_{i-1} \in \boldsymbol{\mathcal{L}}_{i-1}$. $\boldsymbol{\mathcal{L}}_i$ is the local cone of $\psi_i^* \boldsymbol{\mathcal{L}}_{i-1}$ at the point $a_i \in A_i$. Thus, we have $g_i,\ h_i \in \boldsymbol{\mathcal{L}}_i$, by Proposition 2.7.

(2) By (1), we have $g,\ h \in \boldsymbol{\mathcal{P}}_1 \cap \cdots \cap \boldsymbol{\mathcal{P}}_r = \mathbb{R}_+ \cdot f$. Thus, $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}})$. ∎

Let $f \in \boldsymbol{\mathcal{H}}_{3,d}$, $g \in \boldsymbol{\mathcal{H}}_{3,e}$ and $P \in \mathbb{P}^2_{\mathbb{C}}$. Take a local coordinate system $(x,\ y)$ on an affine open subset $U \subset \mathbb{P}^2_{\mathbb{C}}$ whose origin is P. We consider $f,\ g \in \mathbb{C}[[x,\ y]]$ and we denote the local intersection number of $V_{\mathbb{C}}(f)$ and $V_{\mathbb{C}}(g)$ at P by $I_P(f,g) := \dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(f,g)$. The intersection number of $V_{\mathbb{C}}(f)$ and $V_{\mathbb{C}}(g)$ is denoted by $I(f,g)$.

Example 2.14. Consider the case $A = \mathbb{P}^2_+$ and $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}^+_{3,d}$ $(d \ge 2)$. We denote the homogeneous coordinate system of A by $(x_0\colon x_1\colon x_2)$.

(1) Consider $\boldsymbol{\mathcal{P}}_P$ when $P = (1\colon p\colon q) \in \mathbb{P}^2_+$. Put $x := (x_1 - px_0)/x_0$ and $y := (x_2 - qx_0)/x_0$. Take an arbitrary $f \in \boldsymbol{\mathcal{P}}_P$. Then $f(x,y) = ax^2 + 2bxy + cy^2 + (\text{higher terms})$ such that $C_f := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive semidefinite. Let $\psi_1\colon A_1 \to A$ be the blowing up at P and put $t := x/y$. Then $f_1 := \psi_1^*(f) = y^2(at^2 + 2bt + c + \cdots)$ and $\mathrm{mult}_Q f_1 \ge 2$ for all $Q \in \psi_1^{-1}(P)$. In this case

$$\begin{aligned}\boldsymbol{\mathcal{P}}_P &= \bigl\{f(x,y) \in \boldsymbol{\mathcal{P}} \bigm| f(P) = f_x(P) = f_y(P) = 0\bigr\}\\ &= \bigl\{F(x_0,x_1,x_2) \in \boldsymbol{\mathcal{P}} \bigm| F(P) = F_{x_1}(P) = F_{x_2}(P) = 0\bigr\}\end{aligned}$$

where $f_x = \partial f(x,y)/\partial x$, $F_{x_0} = \partial F(x_0,x_1,x_2)/\partial x_0$ and so on.

If C_f is a positive definite matrix, Then $\mathrm{mult}_Q f_1 = 2$ for all $Q \in \psi_1^{-1}(P)$, and $\boldsymbol{\mathcal{Z}}_f(\psi_1^* \boldsymbol{\mathcal{P}}_P) = \emptyset$. Then $\mathrm{length}_P f = 1$.

(2) Assume that $d \ge 4$ and the leading term of $f(x,\ y)$ is equal to $x^2 + y^4$. Then $f_1 = y^2(t^2 + y^2 + \cdots)$ and $\boldsymbol{\mathcal{Z}}_f(\psi_1^* \boldsymbol{\mathcal{P}}_P)$ consists of a single point P_1 defined by $(t,\ y) = (0,0)$. Let $\psi_2\colon A_2 \to A_1$ be the blowing up at P_1 and put $t_2 := x/t$. Then $f_2 := \psi_2^* f_1 = y^4(t_2^2 + 1 + \cdots)$. Thus $\mathrm{length}_P f = 2$. Let $\boldsymbol{\mathcal{P}}_{P_1}$ be the infinitesimal local cone of $\boldsymbol{\mathcal{P}}$ at P_1. Then

$$\boldsymbol{\mathcal{P}}_{P_1} = \bigl\{g(x,y) \in \boldsymbol{\mathcal{P}} \bigm| g(P) = g_x(P) = g_y(P) = g_{yy}(P) = g_{xy}(P) = g_{yyy}(P) = 0\bigr\}.$$

(3) Let $d := 3$, $P = (1\colon p\colon 0) \in \mathbb{P}^2_+$ $(p > 0)$, $x := (x_1 - px_0)/x_0$ and $y := x_2/x_0$. Note that $y \ge 0$ on A. Take a general $f \in \boldsymbol{\mathcal{P}}_P$. Then the leading term of f is equal to $ay + bx^2$

with $a > 0$, $b > 0$. Let $\psi_1 : A_1 \to A$ be the blowing up at P and put $t := y/x$. Then $f_1 = \psi_1^*(f) = x(at + bx + \cdots)$. In this case

$$\begin{aligned}\boldsymbol{\mathcal{P}}_P &= \{f(x, y) \in \boldsymbol{\mathcal{P}} \mid f(P) = f_x(P) = 0\}\\ &= \{F(x_0, x_1, x_2) \in \boldsymbol{\mathcal{P}} \mid F(P) = F_{x_1}(P) = 0\}.\end{aligned}$$

Fix $f \in \boldsymbol{\mathcal{P}}_P$. Then $\boldsymbol{\mathcal{Z}}_f(\psi_1^*\boldsymbol{\mathcal{P}}_P) = \emptyset$. Thus $\mathrm{length}_P\, f = 1$.

(4) Let $P = (1 : 0 : 0) \in \mathbb{P}^2_+$, $x := x_1/x_0$ and $y := x_2/x_0$. Note that $x \geq 0$ and $y \geq 0$ on A. Assume that $f \in \boldsymbol{\mathcal{P}}_P$, $I_P(f,\ x) = 1$ and $I_P(f,\ y) = m$. Then $f(x,\ y) = ax + by^m +$ (higher terms) with $a > 0$, $b > 0$. Let $\psi_1 : A_1 \to A$ be the blowing up at P, and put $t_1 := x/y$. Then $f_1 = y(at_1 + by^{m-1} + \cdots)$. Let $P_1 \in A_1$ be the point defied by $(t_1,\ y) = (0,\ 0)$.

Similarly, let $\psi_i : A_i \to A_{i-1}$ be the blowing up at P_i, and put $t_i := t_{i-1}/y$. Let $P_i \in A_i$ be the point defied by $(t_i,\ y) = (0,\ 0)$. Then $f_i = y^i(at_i + by^{m-i} + \cdots)$ $(i \leq m)$. It is easy to see that $\mathrm{length}_P\, f = m$. If $m = 1$, then

$$\begin{aligned}\boldsymbol{\mathcal{P}}_P &= \{g(x, y) \in \boldsymbol{\mathcal{P}} \mid g(P) = 0\}\\ &= \{F(x_0, x_1, x_2) \in \boldsymbol{\mathcal{P}} \mid F(P) = 0\}.\end{aligned}$$

If $m = 2$, then

$$\begin{aligned}\boldsymbol{\mathcal{P}}_{P_1} &= \{g(x, y) \in \boldsymbol{\mathcal{P}} \mid g(P) = g_y(P) = 0\}\\ &= \{F(x_0, x_1, x_2) \in \boldsymbol{\mathcal{P}} \mid F(P) = F_{x_2}(P) = 0\}.\end{aligned}$$

If $m = 3$, then

$$\begin{aligned}\boldsymbol{\mathcal{P}}_{P_2} &= \{g(x, y) \in \boldsymbol{\mathcal{P}} \mid g(P) = g_y(P) = g_{yy}(P) = 0\}\\ &= \{F(x_0, x_1, x_2) \in \boldsymbol{\mathcal{P}} \mid F(P) = F_{x_2}(P) = F_{x_2x_2}(P) = 0\},\end{aligned}$$

where $\boldsymbol{\mathcal{P}}_{P_i}$ be the infinitesimal local cone of $\boldsymbol{\mathcal{P}}$ at P_i. Thus $\mathrm{length}_P\, f = m$.

(5) Let $P = (1 : p : 0) \in \mathbb{P}^2_+$ $(p > 0)$, $x := (x_1 - px_0)/x_0$ and $y := x_2/x_0$. Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ has the leading term $((y-1)+ax)^2 + b(x+c(y-1))^3$ $(b \neq 0)$. Let's determine all zeros of f infinitely near to P. Put $v := (y-1)+ax$ and $u := x + c(y-1)$. Let $\psi_1 : A_1 \to A$ be the blowing up at P and put $t := v/u$. Then $f_1 = u^2((t^2 + u) + \cdots)$. Let $P_1 \in A_1$ be the point defined by $(u,\ t) = (0,\ 0)$. Since $\mathrm{mult}_{P_1} f_1 = 3 > 2 = \mathrm{mult}_{P_1}(\psi_1^*\boldsymbol{\mathcal{P}}^+_{3,3})_{P_1}$, we need to blow up $\psi_2 : A_2 \to A_1$ at P_1. Put $s := u/t$. Then $f_2 = u^2t((t + s) + \cdots)$. Let $P_2 \in A_2$ be the point defined by $(u,\ s) = (0,\ 0)$. Then $\mathrm{mult}_{P_2} f_2 = 4 > 3 = \mathrm{mult}_{P_2}(\overline{\psi_2}^*\boldsymbol{\mathcal{P}}^+_{3,3})_{P_2}$. There exists no more zero of f infinitely near to P. In this case,

$$\begin{aligned}\boldsymbol{\mathcal{P}}_P &= \{F(x_0, x_1, x_2) \in \boldsymbol{\mathcal{P}} \mid F(P) = F_{x_1}(P) = 0\},\\ \boldsymbol{\mathcal{P}}_{P_1} &= \{g(u, v) \in \boldsymbol{\mathcal{P}} \mid g(P) = g_u(P) = g_v(P) = 0\}\\ &= \{F(x_0, x_1, x_2) \in \boldsymbol{\mathcal{P}} \mid F(P) = F_{x_1}(P) = F_{x_2}(P) = 0.\},\\ \boldsymbol{\mathcal{P}}_{P_2} &= \{g(u, v) \in \boldsymbol{\mathcal{P}} \mid g(P) = g_u(P) = g_v(P) = g_{uu}(P) = g_{uv}(P) = 0\}\\ &= \{F(x_0, x_1, x_2) \in \boldsymbol{\mathcal{P}} \mid F(P) = F_{x_1}(P) = F_{x_2}(P) = F_{x_1x_1}(P) = F_{x_1x_2}(P) = 0\}.\end{aligned}$$

Thus $\mathrm{length}_P\, f = 3$.

Remark 2.15. (1) Let $F(x_0,\ x_1,\ x_2) \in \boldsymbol{\mathcal{H}}_{3,m}$ and $P \notin V_{\mathbb{C}}(x_0)$. Put $x := x_1/x_0$, $y := x_2/x_0$ and $f(x,\ y) := F(1,\ x,\ y)$. Since $f_x(x,\ y) = F_{x_1}(1,\ x,\ y)$, we have

$$f_x(P) = 0 \Longleftrightarrow F_{x_1}(P) = 0.$$

Assume that $F(P) = 0$ and $P = (1\colon p\colon 0)$ with $p \neq 0$. Since $x_0F_{x_0} + x_1F_{x_1} + x_2F_{x_2} = mF$, we have

$$f_x(P) = 0 \Longleftrightarrow F_{x_1}(P) = 0 \Longleftrightarrow F_{x_0}(P) = 0,$$

Assume that $F(P) = F_{x_1}(P) = 0$ and $P = (1\colon 0\colon 0)$. Since $x_0F_{x_0x_1} + x_1F_{x_1x_1} + x_2F_{x_1x_2} = (m-1)F_{x_1}$, $x_0F_{x_0x_0} + x_1F_{x_0x_1} + x_2F_{x_0x_2} = (m-1)F_{x_0}$, we have

$$f_{xx}(P) = 0 \Longleftrightarrow F_{x_1x_1}(P) = 0 \Longrightarrow F_{x_0x_1}(P) = 0,\ F_{x_0x_0}(P) = 0.$$

(2) Let $a \neq b \in A$. As is well known of a property of vector spaces,

$$\dim(\mathcal{H}_a \cap \mathcal{H}_b) = \dim \mathcal{H}_a + \dim \mathcal{H}_b - \dim \mathcal{H}.$$

In the case of convex cones,

$$\dim(\mathcal{P}_a \cap \mathcal{P}_b) \leq \dim \mathcal{P}_a + \dim \mathcal{P}_b - \dim \mathcal{P}$$

is true, but $\leq$ cannot be replaced by $=$ in general. So, we must be careful to compute $\dim\big(\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r\big)$.

Definition 2.16. For an irreducible curve $C = V_{\mathbb{C}}(f)$ with $f \in \mathcal{H}_{3,d}$, we say C has a (simple) node at P if two analytic branches of C intersect at P transversally. We say C has an *acnode* at P, if P is a simple node of $V_{\mathbb{C}}(f)$ and P is an isolated point of $V_{\mathbb{R}}(f)$. In other words, an acnode is a real node whose tangents are non-real complex conjugates.

Note that if $f \in \mathcal{P}(A,\ \mathcal{H})$, $f(P) = 0$, f is irreducible, and $V_{\mathbb{C}}(f)$ has a node at $P \in \mathrm{Int}(A)$, then P is an acnode of $V_{\mathbb{C}}(f)$. Theorem 1.1 can be restated as the following using the notion of infinitely near zeros.

Theorem 2.17. *Assume that $f \in \mathcal{P}_{3,6}$ is not a square of a cubic polynomial. Then, $f \in \mathcal{E}(\mathcal{P}_{3,6})$ if and only if f has just 10 zeros on $\mathbb{P}^2_{\mathbb{R}}$ including all the infinitely near zeros.*

Proof. Assume that $f \in \mathcal{E}(\mathcal{P}_{3,6})$ is not a square of cubic polynomial. Then f is a limit of a sequence $\{f_n\}$ of exposed extremal elements in $\mathcal{P}_{3,6}$ (see [6]). Each f_n has distinct 10 zeros. Any infinitely near zero of f is a limit of a squence of a certain zero of f_n. Thus, f also has just 10 zeros including all infinitely near zeros.

Assume that f has 10 zeros $P_1,\ldots, P_{10}$ including infinitely near zeros. Then f cannot be a product of a quadratic and a quartic, since their intersection consists of 8 points. Similarly, f cannot be a product of two cubics, since their intersection consists of 9 points. Thus f is irreducible. Let $\psi\colon X \to \mathbb{P}^2_{\mathbb{C}}$ be a proper birational morphism such that we can regard $P_1,\ldots, P_{10}$ are distinct points in X in Noether's sense. As is well known in algebraic geometry, there exists a unique irreducible sextic curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ such that the strict transform of C to X has nodes at $P_1,\ldots, P_{10}$. Thus $C = V_{\mathbb{C}}(f)$ and f is extremal. □

Using the ideas in [6, 7], we also obtain the following theorem.

Theorem 2.18. *Assume $d \geq 3$, and $f \in \mathcal{E}(\mathcal{P}_{3,2d})$ is irreducible. Let N be the numbers of zeros of f in $\mathbb{P}^2_{\mathbb{R}}$ including all the infinitely near zeros. Then*

$$\frac{(d+1)(d+2)}{2} \leq N \leq (2d-1)(d-1).$$

Proof. Let $P_1,\ldots, P_N$ be all the zeros of f on $\mathbb{P}^2_{\mathbb{R}}$ including infinitely near zeros. There exists local cones or infinitesimal local cones $\mathcal{L}_1,\ldots, \mathcal{L}_r \subset \mathcal{P}_{3,2d}$ such that $\mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r = \mathbb{R}_+ \cdot f$. We may assume that $r = N$ and $\mathcal{L}_i$ corresponds to P_i.

If $N < (d+1)(d+2)/2$, then there exists $g \in \mathcal{H}_{3,d}$ such that $P_1, \ldots, P_N \in V_{\mathbb{R}}(g)$ in the sense of Noether. Then $g^2 \in \mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r = \mathbb{R}_+ \cdot f$. This implies $g^2 = cf$ ($\exists c \in \mathbb{R}_+$), and f is reducible. Thus $N \geq (d+1)(d+2)/2$.

Let $C := V_{\mathbb{C}}(f) \subset \mathbb{P}^2_{\mathbb{C}}$. Then $p_a(C) \geq \sum_P \nu(P)(\nu(P)-1)/2 + g(C') \geq N + g(C')$, where $p_a(C) = (2d-1)(2d-2)/2$ is the arithmetic (or virtual) genus, $g(C')$ is the genus of the normalization C' of C, and $\nu(P_i)$ is the multiplicity of C at P_i. Thus, we have $N \leq (2d-1)(d-1)$. □

§3. Extremal elements of $\mathcal{P}^+_{3,3}$.

In §3 and §4, we usually use the symbol $(x_0\colon x_1\colon x_2)$ to denote the standard homogeneous coordinate system of $\mathbb{P}^2_+$, $\mathbb{P}^2_{\mathbb{R}}$ or $\mathbb{P}^2_{\mathbb{C}}$. We sometime rewrite $(x_0\colon x_1\colon x_2)$ by $(x\colon y\colon z)$. But we also often use (x, y) to denote a local coordinate system when there is no fear of confusion.

3.1. Reducible elements of $\mathcal{E}(\mathcal{P}^+_{3,3})$.

Let $\mathcal{X} := \{cx \mid c > 0\} \cup \{cy \mid c > 0\} \cup \{cz \mid c > 0\} \subset \mathcal{H}_{3,1}$.

Proposition 3.1. *If $f \in \mathcal{X}$, $g \in \mathcal{H}_{3,d}$ and $fg \in \mathcal{E}(\mathcal{P}^+_{3,d+1})$, then $g \in \mathcal{E}(\mathcal{P}^+_{3,d})$. Conversely, if $f \in \mathcal{X}$ and $g \in \mathcal{E}(\mathcal{P}^+_{3,d})$, then $fg \in \mathcal{E}(\mathcal{P}^+_{3,d+1})$.*

Proof. Assume that $fg \in \mathcal{E}(\mathcal{P}^+_{3,d+1})$ and $g = h_1 + h_2$ for certain $h_1, h_2 \in \mathcal{P}^+_{3,d} - \{0\}$. Since fg is extremal, there exists $c_1, c_2 \in \mathbb{R}_+$ such that $fh_1 = c_1 fg$ and $fh_2 = c_2 fg$. Thus, $h_1 = c_1 g$ and $h_2 = c_2 g$. That is, g is extremal.

Conversely, we assume $f \in \mathcal{X}$, $g \in \mathcal{E}(\mathcal{P}^+_{3,d})$, and $fg = h_1 + h_2$ for certain $h_1, h_2 \in \mathcal{P}^+_{3,d+1} - \{0\}$. Note that $V_+(h_1 + h_2) \subset V_+(h_1) \cap V_+(h_2)$, because $h_1 \geq 0$ and $h_2 \geq 0$ on $\mathbb{P}^2_+$. Thus $V_+(f) \subset V_+(h_1 + h_2) \subset V_+(h_1) \cap V_+(h_2)$. Since $\dim V_+(f) = 1$, we have $V_{\mathbb{C}}(f) \subset V_{\mathbb{C}}(h_i)$ ($i = 1, 2$). Since f is irreducible, h_1 and h_2 must be multiples of f. Let $h_1 = fg_1$ and $h_2 = fg_2$ ($g_1, g_2 \in \mathcal{H}_{3,d}$). Since $h_1, h_2 \in \mathcal{P}^+_{3,d+1} - \{0\}$, we have $g_1, g_2 \in \mathcal{P}^+_{3,d} - \{0\}$. Since $g = g_1 + g_2$ is extremal, there exists $c_1, c_2 \in \mathbb{R}_+$ such that $g_1 = c_1 g$ and $g_2 = c_2 g$. Thus, fg is extremal. □

Proposition 3.2. *If $f \in \mathcal{E}(\mathcal{P}^+_{3,2})$, then one of the following statements holds.*

(1) *$f = f_1 f_2$ where $f_1, f_2 \in \mathcal{X}$.*

(2) *There exists $f_1 \in \mathcal{H}_{3,1}$ such that $f = f_1^2$ and $V_{\mathbb{R}}(f_1) \cap \operatorname{Int}(\mathbb{P}^2_+) \neq \emptyset$.*

Proof. Since f is extremal, there exists $P \in \mathbb{P}^2_+$ such that $f(P) = 0$ by Proposition 2.8(1).

(i) Assume that $P \in \operatorname{Int}(\mathbb{P}^2_+)$ and $V_+(f) = \{P\}$. By the classification of quadratic curves, this occurs only in the case $f = g_1^2 + g_2^2$ where $g_1, g_2 \in \mathcal{H}_{3,1}$. In this case $V_{\mathbb{R}}(g_1)$ and $V_{\mathbb{R}}(g_2)$ are distinct lines which intersect at P. Thus, f is not extremal.

(ii) Assume that $P \in \operatorname{Int}(\mathbb{P}^2_+)$ and $V_+(f) \supsetneqq \{P\}$. Then, there exists $Q \in \mathbb{P}^2_+$ such that $f(Q) = 0$ and $P \neq Q$. Since f is PSD, $V_+(f)$ cannot be a real conic. Thus $V_+(f)$ must be a line passing through P and Q. Thus (2) occurs.

(iii) Assume that $V_{\mathbb{R}}(f) \cap \operatorname{Int}(\mathbb{P}^2_+) = \emptyset$ and $P \in \partial\mathbb{P}^2_+$. We may assume $P = (a\colon 0\colon 1)$ where $a \geq 0$.

It is easy to see that $\dim(\mathcal{P}^+_{3,2})_P \geq \dim \mathcal{P}^+_{3,2} - 2 = 4$. Since $f \in (\mathcal{P}^+_{3,3})_P$ is extremal, there exists $Q \in \partial\mathbb{P}^2_+$ such that $f(Q) = 0$.

If $Q = (b\colon 0\colon 1)$ $(a \neq b,\ b > 0)$, then, $f = cy^2$ $(\exists c > 0)$. If $Q = (0\colon b\colon 1)$ $(b > 0)$, then, $f = cxy$ $(\exists c > 0)$. If $Q = (1\colon b\colon 0)$ $(b > 0)$, then, $f = cyz$ $(\exists c > 0)$. □

Proposition 3.3. *Let $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$. If f is reducible, then one the following statements holds.*

(1) $f = f_1 f_2 f_3$ *and* $f_1,\ f_2,\ f_3 \in \mathcal{X}$.

(2) $f = f_1 f_2^2$, $f_1 \in \mathcal{X}$, $f_2 \in \mathcal{H}_{3,1}$ *and* $V_{\mathbb{R}}(f_2) \cap \mathrm{Int}(\mathbb{P}^2_+) \neq \emptyset$.

Conversely, if $f \in \mathcal{H}_{3,3}$ satisfies (1) or (2), then $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$.

Proof. (i) Since f is reducible, we can write as $f = f_1 g$, where $f_1 = ax + by + cz$ $(a,\ b,\ c \in \mathbb{R})$ and $g \in \mathcal{H}_{3,2}$.

(i-1) Consider the case $V_{\mathbb{R}}(f_1) \cap \mathrm{Int}(\mathbb{P}^2_+) = \emptyset$. We may assume that $f_1 \geq 0$ on $\mathbb{P}^2_+$. Then $a \geq 0$, $b \geq 0$, $c \geq 0$ and $g \in \mathcal{P}^+_{3,2}$. Assume that $a > 0$ and $b > 0$. Then $f = axg + byg + czg$ and $axg,\ byg,\ czg \in \mathcal{P}^+_{3,3}$. Thus f is not extremal. This implies $f_1 \in \mathcal{X}$. We may assume $f_1 = x$. Since f is extremal in $\mathcal{P}^+_{3,3}$, g must be extremal in $\mathcal{P}^+_{3,2}$ by Proposition 3.1. Then, we have the conclusion by Proposition 3.2.

(i-2) Consider the case $V_{\mathbb{R}}(f_1) \cap \mathrm{Int}(\mathbb{P}^2_+) \neq \emptyset$. Then f must be divisible by f_1^2, because $f \geq 0$ on $\mathbb{P}^2_+$. So, $f = f_1^2 f_2$ where $f_2 \in \mathcal{H}_{3,1}$ with $V_{\mathbb{R}}(f_2) \cap \mathrm{Int}(\mathbb{P}^2_+) = \emptyset$. If we put $(f_2,\ f_1^2)$ as new $(f_1,\ g)$, we can apply (i-1).

(ii) We prove the converse part. For $c > 0$, the forms cx^3, cx^2y, $cxyz$ are also extremal in $\mathcal{P}^+_{3,3}$. Thus if f satisfies (1), then f is extremal.

Consider the case f satisfies (2). Assume that $f = g_1 + g_2$ where $g_1,\ g_2 \in \mathcal{P}^+_{3,3}$. Then $V_+(f_1) \subset V_+(g_1) \cap V_+(g_2)$. Since $\dim V_+(f_1) = 1$ and f_1 is irreducible, there exists $h_1,\ h_2 \in \mathcal{P}^+_{3,2}$ such that $g_1 = f_1 h_1$, $g_2 = f_1 h_2$. Since $V_+(f_2) \subset V_+(h_1) \cap V_+(h_2)$, there exits $c_1,\ c_2 \in \mathbb{R}_+$ such that $h_1 = c_1 f_2^2$, $h_2 = c_2 f_2^2$. Thus $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$. □

Proposition 3.4. *If $f \in \mathcal{H}_{3,3}$ is irreducible in $\mathbb{R}[x, y, z]$, then f is irreducible in $\mathbb{C}[x, y, z]$.*

Proof. Assume that f is divisible by $f_1 \in \mathbb{C}[x, y, z]$ with $f_1 \notin \mathbb{R}[x, y, z]$. Let $\overline{f_1}$ be the complex conjugate of f_1. Then, f is divisible by $\overline{f_1}$. Thus $f = f_1 \overline{f_1} f_2$. Since $f_1 \overline{f_1} \in \mathbb{R}[x, y, z]$, we have $f_2 \in \mathbb{R}[x, y, z]$. □

3.2. Basic lemmata for irreducible elements of $\mathcal{E}(\mathcal{P}^+_{3,3})$.

In §3.2 — §4, we use the symbols $P_x = P_{x_0} := (1\colon 0\colon 0)$, $P_y = P_{x_1} := (0\colon 1\colon 0)$, $P_z = P_{x_2} := (0\colon 0\colon 1) \in \mathbb{R}^2_+$, $L_x = L_{x_0} := V_+(x) - \{P_y,\ P_z\}$, $L_y = L_{x_1} := V_+(y) - \{P_z,\ P_x\}$, and $L_z = L_{x_2} := V_+(z) - \{P_x,\ P_y\}$. For an irreducible complex algebraic curve C, we denote

$$\mathrm{Sing}(C) := \{P \in C \mid P \text{ is a singular point of } C\},$$
$$\mathrm{Reg}(C) := \{P \in C \mid P \text{ is a non-singular point of } C\}.$$

Lemma 3.5. *Assume that $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ is irreducible. Then, $V_{\mathbb{C}}(f)$ is a rational curve on $\mathbb{P}^2_{\mathbb{C}}$ whose unique singular point lies on $\mathbb{P}^2_+$.*

Proof. Assume that $\mathrm{Sing}(V_{\mathbb{C}}(f)) \cap \mathbb{P}^2_+ = \emptyset$. Then $f(P) > 0$ for all $P \in \mathrm{Int}(\mathbb{P}^2_+)$. Let

$$D_i := \{(x_0\colon x_1\colon x_2) \in \mathbb{P}^2_+ \mid x_0^2 + x_1^2 + x_2^2 \leq 5x_i^2\}$$

$(i = 0, 1, 2)$. Note that $\mathbb{P}^2_+ = D_0 \cup D_1 \cup D_2$. On D_0, we put $x := x_1/x_0$, $y := x_2/x_0$, and $f_0(x, y) := f(1, x, y)$. Let

$$D := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leq 4,\ x \geq y \geq 0\right\} \subset D_0.$$

(1) We shall prove that there exists $c_0 > 0$ such that $f_0(x, y) \geq c_0 xy$ for all $(x, y) \in D$.

Note that if $f_0(a, 0) = 0$ for a certain $a > 0$, then x-axis is a tangent to $V_{\mathbb{C}}(f_0)$ at $(a, 0)$. Since any cubic curve has no bitangent line, there exists at most one $a_0 > 0$ such that $f_0(a_0, 0) = 0$.

(1-1) Assume that $f_0(x, 0) > 0$ for all $0 \leq x \leq 2$. Then there exists no $(x, y) \in D$ such that $f_0(x, y) = 0$. Thus $m := \min\left\{f_0(x, y) \;\middle|\; (x, y) \in D\right\} > 0$. Put $c_0 := m/4$, then $f_0(x, y) \geq c_0 xy$ for all $(x, y) \in D$.

(1-2) Assume that $f_0(a, 0) = 0$ for a certain $0 \leq a \leq 2$. There exist $c_1, \ldots, c_8 \in \mathbb{R}$ such that

$$f_0(x, y) = c_1 y + c_2(x - a)^2 + 2c_3(x - a)y + c_4 y^2 + g(x - a, y)$$

where $g(s, t) = c_5 s^3 + c_6 s^2 t + c_7 s t^2 + c_8 t^3$. Since $f_0(a, y) \geq 0$ for all $y \geq 0$, we have $c_1 \geq 0$. If $c_1 = 0$, then $(a, 0)$ is a singular point of $V_{\mathbb{C}}(f)$. Thus $c_1 > 0$. Then, there exists an open neighborhood $(a, 0) \in U_a \subset D$ such that $f_0(x, y) \geq (c_1/4)y$ for all $(x, y) \in U_a$. Then $f_0(x, y) \geq (c_1/8)xy$ for all $(x, y) \in U_a$. So, we put $m_1(a) := c_1/8$.

Let $V := \left\{a \in [0, 2] \;\middle|\; f_0(a, 0) = 0\right\} \subset \{0, a_0\}$, $U := \displaystyle\bigcup_{a \in V} U_a$, and $m_2 := \displaystyle\min_{a \in V} m_1(a)$. Then $f_0(x, y) \geq m_2 xy$ for all $(x, y) \in U$. Note that

$$m_3 := \min\left\{f_0(x, y) \;\middle|\; (x, y) \in \mathrm{Cls}(D - U)\right\} > 0.$$

So, put $c_0 := \min\{m_2, m_3/4\}$. Then $f_0(x, y) \geq c_0 xy$ for all $(x, y) \in D$.

By (1), there exists $c > 0$ such that $F(x_0, x_1, x_2) \geq c x_0 x_1 x_2$ for all $(x_0{:}\, x_1{:}\, x_2) \in \mathbb{P}^2_+$. So, $f(x_0, x_1, x_2) - c x_0 x_1 x_2 \in \boldsymbol{\mathcal{P}}^+_{3,3}$ and f is not extremal. A contradiction. Thus $\mathrm{Sing}(V_{\mathbb{C}}(f)) \cap \mathbb{P}^2_+ \neq \emptyset$.

Since $V_{\mathbb{C}}(f)$ is a cubic curve, $p_a(C) = 1$. Since $\mathrm{Sing}(V_{\mathbb{C}}(f)) \cap \mathbb{P}^2_+ \neq \emptyset$, $V_{\mathbb{C}}(f)$ has just one singular point P and $V_{\mathbb{C}}(f)$ is a rational curve, by Riemann genus formula. Thus, $P \in \mathbb{P}^2_+$. □

Lemma 3.6. *Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ is irreducible, and P is the unique singular point of $V_{\mathbb{C}}(f)$. Then $P \notin \left\{P_x, P_y, P_z\right\}$.*

Proof. Assume that $P \in \left\{P_x, P_y, P_z\right\}$. We may assume $P = P_x$. We use the same notation as in the proof of the previous lemma. Then $P = (0, 0) \in D \subset D_0 \subset \mathbb{P}^2_+$. Then $f_0(x, y) = g_2(x, y) + g_3(x, y)$ where $g_d(x, y)$ is a homogeneous polynomial of degree d. Then $f(x_0, x_1, x_2) = g_2(x_1, x_2)x_0 + g_3(x_1, x_2)$. Considering the cases $x_0 = 0$ and $x_0 \to +\infty$, we conclude that $g_2(x_1, x_2) \in \boldsymbol{\mathcal{P}}^+_{2,2}$ and $g_3(x_1, x_2) \in \boldsymbol{\mathcal{P}}^+_{2,3}$. If $g_3 = 0$, then $f = x_0 g_2$ is not irreducible. So, $g_3 \neq 0$. If $g_2 = 0$, the cubic $g_3(x_1, x_2)$ is reducible in $\mathbb{C}[x_1, x_2]$. Thus, $g_3 \neq 0$. Therefore, f is not extremal. □

Lemma 3.7. *Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ is irreducible. Then $I_{P_y}(f, x) + I_{P_z}(f, x) \leq 3$. If $V_+(f) \cap L_x \neq \emptyset$, then $\#(V_+(f) \cap L_x) = 1$ and $I_{P_y}(f, x) + I_{P_z}(f, x) \leq 1$.*

Proof. Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ is irreducible. Then $I_{P_y}(f, x) + I_{P_z}(f, x) \leq I(f, x) = \deg f = 3$. If $Q \in V_+(f) \cap L_x \neq \emptyset$, then, $I_{P_y}(f, x) + I_{P_z}(f, x) + I_Q(f, x) \leq I(f, x) = 3$. Since $I_Q(f, x) \geq 2$, we have $I_{P_y}(f, x) + I_{P_z}(f, x) \leq 1$. □

Lemma 3.8. *Let $\mathcal{P} = \mathcal{P}^+_{3,3}$. Assume that $f(x, y, z) \in \mathcal{E}(\mathcal{P})$ is irreducible in $\mathbb{C}[x, y, z]$. Then, $V_+(f)$ is a finite set, and we can choose local cones or infinitesimal local cones $\mathcal{L}_i$ of $\mathcal{P}$ at P_i which satisfy the following conditions.*
(1) $\mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r = \mathbb{R}_+ \cdot f$.
(2) $P_i \neq P_j$ *if* $i \neq j$.
(3) $\{P_1, \ldots, P_r\} = V_+(f)$.
(4) *If* $P_i \in V_+(f) \cap \mathrm{Reg}(V_{\mathbb{C}}(f))$, *then* $P_i \in \partial\mathbb{P}^2_+$, *and one of the following cases occurs.*
(4-1) *If* $P \in L_{x_j}$ *and* $k \in \{0, 1, 2\} - \{j\}$, *then*

$$\mathcal{L}_i = \{F \in \mathcal{P} \mid F(P) = F_{x_k}(P) = 0\}.$$

(4-2) *Assume that* $P = P_{x_j}$, $\{j, k, l\} = \{0, 1, 2\}$ *and* $m := I_P(f, x_k) \geq I_P(f, x_l) = 1$. *If* $m = 1$, *then*

$$\mathcal{L}_i = \{F \in \mathcal{P} \mid F(P) = 0\}.$$

If $m = 2$, *then*

$$\mathcal{L}_i = \{F \in \mathcal{P} \mid F(P) = F_{x_k}(P) = 0\}.$$

If $m = 3$, *then*

$$\mathcal{L}_i = \{F \in \mathcal{P} \mid F(P) = F_{x_k}(P) = F_{x_k x_k}(P) = 0\}.$$

(5) *If* $P_i \in V_+(f)$ *is an acnode, then*

$$\mathcal{L}_i = \{F(x, y, z) \in \mathcal{P} \mid F(P) = F_x(P) = F_y(P) = 0\}.$$

Proof. (1) By Theorem 2.11, there exist local cones or infinitesimal local cones $\mathcal{L}_i$ of $\mathcal{P}$ at P_i which satisfy $\mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r = \mathbb{R}_+ \cdot f$. If $\mathcal{L}_i \subset \mathcal{L}_j$ for $i \neq j$, we get rid of $\mathcal{L}_j$. So, we may assume that $\mathcal{L}_i \not\subset \mathcal{L}_j$ if $i \neq j$.

(2) By our observation in Example 2.14, the local cone $(\mathcal{P}^+_{3,3})_{P_i}$ does not have two distinct infinitesimal local cones, if f is cubic and P is an acnode, a cusp or a non-singular point in $\partial\mathbb{P}^2_+$. Thus, $P_i \neq P_j$ if $i \neq j$.

(3) $P_i \in V_+(f)$ by the definition. Assume that $P_{r+1} \in V_+(f) - \{P_1, \ldots, P_r\}$. Let $\mathcal{L}_{P_{r+1}}$ be the local cone or the infinitesimal local cone of $\mathcal{P}^+_{3,3}$ at P_{r+1} with respect to f. $\mathcal{L}_{P_{r+1}}$ is unique by Example 2.14. Then $\mathcal{L}_{P_1} \cap \cdots \cap \mathcal{L}_{P_{r+1}} = \mathbb{R}_+ \cdot f$ still holds. After repeating this process, we may assume $V_+(f) = \{P_1, \ldots, P_{r+k}\}$.

(4) Assume that $P \in V_+(f) \cap \mathrm{Reg}(V_{\mathbb{C}}(f))$. If $P \in \mathrm{Int}(\mathbb{P}^2_+)$, then $V_+(f)$ must contain a locus of analytic curve C_P near P, and $\mathrm{sign}(f)$ is opposite on both sides of C_P. Thus $P \in \partial\mathbb{P}^2_+$.

(4-1) Consider the case $P \in L_z$. Note that $I_P(f, z) \leq I(f, z) = 3$. If $I_P(f, z)$ is odd, then $V_+(f)$ must contain a locus of analytic curve C_P near P, and $\mathrm{sign}(f)$ changes across C_P. Thus $I_P(f, z) = 2$. By Example 2.14(3), we have (4-1).

(4-2) follows from Example 2.14(4).

(5) follows from Example 2.14(2). □

Notation 3.9. Throughout the remaining part of §3, we use the following notation. We denote a local cone or an infinitesimal local cone $\mathcal{L}_i$ in the above lemma by $\mathcal{L}_{P_i}$. Assume that $f(x, y, z) \in \mathcal{E}(\mathcal{P})$ is irreducible in $\mathbb{C}[x, y, z]$. Lemma 3.8(1) is represented as

$$\mathcal{L}_{P_1} \cap \cdots \cap \mathcal{L}_{P_r} = \mathbb{R}_+ \cdot f$$

where $\{P_1, \ldots, P_r\} = V_+(f)$ and $P_i \neq P_j$ if $i \neq j$. This $\mathcal{L}_{P_1} \cap \cdots \cap \mathcal{L}_{P_r}$ is denoted by $\mathcal{L}_f$.

Lemma 3.10. *Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ is irreducible. Let $\pi: \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_+$ be the surjective morphism defined by $\pi(x_0 \colon x_1 \colon x_2) = (x_0^2 \colon x_1^2 \colon x_2^2)$, and let $g := \pi^* f$. That is $g(x_0, x_1, x_2) = f(x_0^2, x_1^2, x_2^2)$. Take $P \in V_+(f)$. We define*

$$N_P(f) := \begin{cases} 4 & (\text{If } P \text{ is an acnode of } V_{\mathbb{C}}(f).) \\ 12 & (\text{If } P \in \operatorname{Int}(\mathbb{P}^2_+) \text{ and } P \text{ is a cusp of } V_{\mathbb{C}}(f).) \\ 6 & (\text{If } P \in \partial\mathbb{P}^2_+ \text{ and } P \text{ is a cusp of } V_{\mathbb{C}}(f).) \\ 2 & (\text{If } P \in L_{x_0} \cup L_{x_1} \cup L_{x_2} \text{ and } P \in \operatorname{Reg}(V_{\mathbb{C}}(f)).) \\ \operatorname{length}_P f & (\text{If } P \in \{P_{x_0}, P_{x_1}, P_{x_2}\} \text{ and } P \in \operatorname{Reg}(V_{\mathbb{C}}(f)).) \end{cases}$$

and $N(f) := \sum_{Q \in V_+(f)} N_Q(f)$. Then $N(f) \leq 10$. Moreover, if $N(f) = 10$, then g is irreducible in $\mathbb{C}[x_0, x_1, x_2]$ and $g \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{3,6})$. In particular, $V_{\mathbb{C}}(f)$ does not have a cusp in $\operatorname{Int}(\mathbb{P}^2_+)$.

Proof. We will separate 8 cases according to the type of the point P.

(1) Consider the case $P \in \operatorname{Int}(\mathbb{P}^2_+)$ and P is the acnode of $V_{\mathbb{C}}(f)$. If $P = (1 \colon p^2 \colon q^2)$ ($p > 0$, $q > 0$), then $\pi^{-1}(P) = \{(1 \colon p \colon q), (1 \colon p \colon -q), (1 \colon -p \colon q), (1 \colon -p \colon -q)\}$. Put $x := (x_1 - p^2 x_0)/x_0$ and $y := (x_2 - q^2 x_0)/x_0$. Then $f(x, y) = ax^2 + 2bxy + cy^2 +$ (higher terms) as in Example 2.14(1). Put $x' := (x_1 - px_0)/x_0$ and $y' := (x_2 - qx_0)/x_0$ around $Q := (1 \colon p \colon q) \in \mathbb{P}^2_{\mathbb{R}}$. Then $g(x', y') = ax'^2 + 2bx'y' + cy'^2 +$ (higher terms). Thus, $\operatorname{length}_Q g = 1$ for all $Q \in \pi^{-1}(P)$. Therefore, $V_{\mathbb{C}}(g)$ has 4 acnodes in $\pi^{-1}(P)$.

(2) Consider the case $P \in L_x \cup L_y \cup L_z$ is the acnode of $V_{\mathbb{C}}(f)$. We may assume that $P = (1 \colon p^2 \colon 0)$ ($p > 0$). Let $x := (x_1 - p^2 x_0)/x_0$, $y := x_2/x_0$, $x' := (x_1 - px_0)/x_0$ and $y' := x_2/x_0$. Take $Q := (1 \colon p \colon 0) \in \pi^{-1}(P) = \{(1 \colon \pm p \colon 0)\}$. Then $g(x', y') = ax'^2 + 2bx'y'^2 + cy'^4 +$ (higher terms). Thus $\operatorname{length}_Q g = 2$. Therefore, $V_C(g)$ has 4 zeros or infinitely near zeros in $\pi^{-1}(P)$.

(3) Consider the case $P \in \operatorname{Int}(\mathbb{P}^2_+)$ and P is the cusp of $V_{\mathbb{C}}(f)$. If $P = (1 \colon p^2 \colon q^2)$ ($p > 0$, $q > 0$), then $f(x, y) = x^3 + y^2 +$ (higher terms) as in Example 2.14(5). Then $g(x', y') = x'^3 + y'^2 +$ (higher terms). Thus $\operatorname{length}_Q g = 3$. Therefore, $V_{\mathbb{C}}(g)$ has 12 zeros or infinitely near zeros in $\pi^{-1}(P)$.

(4) Consider the case $P \in L_x \cup L_y \cup L_z$ and P is the cusp of $V_{\mathbb{C}}(f)$. Assume that $P = (1 \colon p^2 \colon 0)$ ($p > 0$), Then $g(x', y') = x'^6 + y'^2 +$ (higher terms). Thus $\operatorname{length}_Q g = 3$. Therefore, $V_C(g)$ has 6 zeros or infinitely near zeros in $\pi^{-1}(P)$.

(5) Consider the case $P \in L_x \cup L_y \cup L_z$ and $P \in \operatorname{Reg}(V_{\mathbb{C}}(f))$. Then $f(x, y) = ay + bx^2 +$ (higher terms) ($a > 0$, $b > 0$) as in Example 2.14(3). Since $g(x', y') = ay'^2 + bx'^2 +$ (higher terms), we have $\operatorname{length}_Q g = 1$. Thus, $V_C(g)$ has 2 acnodes in $\pi^{-1}(P)$.

(6) Consider the case $P \in \{P_{x_0}, P_{x_1}, P_{x_2}\}$ and $P \in \operatorname{Reg}(V_{\mathbb{C}}(f))$. Let $x := x_1/x_0$ and $y := x_2/x_0$. We may assume that $f(x, y) = ax + by^m +$ (higher terms) with $a > 0$, $b > 0$. Then $g(x, y) = ax^2 + by^{2m} +$ (higher terms). Thus, $\operatorname{length}_Q g = \operatorname{length}_P f$.

(7) Consider the case that $g = g_1 g_2$ is reducible in $\mathbb{C}[x, y, z]$.

Then $\pi(V_{\mathbb{C}}(g_1)) \cup \pi(V_{\mathbb{C}}(g_2)) = \pi(V_{\mathbb{C}}(g)) = V_{\mathbb{C}}(f)$. Since $V_{\mathbb{C}}(f)$ is an irreducible cubic curve, $V_{\mathbb{C}}(g_1)$ and $V_{\mathbb{C}}(g_2)$ are irreducible cubic curves. Thus, we may assume that there exists $g_3 \in \mathbb{C}[x, y, z]_{(3)} - \mathbb{R}[x, y, z]$ such that $g_1 = g_3$ and $g_2 = \overline{g_3}$. g_3 is irreducible in $\mathbb{C}[x, y, z]$, because $V_{\mathbb{C}}(g_3)$ is irreducible.

The intersection number of $V_{\mathbb{C}}(g_3)$ and $V_{\mathbb{C}}(\overline{g_3})$ is equal to 9. So, $N(f) \leq 9$.

(8) Consider the case $N(f) \geq 10$.

Then, g is irreducible, and the arithmetic genus of $V_{\mathbb{C}}(g)$ satisfies

$$10 = \frac{(\deg g - 1)(\deg g - 2)}{2} \geq \sum_{i=1}^{n} \frac{\nu(Q_i')(\nu(Q_i') - 1)}{2} \geq n.$$

So, $N(f) = 10$. Then $g \in \mathcal{E}(\mathcal{P}_{3,6})$ by Theorem 2.17. $\square$

Lemma 3.11. *Assume that $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ is irreducible, and P is the acnode of $V_{\mathbb{C}}(f)$. Then $P \notin \partial\mathbb{P}^2_+$.*

Proof. Assume that P is an acnode of $V_{\mathbb{C}}(f)$ and $P \in \partial\mathbb{P}^2_+$. Then, $P \in L_x \cup L_y \cup L_z$ by Lemma 3.5 and 3.6. We may assume $P = (0\colon a\colon 1) \in L_x$ $(a > 0)$. If we replace y by y/a, we may assume $P = (0\colon 1\colon 1)$. $f(P_y) = f(P_z) = 0$ is impossible by Lemma 3.7. So, we may further assume $f(P_y) > 0$ by the symmetry.

(1) Consider the case $Q_z = (1\colon r\colon 0) \in V_+(f)$ $(r > 0)$.

Then $V_+(f) \subset \{P, Q_z\} \cup V_+(y)$ by Lemma 3.7. In Notation 3.9, we put $P_1 = P$ and $P_2 = Q_z$. Let $G(x, y, z) := y(rx - y - z)^2$. Then, $G_x = 2ry(rx - y - z)$, $G_{xx} = 2r^2y$, $G_y = (rx - y - z)(rx - 3y - z)$, $G_z = -2y(rx - y - z)$ and $G_{zz} = 2y$.

For $Q \in \{P, Q_z\} = \{P_1, P_2\}$, we have $G(Q) = G_x(Q) = G_y(Q) = G_z(Q) = 0$. Thus, $G \in \mathcal{L}_{P_1} \cap \mathcal{L}_{P_2}$.

For any point $Q \in V_+(y)$, we have $G(Q) = G_x(Q) = G_{xx}(Q) = G_z(Q) = G_{zz}(Q) = 0$. Note that $I_{P_x}(f, z) \leq 1$ by Lemma 3.7. Thus, if $P_x \in V_+(y)$, $\mathcal{L}_{P_x}$ is one of the following:

$$\begin{aligned}
\mathcal{L}_1 &:= \left\{g \in \mathcal{P}^+_{3,3} \;\middle|\; g(P_x) = 0\right\},\\
\mathcal{L}_2 &:= \left\{g \in \mathcal{P}^+_{3,3} \;\middle|\; g(P_x) = g_z(P_x) = 0\right\},\\
\mathcal{L}_3 &:= \left\{g \in \mathcal{P}^+_{3,3} \;\middle|\; g(P_x) = g_z(P_x) = g_{zz}(P_x) = 0\right\}.
\end{aligned}$$

Since $G(P_x) = G_z(P_x) = G_{zz}(P_x) = 0$, we have $G \in \mathcal{L}_i$ $(i = 1, 2, 3)$. Thus, $G \in \mathcal{L}_{P_x}$ if $P_x \in V_+(f)$.

Similarly, since $I_{P_z}(f, x) \leq 1$, we have $G \in \mathcal{L}_{P_z}$ if $P_z \in V_+(f)$.

If $Q \in L_y \cap V_+(f)$, then

$$\mathcal{L}_Q = \left\{g \in \mathcal{P}^+_{3,3} \;\middle|\; g(Q) = g_z(Q) = 0\right\}.$$

Thus, $G \in \mathcal{L}_Q$.

Since $V_+(f) \subset \{P, Q_z\} \cup V_+(y)$, we have $G \in \mathcal{L}_{P_1} \cap \cdots \cap \mathcal{L}_{P_r} = \mathbb{R}_+ \cdot f$. This implies f is a multiple of G and f is reducible. A contradiction.

(2) Consider the case $V_+(f) \cap L_z = \emptyset$.

Then $V_+(f) \subset \{P\} \cup V_+(y)$. Let $H(x, y, z) := y(y - z)^2$. Then, $H_x = 0$, $H_y = (y - z)(3y - z)$, $H_{yy} = 2(3y - 2z)$, $H_z = -2y(y - z)$ and $H_{zz} = 2y$. Thus, for any point $Q \in V_+(y)$, we have $H(Q) = H_x(Q) = H_{xx}(Q) = H_z(Q) = H_{zz}(Q) = 0$. Thus, if $Q \in V_+(f) \cap (L_y \cup \{P_z\})$, then $H \in \mathcal{L}_Q$.

Moreover, $H_y(P_x) = H_{yy}(P_x) = 0$. So, $H \in \mathcal{L}_{P_x}$. Therefore $H \in \mathcal{L}_{P_1} \cap \cdots \cap \mathcal{L}_{P_r} = \mathbb{R}_+ \cdot f$. A contradiction. $\square$

By Lemma 3.5— 3.11, if $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ is irreducible, $V_{\mathbb{C}}(f)$ has an acnode in $\mathrm{Int}(\mathbb{P}^2_+)$ or a cusp on $\partial\mathbb{P}^2_+ - \{P_x, P_y, P_z\}$.

3.3. The case $V_{\mathbb{C}}(f)$ has a cusp.

From this subsection, many complicated calculations appear. In most of them, we use the software Mathematica. The code for Mathematica can be found on the link of the authors WEB or in arXiv's anc folder.

Lemma 3.12. *Let $p \geq 0$, $q \geq 0$ be constants with $(p,\, q) \neq (0,\, 0)$, and $P := (0\colon 1\colon 1) \in L_x \subset \mathbb{P}^2_+$. Let*

$$\mathcal{H}_c := \left\{ g(x,y,z) \in \mathcal{H}_{3,3} \;\middle|\; \begin{array}{l} g(P) = g_x(P) = g_y(P) = g_{xx}(P) = g_{xy}(P) = 0, \\ g(p,0,1) = g_x(p,0,1) = g(q,1,0) = g_y(q,1,0) = 0 \end{array} \right\}.$$

Then $\mathcal{H}_c = \mathbb{R} \cdot \mathfrak{h}_{pq}$, where

$$\begin{aligned} \mathfrak{h}_{pq}(x,y,z) := {} & 2x^3 + 3(p-q)x^2y - 6pqxy^2 + q^2(3p+q)y^3 + 3(q-p)x^2z - 6pqxz^2 \\ & + (p^3 + 36p^2q - 6pq^2 - 2q^3)y^2z + (-2p^3 - 6p^2q + 3pq^2 + q^3)yz^2 \\ & + p^2(p+3q)z^3 + 12pqxyz. \end{aligned}$$

Moreover, $\mathfrak{h}_{pq} \in \mathcal{E}(\mathcal{P}^+_{3,3})$ and $V_{\mathbb{C}}(\mathfrak{h}_{pq})$ is irreducible. Moreover, $\mathfrak{h}_{pq}(x^2,\, y^2,\, z^2) \in \mathcal{E}(\mathcal{P}_{3,6})$.

Proof. Let $\{e_1, \ldots, e_{10}\} = \{x^3, \ldots, xyz\}$ be all the monomials of $\mathcal{H}_{3,3}$. Take $g = a_1e_1 + \cdots + a_{10}e_{10} \in \mathcal{H}_{3,3}$, and let $\mathbf{a}$ be the column vector ${}^t(a_1, \ldots, a_{10})$. A differential equation $g(0,\, 1,\, 1) = g_x(0,\, 1,\, 1) = g_y(0,\, 1,\, 1) = g_{xx}(0,\, 1,\, 1) = g_{xy}(0,\, 1,\, 1) = g(p,\, 0,\, 1) = g_x(p,\, 0,\, 1) = g(q,\, 1,\, 0) = g_y(q,\, 1,\, 0) = 0$ can be written as $A\mathbf{a} = \mathbf{0}$ for a certain 9×10 matrix A. Using Mathematica, we can check that $\operatorname{Ker} A = \mathbb{R} \cdot \mathfrak{h}_{pq}$. If $\mathfrak{h}_{pq}$ is PSD on $\mathbb{P}^2_+$, then $\mathfrak{h}_{pq} \in \mathcal{E}(\mathcal{P}^+_{3,3})$ by Proposition 2.13.

(1) Solving the equation $\mathfrak{h}_{pq}(x(t),\, 1,\, 1 + tx(t)) = 0$, then we have

$$x(t) = -\frac{t^2(p+q)^3}{(pt-1)^2((p+3q)t+2)}.$$

Put $z(t) := 1 + tx(t) = -\dfrac{(qt+1)^2((3p+q)t-2)}{(pt-1)^2((p+3q)t+2)}$. Then $\mathfrak{h}_{pq}(x(t),\, 1,\, z(t)) = 0$ for all $t \in \mathbb{C}$. Thus $V_{\mathbb{C}}(\mathfrak{h}_{pq})$ is an irreducible rational curve which has a cusp at P.

(2) We shall show that $V_{\mathbb{R}}(\mathfrak{h}_{pq}) \cap \operatorname{Int}(\mathbb{P}^2_+) = \emptyset$. It is enough to show that if $x(t) > 0$ then $y(t) \leq 0$.

If $x(t) > 0$, then $(p+3q)t + 2 < 0$ and $t \leq -\dfrac{2}{p+3q}$. Then

$$(3p+q)t - 2 \leq -(3p+q)\frac{2}{p+3q} - 2 = -\frac{8(p+q)}{p+3q} < 0.$$

Thus $z(t) \leq 0$.

(3) Since $\mathfrak{h}_{pq}(1,\, 1,\, 1) = 2p^3q^3 > 0$, we have $\mathfrak{h}_{pq} \in \mathcal{P}^+_{3,3}$. Put $P_2 = (p\colon 0\colon 1)$ and $P_3 = (q\colon 1\colon 0)$. Then $N_P(\mathfrak{h}_{pq}) + N_{P_1}(\mathfrak{h}_{pq}) + N_{P_2}(\mathfrak{h}_{pq}) = 6 + 4 + 4 = 10$. Thus $\mathfrak{h}_{pq}(x^2,\, y^2,\, z^2) \in \mathcal{E}(\mathcal{P}_{3,6})$ by Lemma 3.10. $\square$

Theorem 3.13. *Assume that $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ is irreducible, and $V_{\mathbb{C}}(f)$ has a cusp at $P \in \mathbb{P}^2_{\mathbb{C}}$. Then $P \in L_x \cup L_y \cup L_z$.*

Assume that $P = (0\colon 1\colon 1)$. Then, there exists $p \geq 0$, $q \geq 0$ and $c > 0$ such that $(p,\, q) \neq (0,\, 0)$ and $f = c\mathfrak{h}_{pq}$.

Proof. Let P be a cusp of $V_{\mathbb{C}}(f)$. Then $P \in L_x \cup L_y \cup L_z$, by Lemmmas 3.5, 3.6 and 3.10. Assume that $P = (0\colon 1\colon 1)$. By Example 2.14(5),

$$\mathfrak{L}_P = \{g(x,y,z) \in \mathfrak{P}^+_{3,3} \mid g(P) = g_x(P) = g_y(P) = g_{xx}(P) = g_{xy}(P) = 0\}.$$

We consider conditions :

(Cy) $\exists P_2 = (p\colon 0\colon 1) \in V_+(f)$ or "$I_{P_2}(f,\, y) \geq 2$ for $P_2 = (0\colon 0\colon 1)$ $(p := 0)$".

(Cz) $\exists P_3 = (q\colon 1\colon 0) \in V_+(f)$ or "$I_{P_3}(f,\, y) \geq 2$ for $P_3 = (0\colon 1\colon 0)$ $(q := 0)$".

(1) Consider the case (Cy) and (Cz) are true. Then, $f = c\mathfrak{h}_{pq}$ $(p \geq 0,\, q \geq 0,\, c > 0)$ by Lemma 3.12.

(2) Consider the case (Cy) is true, (Cz) is false and $f(P_y) > 0$.

Let $G(x,y,z) := y(y-z)^2$. Then, $G_x = 0$, $G_{xx} = 0$, $G_{xy} = 0$, $G_z = -2y(y-z)$ and $G_{zz} = 2y$. Thus $G \in \mathfrak{L}_P$. Since $G(P_2) = G_x(P_2) = 0$, we have $G \in \mathfrak{L}_{P_2}$.

Note that $V_+(f) \subset \{P,\, P_2,\, P_x,\, P_z\}$ and $I_{P_z}(f,\, x) \leq 1$. Since $G(P_z) = G_x(P_z) = G_{xx}(P_z) = 0$, we have $G \in \mathfrak{L}_{P_z}$. Since $G(P_x) = G_z(P_x) = G_{zz}(P_x) = 0$, we have $G \in \mathfrak{L}_{P_x}$. Thus $G \in \mathfrak{L}_f = \mathbb{R}_+ \cdot f$. A contradiction.

(3) Consider the case (Cy) is true, (Cz) is false and $f(P_y) = 0$. Let

$$\mathfrak{H}_1 := \left\{ g(x,y,z) \in \mathfrak{H}_{3,3} \;\middle|\; \begin{array}{l} g(P) = g_x(P) = g_y(P) = g_{xx}(P) = g_{xy}(P) = 0, \\ g(p,0,1) = g_x(p,0,1) = g(P_y) = 0 \end{array} \right\}.$$

Then $\dim_{\mathbb{R}} \mathfrak{H}_1 = 2$ and $\mathfrak{P}^+_{3,3} \cap \mathfrak{H}_1 = \mathbb{R}_+ \cdot \mathfrak{h}_{p,0} + \mathbb{R}_+ \cdot x(x+py-pz)^2$. Since f is irreducible, $\mathfrak{L}_f = \mathbb{R}_+ \cdot \mathfrak{h}_{p,0}$.

(4) Consider the case (Cy) and (Cz) are false.

$f(P_y) = f(P_z) = 0$ is impossible by Lemma 3.7. We may assume $f(P_y) > 0$. Then $G(x,y,z) = y(y-z)^2 \in \mathfrak{L}_f = \mathbb{R}_+ \cdot f$ as (2). A contradiction. $\square$

3.4. The case $V_{\mathbb{C}}(f)$ has a node.

Assume that $f \in \mathfrak{E}(\mathfrak{P}^+_{3,3})$ is irreducible and $V_{\mathbb{C}}(f)$ does not have a cusp. Then $V_{\mathbb{C}}(f)$ has the unique node P on $\mathbb{P}^2_+$ by Lemma 3.5. P must be an acnode. By Lemma 3.11, $P \in \mathrm{Int}(\mathbb{P}^2_+)$. Let $P = (a\colon b\colon 1)$ $(a > 0,\, b > 0)$. Note that if we replace the coordinate system $(x\colon y\colon z)$ by $(x/a\colon y/b\colon z)$, we can assume $P = (1\colon 1\colon 1)$.

Lemma 3.14. *Put* $Q_x := (0\colon 1\colon p)$, $Q_y := (1\colon 0\colon q)$, $P := (1\colon 1\colon 1)$ *and*

$$\begin{aligned} \mathfrak{g}_{pq}(x,y,z) := {} & z^3 + q^2x^2z + p^2y^2z - 2qxz^2 - 2pyz^2 - (p^2+q^2-4p-4q+3)xyz \\ & + (1-p+q)(1-p-q)x^2y + (1+p-q)(1-p-q)xy^2. \end{aligned}$$

(1) *Assume that* $p > 0$ *and* $q > 0$. *If* $f \in \mathfrak{P}^+_{3,3}$ *satisfies*

$$f(P) = f(Q_x) = f(Q_y) = f(P_x) = f(P_y) = 0,$$

then there exists $\alpha \in \mathbb{R}$ *such that* $f = \alpha\mathfrak{g}_{pq}$.

(2) *Assume that* $p > 0$ *and* $q = 0$. *If* $f \in \mathfrak{H}_{3,3}$ *satisfies*

$$f(P) = f(Q_x) = f(P_x) = f(P_y) = f_z(P_y) = f_{zz}(P_y) = 0,$$

then there exists $\alpha \in \mathbb{R}$ *such that* $f = \alpha\mathfrak{g}_{p0}$.

(3) *Assume that* $p = 0$ *and* $q = 0$. *If* $f \in \mathfrak{H}_{3,3}$ *satisfies*

$$f(P) = f(P_x) = f_z(P_x) = f_{zz}(P_x) = f(P_y) = f_z(P_y) = f_{zz}(P_y) = 0,$$

then there exists $\alpha \in \mathbb{R}$ *such that* $f = \alpha\mathfrak{g}_{00}$.

Proof. (1) Since $f(1,1,1) = f(0,1,p) = f(1,0,q) = f(1,0,0) = f(0,1,0) = 0$ and $f \in \boldsymbol{\mathcal{H}}_{3,3}$, f must satisfies $f(1,1,1) = 0$, $f_x(1,1,1) = 0$, $f_y(1,1,1) = 0$, $f(0,1,p) = 0$, $f_y(0,1,p) = 0$, $f(1,0,q) = 0$, $f_x(1,0,q) = 0$, $f(1,0,0) = 0$, and $f(0,1,0) = 0$. Using Mathematica, we have that the solution space is equal to $\mathbb{R} \cdot \mathfrak{g}_{pq}(x,y,z)$. This implies

$$\boldsymbol{\mathcal{L}}_P \cap \boldsymbol{\mathcal{L}}_{Q_x} \cap \boldsymbol{\mathcal{L}}_{Q_y} \cap \boldsymbol{\mathcal{L}}_{P_x} \cap \boldsymbol{\mathcal{L}}_{P_y} \subset \mathbb{R}_+ \cdot \mathfrak{g}_{pq}.$$

If $\mathfrak{g}_{pq} \in \boldsymbol{\mathcal{P}}^+_{3,3}$, then $\boldsymbol{\mathcal{L}}_f = \mathbb{R}_+ \cdot \mathfrak{g}_{pq}$, and $\mathfrak{g}_{pq} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$.

We can prove (2) and (3) by the similay way as (1). ◻

Theorem 3.15. *Assume that $p \geq 0$ and $q \geq 0$.*
(1) *If $p+q > 1$, then $\mathfrak{g}_{pq} \notin \boldsymbol{\mathcal{P}}^+_{3,3}$.*
(2) *If $p+q = 1$, then $\mathfrak{g}_{pq}(x,y,z) = (px+qy-z)^2 z$.*
(3) *If $p+q < 1$, $p \geq 0$ and $q \geq 0$, then $\mathfrak{g}_{pq} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$. Moreover, $\mathfrak{g}_{pq}(x^2,\, y^2,\, z^2) \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{3,6})$.*

Proof. (1) Assume that $p+q > 1$. Let $a := 1/(2(p+q-1)) > 0$. Since

$$\mathfrak{g}_{pq}(x,x,1) = (x-1)^2\big(1 - 2(p+q-1)x\big),$$

we have $\mathfrak{g}_{pq}(x,x,1) < 0$ if $x > a$. Thus $\mathfrak{g}_{pq} \notin \boldsymbol{\mathcal{P}}^+_{3,3}$.

(2) is easy to see.

(3) We shall show that $\mathfrak{g}_{pq} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ if $p+q < 1$.

Let $g(x,y,z) := (1-p+q)x + (1+p-q)y - 2z$. Note that $g(1,1,1) = 0$. Fix $(x\colon y\colon z) \in \mathbb{P}^2_+$.

Consider the case $g(x,y,z) \geq 0$. Then $x \geq z$ or $y \geq z$, since $g(1,1,1) = 0$. If $x \geq z$, then

$$\mathfrak{g}_{pq}(x,y,z) = (1-p-q)y(x-z)g(x,y,z) + \big(qx + (1-q)y - z\big)^2 z \geq 0. \tag{3.15.1}$$

If $y \geq z$, then

$$\mathfrak{g}_{pq}(x,y,z) = (1-p-q)x(y-z)g(x,y,z) + \big((1-p)x + py - z\big)^2 z \geq 0. \tag{3.15.2}$$

Consider the case $g(x,y,z) < 0$. Then $x \leq z$ or $y \leq z$, since $g(1,1,1) = 0$. If $x \leq z$, then $\mathfrak{g}_{pq}(x,y,z) \geq 0$ by (3.15.1). If $y \leq z$, then $\mathfrak{g}_{pq}(x,y,z) \geq 0$ by (3.15.2). Thus $\mathfrak{g}_{pq}(x,y,z) \geq 0$ for all $(x\colon y\colon z) \in \mathbb{P}^2_+$, if $p+q < 1$.

It $p+q \leq 1$, $p \geq 0$ and $q \leq 0$, then $\mathfrak{g}_{pq} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ by Lemma 3.14.

(3-2) We shall show that $\mathfrak{g}_{pq}$ is irreducible if $p+q < 1$. Then $V_{\mathbb{R}}(f) - \{P\}$ has a parametrization

$$x(t) := -\frac{(1-q+qt)^2}{t(1-p-q)((1+p-q)+t(1-p+q))},$$
$$y(t) := -\frac{(p+t-pt)^2}{(1-p-q)((1+p-q)+t(1-p+q))}.$$

This implies $V_{\mathbb{C}}(f)$ is an irreducible rational cubic curve.

(3-3) $\mathfrak{g}_{pq}(x^2, y^2, z^2) \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{3,6})$ follows from $N_P(\mathfrak{g}_{pq}) + N_{Q_x}(\mathfrak{g}_{pq}) + N_{Q_y}(\mathfrak{g}_{pq}) + N_{P_x}(\mathfrak{g}_{pq}) + N_{P_y}(\mathfrak{g}_{pq}) = 4+2+2+1+1 = 10$. ◻

Note that

$\mathfrak{g}_{p0}(x,y,z) = (1-p)^2x^2y + (1-p)^2xy^2 + p^2x^2z - 2pxz^2 - (1-p)(3-p)xyz + z^3$,
$\mathfrak{g}_{00}(x,y,z) = x^2y + xy^2 + z^3 - 3xyz$.

Definition 3.16. We define $\mathfrak{f}_{pqr}(x,y,z)$ as the following:

$$
\begin{aligned}
&a_1(p,q) := pq - p + 1,\\
&a_2(p,q,r) := p^2qr - p^2r + pqr - pq + 2pr + p - r + 1,\\
&c_1(p,q,r) := q^2 a_1(p,q)a_1(r,p)a_2(p,q,r),\\
&c_2(p,q,r) := -a_1(p,q)(2p^3q^3r^3 - 2p^3q^2r^3 + 6p^2q^2r^3 - 2pq^3r^3 + 3pq^3r^2 - 6pq^2r^3\\
&\qquad\qquad + 3pq^2r^2 + 2q^2r^3 - pq^3 - 3q^2r^2 + 3pq^2 - 3pq + q^2 + p - 1),\\
&c_3(p,q,r) := ra_1(p,q)(p^3q^3r^3 - p^3q^2r^3 + 3p^2q^2r^3 - pq^3r^3 - 3pq^2r^3 + 3pq^3r + q^2r^3\\
&\qquad\qquad - 2pq^3 - 3pq^2r + 6pq^2 - 3q^2r - 6pq + 2q^2 + 2p - 2),\\
&c_4(p,q,r) := -c_1(p,q,r) - c_1(q,r,p) - c_1(r,p,q) - c_2(p,q,r) - c_2(q,r,p) - c_2(r,p,q)\\
&\qquad\qquad - c_3(p,q,r) - c_3(q,r,p) - c_3(r,p,q),\\
&\mathfrak{f}_{pqr}(x,y,z) := c_1(p,q,r)x^3 + c_1(q,r,p)y^3 + c_1(r,p,q)z^3\\
&\qquad\qquad + c_2(p,q,r)x^2y + c_3(p,q,r)xy^2 + c_2(q,r,p)y^2z + c_3(q,r,p)yz^2\\
&\qquad\qquad + c_2(r,p,q)z^2x + c_3(r,p,q)zx^2 + c_4(p,q,r)xyz.
\end{aligned}
$$

Lemma 3.17. *Put* $Q_x := (0\colon p\colon 1)$, $Q_y := (1\colon 0\colon q)$, $Q_z := (r\colon 1\colon 0)$, $P := (1\colon 1\colon 1)$.

(1) *Assume that* $p > 0$, $q > 0$ *and* $r > 0$. *If* $f \in \mathfrak{P}^+_{3,3}$ *satisfies*
$$f(P) = f(Q_x) = f(Q_y) = f(Q_z) = 0,$$
then there exists $\alpha \in \mathbb{R}$ *such that* $f = \alpha\mathfrak{f}_{pqr}$.

(2) *Assume that* $p > 0$, $q > 0$ *and* $r = 0$. *If* $f \in \mathfrak{P}^+_{3,3}$ *satisfies*
$$f(P) = f(Q_x) = f(Q_y) = f(P_y) = f_x(P_y) = 0,$$
then there exists $\alpha \in \mathbb{R}$ *such that* $f = \alpha\mathfrak{f}_{p,q,0}$.

(3) *Assume that* $p > 0$, $q = 0$ *and* $r = 0$. *If* $f \in \mathfrak{P}^+_{3,3}$ *satisfies*
$$f(P) = f(Q_x) = f(P_x) = f_z(P_x) = f(P_y) = f_x(P_y) = 0,$$
then there exists $\alpha \in \mathbb{R}$ *such that* $f = \alpha\mathfrak{f}_{p,0,0}$.

(4) *Assume that* $p = 0$, $q = 0$ *and* $r = 0$. *If* $f \in \mathfrak{P}^+_{3,3}$ *satisfies*
$$f(P) = f(P_z) = f_y(P_z) = f(P_x) = f_z(P_x) = f(P_y) = f_x(P_y) = 0,$$
then there exists $\alpha \in \mathbb{R}$ *such that* $f = \alpha\mathfrak{f}_{0,0,0}$.

Proof. $f \in \mathfrak{P}^+_{3,3}$ and $f(P) = 0$ implies $f_x(P) = f_y(P) = 0$. If $r > 0$, $f(Q_z) = 0$ implies $f(Q_z) = f_x(Q_z) = 0$. So, in any case of (1), (2), (3), (4), f must satisfy $f(1,1,1) = 0$, $f_x(1,1,1) = 0$, $f_y(1,1,1) = 0$, $f(0,p,1) = 0$, $f_y(0,p,1) = 0$, $f(1,0,q) = 0$, $f_z(1,0,q) = 0$, $f(r,1,0) = 0$, $f_x(r,1,0) = 0$, where $p \geq 0$, $q \geq 0$ and $r \geq 0$. We know that the solution of the above equations is a multiple of $\mathfrak{f}_{pqr}(x,y,z)$, using Mathematica. □

We shall study the conditions on p, q, r for $\mathfrak{f}_{pqr} \in \mathfrak{P}^+_{3,3}$.

Lemma 3.18. (I) *Assume that* $p > 0$, $q > 0$ *and* $r > 0$. *Then,* $\mathfrak{f}_{pqr}(1,0,0) > 0$, $\mathfrak{f}_{pqr}(0,1,0) > 0$, *and* $\mathfrak{f}_{pqr}(0,0,1) > 0$, *if and only if* $a_1(p,q) > 0$, $a_1(q,r) > 0$ *and* $a_1(r,p) > 0$.

(II) *Assume that* $p \geq 0$, $q \geq 0$ *and* $r \geq 0$. *Then,* $\mathfrak{f}_{pqr}(1,0,0) \geq 0$, $\mathfrak{f}_{pqr}(0,1,0) \geq 0$, *and* $\mathfrak{f}_{pqr}(0,0,1) \geq 0$, *if and only if* $a_1(p,q) \geq 0$, $a_1(q,r) \geq 0$ *and* $a_1(r,p) \geq 0$.

Proof. (1) Assume that $a_1(p,q) > 0$, $a_1(q,r) > 0$, and $a_1(r,p) > 0$.

Then $a_2(p,q,r) = pra_1(p,q) + pa_1(q,r) + a_1(r,p) > 0$. Thus,

$$\mathfrak{f}_{pqr}(1,0,0) = c_1(p,q,r) = q^2 a_1(p,q) a_1(r,p) a_2(p,q,r) > 0.$$

Similarly, we have $\mathfrak{f}_{pqr}(0,1,0) > 0$ and $\mathfrak{f}_{pqr}(0,0,1) > 0$.

(2) Assume that $\mathfrak{f}_{pqr}(1,0,0) > 0$, $\mathfrak{f}_{pqr}(0,1,0) > 0$, and $\mathfrak{f}_{pqr}(0,0,1) > 0$. We shall derive a contradiction assuming $a_1(p,q) < 0$.

(2-1) we shall show that $p > 1$, $0 < q < 1$, $a_1(q,r) > 0$ and $a_1(r,p) > 0$.

If $p \leq 1$, then $a_1(p,q) = pq + (1-p) \geq 0$. If $q \geq 1$, then $a_1(p,q) = p(q-1) + 1 \geq 0$. Thus, if $a_1(p,q) < 0$, then $p > 1$ and $0 < q < 1$. Then, $a_1(q,r) = qr + (1-q) > 0$, and $a_1(r,p) = r(p-1) + 1 > 0$.

Let

$$\begin{aligned}
b_1(p,q) &:= -p^2 q + p^2 - pq - 2p + 1 = (p-1)^2 - p(p+1)q, \\
b_2(p,q) &:= p^2 q^2 - 2p^2 q - 2pq + p^2 - 2p + 1, \\
r_0(p,q) &:= \frac{1 + p(1-q)}{b_1(p,q)}, \\
r_2(p,q) &:= \frac{p(1-q)^2 - 1 - q}{q(pq + (p-1))}.
\end{aligned}$$

(2-2) We shall show that $b_1(p,q) > 0$ and $r_0(p,q) < r < r_2(p,q)$. Since

$$\begin{aligned}
0 &< \mathfrak{f}_{pqr}(1,0,0) = q^2 a_1(p,q) a_1(r,p) a_2(p,q,r), \\
0 &< \mathfrak{f}_{pqr}(0,1,0) = r^2 a_1(q,r) a_1(p,q) a_2(q,r,p), \\
0 &< \mathfrak{f}_{pqr}(0,0,1) = p^2 a_1(r,p) a_1(q,r) a_2(r,p,q),
\end{aligned}$$

we have $a_2(p,q,r) < 0$, $a_2(q,r,p) < 0$ and $a_2(r,p,q) > 0$. Note that $a_2(p,q,0) = 1 + p(1-q) > 0$. Since $0 > a_2(p,q,r) = -b_1(p,q)r + a_2(p,q,0)$, we have $b_1(p,q) > a_2(p,q,0)/r > 0$. Thus, $a_2(p,q,r)$ is monotonically decreasing on r. The equation $a_2(p,q,r) = 0$ on r has just one root $r = r_0(p,q)$. Since $a_2(p,q,r) < 0$, we have $r > r_0(p,q)$.

Since $0 > a_2(q,r,p) = rq(pq + (p-1)) + (-pq^2 + 2pq - p + q + 1)$ is monotonically increasing on r, we have $r < r_2(p,q)$. Since $r_0(p,q) < r < r_2(p,q)$, we have

$$0 < r_2(p,q) - r_0(p,q) = \frac{-a_1(p,q) b_2(p,q)}{q(pq + (p-1)) b_1(p,q)}.$$

Thus $b_2(p,q) > 0$.

Remember that $0 < a_2(r,p,q) = (p-1)qr^2 - ((p-1) - (p+2)q)r + (1-q)$. This is a quadratic function on r with $(p-1)q > 0$. Note that

$$\begin{aligned}
a_2\big(r_0(p,q), p, q\big) &= \frac{2a_1(p,q) b_2(p,q)}{\big((p-1)^2 - pq(p+1)\big)^2} < 0, \\
a_2\big(r_2(p,q), p, q\big) &= \frac{2a_1(p,q) b_2(p,q)}{\big((p-1) + pq\big)^2} < 0.
\end{aligned}$$

Thus, if $r_0(p,q) \leq r \leq r_2(p,q)$, then $a_2(r,p,q) < 0$. A contradiction. Thus we have $a_1(p,q) > 0$.

Similarly, we obtain $a_1(q,r) > 0$ and $a_1(r,p) > 0$.

We can obtain (II), if we consider limits $p \to +0$, $q \to +0$ or $r \to +0$ in (I). $\square$

Lemma 3.19. *$\mathfrak{f}_{ppr}$ has the followinf properties:*

(1) $\mathfrak{f}_{pqr}(z,x,y)=\mathfrak{f}_{qrp}(x,y,z)$, $\mathfrak{f}_{pqr}(y,z,x)=\mathfrak{f}_{rpq}(x,y,z)$.
(2) $\mathfrak{f}_{\frac{1}{p}\frac{1}{q}\frac{1}{r}}(x,y,z)=\dfrac{1}{p^4q^4r^4}\mathfrak{f}_{p,r,q}(x,z,y)$.
(3) *If* $a_1(p,q)=0$, *i.e. if* $q=(p-1)/p$, *then*

$$\mathfrak{f}_{p,(p-1)/p,r}(x,y,z)=\frac{(1+(p-1)r)^4z\big((p-1)x+y-pz\big)^2}{p^2}.$$

Proof. These follow from direct calculations using Mathematica. ∎

Theorem 3.20. *Assume that* $p\ge 0$, $q\ge 0$ *and* $r\ge 0$. *Then:*
(1) $\mathfrak{f}_{pqr}\in\boldsymbol{\mathcal{P}}^+_{3,3}$ *if and only if* $a_1(p,q)\ge 0$, $a_1(q,r)\ge 0$ *and* $a_1(r,p)\ge 0$.
(2) $\mathfrak{f}_{pqr}$ *is irreducible if and only if* $a_1(p,q)>0$, $a_1(q,r)>0$ *and* $a_1(r,p)>0$.
(3) *If* $a_1(p,q)>0$, $a_1(q,r)>0$ *and* $a_1(r,p)>0$, *then* $\mathfrak{f}_{pqr}(x^2,\,y^2,\,z^2)\in\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{3,6})$.

Proof. (i) 'Only if part' of (1): Assume that $\mathfrak{f}_{pqr}\in\boldsymbol{\mathcal{P}}^+_{3,3}$. Then $\mathfrak{f}_{pqr}(1,0,0)\ge 0$, $\mathfrak{f}_{pqr}(0,1,0)\ge 0$, $\mathfrak{f}_{pqr}(0,0,1)\ge 0$. Thus $a_1(p,q)\ge 0$, $a_1(q,r)\ge 0$, and $a_1(r,p)\ge 0$, by Lemma 3.18(II).

(ii) 'Only if part' of (2) follows from Lemma 3.19(3).

(iii) 'If part' of (1) and 'if part' of (2):

If $p=q=r=0$, then $\mathfrak{f}_{000}(x,y,z)=x^2y+y^2z+z^2x-3xyz$. In this case, $\mathfrak{f}_{000}\in\boldsymbol{\mathcal{P}}^+_{3,3}$, and $\mathfrak{f}_{000}$ is irreducible. So, we may assume $p\ge 0$, $q\ge 0$ and $r>0$, by Lemma 3.19(1).

(iii-1) Assume that $a_1(p,q)>0$, $a_1(q,r)>0$, and $a_1(r,p)>0$.

Let $\ell_t\subset\mathbb{P}^2_{\mathbb{R}}$ be the line defined by $y-z=t(x-z)$ where $t\in\mathbb{R}$. The intersection point of $V_{\mathbb{R}}(\mathfrak{f}_{pqr})$ and ℓ_t ($\neq(1{:}\,1{:}\,1)$) is given by $P(t):=\big(x_{pqr}(t):y_{pqr}(t):z_{pqr}(t)\big)$, where

$$\begin{aligned}
x_{pqr}(t) &:= a_1(q,r)\big(t+(p-1)\big)^2\Big(r^2a_1(p,q)a_2(q,r,p)t\\
&\qquad -\big((p^2q^2r^2+1)a_1(q,r)+2qra_1(r,p)+2pqr^2a_1(p,q)\big)\Big),\\
y_{pqr}(t) &:= a_1(r,p)\big((1-q)t+q\big)^2\Big(-\big((p^2q^2r^2+1)a_1(r,p)\\
&\qquad +2pra_1(p,q)+2p^2qra_1(q,r)\big)t+a_1(p,q)a_2(p,q,r)\Big),\\
z_{pqr}(t) &:= (rt-1)^2a_1(p,q)\big(a_1(q,r)a_2(q,r,p)t+q^2a_1(r,p)a_2(p,q,r)\big).
\end{aligned}$$

Note that this also implies that $V_{\mathbb{C}}(\mathfrak{f}_{pqr})$ is irreducible. Thus we obtain the 'if part' of (2). We shall show $\mathfrak{f}_{pqr}\in\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$.

(iii-1-1) Consider the case $t\le 0$. Then $x_{pqr}(t)\le 0$ and $y_{pqr}(t)\ge 0$. Thus $P(t)\notin\mathrm{Int}(\mathbb{P}^2_+)$.

(iii-1-2) Consider the case $t>0$. Let

$$\begin{aligned}
t_1 &:= \frac{(p^2q^2r^2+1)a_1(q,r)+2qra_1(r,p)+2pqr^2a_1(p,q)}{r^2a_1(p,q)a_2(q,r,p)},\\
t_2 &:= \frac{a_1(p,q)a_2(p,q,r)}{(p^2q^2r^2+1)a_1(r,p)+2pra_1(p,q)+2p^2qra_1(q,r)}.
\end{aligned}$$

$z_{pqr}(t)\ge 0$ for all $t>0$. If $t<t_1$ then $x_{pqr}(t)\le 0$, and if $t\ge t_1$ then $x_{pqr}(t)\ge 0$. If $t<t_2$ then $y_{pqr}(t)\ge 0$, and if $t\ge t_2$ then $y_{pqr}(t)\le 0$. Note that

$$t_1-t_2=\frac{a_2(r,p,q)\big(a_1(r,p)+pra_1(p,q)+p^2qra_1(q,r)big\big)}{r^2a_1(p,q)a_2(q,r,p)\Big(a_1(r,p)+2pra_1(p,q)+2p^2qra_1(q,r)+p^2q^2r^2a_1(r,p)\Big)}$$

$> 0.$

Thus, for every $t > 0$, at least one of $x_{pqr}(t)$, $y_{pqr}(t)$, $z_{pqr}(t)$ is non-negative, and at least one of $x_{pqr}(t)$, $y_{pqr}(t)$, $z_{pqr}(t)$ is non-positive. Thus $P(t) \notin \mathrm{Int}(\mathbb{P}^2_+)$. Therefore $\mathfrak{f}_{pqr} \in \boldsymbol{\mathcal{P}}^+_{3,3}$. By Lemma 3.17, we have $\mathfrak{f}_{pqr} \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$.

(iii-2) Consider the cases that at least one of $a_1(p,q)$, $a_1(q,r)$, $a_1(r,p)$ is equal to zero. $\mathfrak{f}_{pqr}(x,y,z)$ is contineous with respect to $a_1(p,q)$, $a_1(q,r)$ and $a_1(r,p)$. Thus, the 'if part' of (2) follows from the above result.

(3) $\mathfrak{f}_{pqr}(x^2, y^2, z^2) \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}_{3,6})$ follows from $N_P(\mathfrak{f}_{pqr}) + N_{Q_x}(\mathfrak{f}_{pqr}) + N_{Q_y}(\mathfrak{f}_{pqr}) + N_{Q_z}(\mathfrak{f}_{pqr}) = 4+2+2+2 = 10$, where $Q_x := (0\colon p\colon 1)$, $Q_y := (1\colon 0\colon q)$ and $Q_z := (r\colon 1\colon 0)$ with $p \geq 0$, $q \geq 0$ and $r \geq 0$. ∎

3.5. Final classification of $\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$.

Throughout §3.5, P will stand for $(1\colon 1\colon 1)$.

Corollary 3.21. *Let $Q_x := (0\colon p\colon 1)$, $Q_y := (1\colon 0\colon q)$ and $Q_z := (r\colon 1\colon 0)$ with $p > 0$, $q > 0$, $r > 0$. Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ is irreducible, and satisfies $f(P) = f(Q_x) = f(Q_y) = f(Q_z) = 0$. Then there exists $\alpha > 0$ such that $f = \alpha\mathfrak{f}_{pqr}$. Moreover, $pq - p + 1 > 0$, $qr - q + 1 > 0$ and $rp - r + 1 > 0$ hold.*

Proof. This follows from Lemma 3.17(1) and 3.20(2). ∎

Theorem 3.22. *Let $Q_x = (0\colon p\colon 1)$ and $Q_y = (q\colon 0\colon 1)$. Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ is irreducible, and satisfies $f(P) = f(Q_x) = f(Q_y) = 0$ and $V_+(f) \cap L_z = \emptyset$. Then, one of the following statements holds:*

(1) *$f = \alpha\mathfrak{g}_{\frac{1}{p},\frac{1}{q}}$ $(\exists\alpha > 0)$ where $\dfrac{1}{p} + \dfrac{1}{q} < 1$. In this case $f(P_x) = f(P_y) = 0$.*

(2) *$f(x,y,z) = \alpha\mathfrak{f}_{\frac{1}{p},0,q}(x,z,y)$ $(\exists\alpha > 0)$ where $p > 1$ and $q < \dfrac{p}{p-1}$. In this case $f(P_x) = 0$.*

(3) *$f = \alpha\mathfrak{f}_{p,\frac{1}{q},0}$ $(\exists\alpha > 0)$ where $q > 1$ and $p < \dfrac{q}{q-1}$. In this case $f(P_y) = 0$.*

Proof. Assume that $f \in \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{P}}^+_{3,3})$ is irreducible, $f(P) = f(Q_x) = f(Q_y) = 0$ and $V_+(f) \cap L_z = \emptyset$. In Notation 3.9, we put $P_1 = P$, $P_2 = Q_x$ and $P_3 = Q_y$. Let

$$\boldsymbol{\mathcal{P}}_1 := \left\{ g \in \boldsymbol{\mathcal{P}}^+_{3,3} \;\middle|\; \begin{array}{l} g(P) = g_x(P) = g_y(P) = 0, \\ g(Q_x) = g_y(Q_x) = g(Q_y) = g_z(Q_y) = 0 \end{array} \right\}.$$

Note that $\boldsymbol{\mathcal{P}}_1 = \boldsymbol{\mathcal{L}}_{P_1} \cap \boldsymbol{\mathcal{L}}_{P_2} \cap \boldsymbol{\mathcal{L}}_{P_3}$. A direct calculation usung Mathematica shows that $\dim \boldsymbol{\mathcal{P}}_1 = 3$. Thus, $\{P_x, P_y, P_z\} \cap V_+(f) \neq \emptyset$. Then, there are following four cases (i)-(iv).

(i) The case $f(P_z) = 0$.

Then $f(P_x) > 0$ and $f(P_y) > 0$ by Lemma 3.7. Moreover $I_{P_z}(f,\, x) = 1$ and $I_{P_z}(f,\, y) = 1$. Thus $V_+(f) = \{P,\, Q_x,\, Q_y,\, P_z\}$. $\boldsymbol{\mathcal{P}}_2 := \boldsymbol{\mathcal{P}}_1 \cap (\boldsymbol{\mathcal{P}}^+_{3,3})_{P_z}$ is a two dimensional fan whose edges are generated by extremal elements $y((p-1)x + y - pz)^2$ and $x(qx + (1-q)y - z)^2$. Thus $\boldsymbol{\mathcal{P}}_2 \not\ni \mathbb{R}_+ \cdot f$. Therefore, we have $f(P_z) > 0$.

Note that $I_{P_x}(f,\, y) \leq 1$, $I_{P_y}(f,\, x) \leq 1$ by Lemma 3.7.

(ii) The case $f(P_x) = f(P_y) = 0$. Then, there exists $\alpha \in \mathbb{R}$ such that $f = \alpha\mathfrak{g}_{\frac{1}{p},\frac{1}{q}}$, by Lemma 3.14(1).

(iii) The case $f(P_x) = 0$ and $f(P_y) > 0$. Let

$$\mathfrak{P}_3 := \{g \in \mathfrak{P}_1 \mid g(P_x) = g_y(P_x) = 0\}.$$

Then $\mathfrak{P}_3 = \mathbb{R}_+ \cdot \mathfrak{f}_{\frac{1}{p},0,q}(x, z, y)$ by Lemma 3.17. Since $I_{P_x}(f, y) \leq 1$, we have $g_z(P_x) \neq 0$. Note that if $g_x(P_x) = 0$ then $g_y(P_x) = 0$ by Remark 2.15. This implies that $\mathbb{R}_+ \cdot f = \mathfrak{L}_f \supset \mathfrak{P}_3$. Therefore, $f(x, y, z) = \alpha\mathfrak{f}_{\frac{1}{p},0,q}(x, z, y)$.

(iv) The case $f(P_x) > 0$ and $f(P_y) = 0$. Same as (ii). ∎

Assume that $f \in \mathfrak{E}(\mathfrak{P}^+_{3,3})$ is irreducible and satisfies $f(0,\, p,\, 1) = 0$ for $p > 0$. Then, $f(P_y) > 0$ or $f(P_z) > 0$ by Lemma 3.7.

Theorem 3.23. *Let $Q_x = (0\colon p\colon 1)$, Assume that $f \in \mathfrak{E}(\mathfrak{P}^+_{3,3})$ is irreducible, and satisfies $f(P) = f(Q_x) = 0$, $f(P_z) > 0$, $V_+(f) \cap L_y = \emptyset$ and $V_+(f) \cap L_z = \emptyset$. Then, $f(P_x) = f(P_y) = 0$ and one of the following statements holds:*
(1) $f = \alpha\mathfrak{g}_{\frac{1}{p},0}$ *($\exists\alpha > 0$).*
(2) $f(x, y, z) = \alpha\mathfrak{f}_{p,0,0}(x, y, z)$ *($\exists\alpha > 0$) where $p < 1$.*

Proof. Assume that $f \in \mathfrak{E}(\mathfrak{P}^+_{3,3})$ is irreducible, and $V_+(f) \subset \{P,\, Q_x,\, P_x,\, P_y\}$.

In Notation 3.9, we put $P_1 = P$ and $P_2 = Q_x$. Let

$$\mathfrak{P}_1 := \left\{ g \in \mathfrak{P}^+_{3,3} \;\middle|\; \begin{array}{l} g(P) = g_x(P) = g_y(P) = 0, \\ g(Q_x) = g_y(Q_x) = 0 \end{array} \right\}.$$

Note that $\mathfrak{P}_1 = \mathfrak{L}_{P_1} \cap \mathfrak{L}_{P_2}$.

For $u \neq v \in \{x,\, y,\, z\}$, we denote $I_{uv} := I_{P_u}(f,\, v)$. For example, $I_{xy} = I_{P_x}(f,\, y)$. Let $I_x := \max\{I_{xy},\, I_{xz}\}$, $I_y := \max\{I_{yz},\, I_{yx}\}$ and $I_z := \max\{I_{zx},\, I_{zy}\}$. Note that $I_x = \operatorname{length}_{P_x} f$, $I_y = \operatorname{length}_{P_y} f$ and $I_z = \operatorname{length}_{P_z} f$.

(i) Consider the case $I_x + I_y = 4$. Then $(I_x,\, I_y) = (3,\, 1)$, $(2,\, 2)$ or $(1,\, 3)$.

(i-1) The case $I_x = 3$ and $I_y = 1$.

Since $I_{xz} + I_{yz} \leq \deg f = 3$ and $I_{yz} \neq 0$, we have $I_{xz} \leq 2$. Thus $I_{xy} = 3$ and $I_{xz} = 1$. Let

$$\mathfrak{P}_2 := \{g \in \mathfrak{P}_1 \mid f(P_x) = f_z(P_x) = f_{zz}(P_x) = f(P_y) = 0\}.$$

Then $\mathfrak{P}_2 = \mathbb{R}_+ \cdot \mathfrak{g}_{1/p,0}$ by Lemma 3.14(2). Since $\mathbb{R}_+ \cdot f = \mathfrak{L}_f \subset \mathfrak{P}_2$, we have $f = \alpha\mathfrak{g}_{1/p,0}$ ($\exists\alpha > 0$).

(i-2) The case $I_x = I_y = 2$.

Since $I_{yx} = 1$, we have $I_{yz} = 2$. Since $I_{yz} + I_{xz} \leq 3$, we have $I_{xz} = 1$ and $I_{xy} = 2$. Let

$$\mathfrak{P}_3 := \{g \in \mathfrak{P}_1 \mid f(P_x) = f_z(P_x) = f(P_y) = f_x(P_y) = 0\}.$$

Then $\mathfrak{P}_3 = \mathbb{R}_+ \cdot \mathfrak{f}_{p,0,0}$ by Lemma 3.17(3). Since $\mathbb{R}_+ \cdot f = \mathfrak{L}_f \subset \mathfrak{P}_3$, we have $f = \alpha\mathfrak{f}_{p,0,0}$ ($\exists\alpha > 0$).

(i-3) The case $I_x = 1$ and $I_y = 3$.

Since $I_{xz} + I_{yz} \leq 3$ and $I_{xz} = I_{xy} = I_x = 1$, we have $I_{yz} = 1$ and $I_{xy} = 3$. Let

$$\mathfrak{P}_4 := \{g \in \mathfrak{P}_1 \mid f(P_x) = f(P_y) = f_x(P_y) = f_{xx}(P_y) = 0\}.$$

Then $\mathfrak{P}_4 = \mathbb{R}_+ \cdot z((p-1)x + y - pz)^2$. Thus, f is reducible.

(ii) Consider the case $I_x + I_y \neq 4$.

If $I_x + I_y \leq 3$, then $\mathfrak{L}_f \supset \mathfrak{P}_2$, $\mathfrak{L}_f \supset \mathfrak{P}_3$ or $\mathfrak{L}_f \supset \mathfrak{P}_4$. If $I_x + I_y \geq 5$, then $\mathfrak{L}_f \subset \mathfrak{P}_2$, $\mathfrak{L}_f \subset \mathfrak{P}_3$ or $\mathfrak{L}_f \subset \mathfrak{P}_4$. In any case, $\mathfrak{L}_f = \mathfrak{P}_2$ or $\mathfrak{L}_f = \mathfrak{P}_3$ or $\mathfrak{L}_f = \mathfrak{P}_4$. ∎

Theorem 3.24. *Assume that $f \in \mathfrak{E}(\mathfrak{P}^+_{3,3})$ is irreducible, and satisfies $f(P) = 0$ where $P = (1\colon 1\colon 1)$. Moreover, we assume that $V_+(f) \subset \{P,\ P_x,\ P_y,\ P_z\}$ and $\mathrm{length}_{P_x} f \geq \mathrm{length}_{P_y} f \geq \mathrm{length}_{P_z} f$. Then one of the following statements holds:*
(1) $f = \alpha\mathfrak{g}_{0,0}$ $(\exists\alpha > 0)$.
(2) $f = \alpha\mathfrak{f}_{0,0,0} = \alpha(x^2y + y^2z + z^2x - 3xyz)$ $(\exists\alpha > 0)$.
(3) $f(x, y, z) = \alpha\mathfrak{f}_{0,0,0}(x, z, y) = \alpha(xy^2 + yz^2 + zx^2 - 3xyz)$ $(\exists\alpha > 0)$.

Proof. Assume that $f \in \mathfrak{E}(\mathfrak{P}^+_{3,3})$ is irreducible, and $V_+(f) \subset \{P,\ P_x,\ P_y,\ P_z\}$. In Notation 3.9, we assume $P_1 = P$. Let I_x, I_y, I_z, I_{xy},..., I_{zy} be same as in the proof of Theorem 3.23. By our assumption, $I_x \geq I_y \geq I_z$. If $I_{xy} \geq 2$, then $I_{xz} = 1$. Note that $I_{yx} + I_{zx} \leq 3$, $I_{xy} + I_{zy} \leq 3$ and $I_{xz} + I_{yz} \leq 3$ by Lemma 3.7. In paticular, $I_x + I_y + I_z \leq 9 - 1 - 1 - 1 = 6$.

(i) Consider the case $I_x + I_y + I_z = 6$. Then $(I_x,\ I_y,\ I_z) = (3,\ 3,\ 0)$, $(3,\ 2,\ 1)$ or $(2,\ 2,\ 2)$. By Lemma 3.7, $(I_x,\ I_y,\ I_z) = (3,\ 2,\ 1)$ is impossible.

(i-1) The case $I_x = I_y = 3$ and $I_z = 0$.
This can possible only if $I_{xy} = I_{yx} = 3$. Thus f is an element of

$$\mathfrak{P}_1 := \left\{g \in \mathfrak{P}^+_{3,3} \;\middle|\; g(P) = g(P_x) = g_z(P_x) = g_{zz}(P_x) = g(P_y) = g_z(P_y) = g_{zz}(P_y) = 0\right\}.$$

$\mathfrak{P}_1 = \mathbb{R}_+ \cdot \mathfrak{g}_{00}$ Lemma 3.14(3). Thus $f = \alpha\mathfrak{g}_{00}$ $(\exists\alpha > 0)$.

(i-2) The case $I_x = I_y = I_z = 2$.
By Lemma 3.14(3), there are two possibilities that $I_{xy} = I_{yz} = I_{zx} = 2$ and $I_{xz} = I_{yx} = I_{zy} = 2$.
If $I_{xy} = I_{yz} = I_{zx} = 2$, then f is an element of

$$\mathfrak{P}_2 := \left\{g \in \mathfrak{P}^+_{3,3} \;\middle|\; g(P) = g(P_x) = g_z(P_x) = g(P_y) = g_x(P_y) = g(P_z) = g_y(P_z) = 0\right\}.$$

Then, $\mathfrak{P}_2 = \mathbb{R}_+ \cdot \mathfrak{f}_{000}$ by Lemma 3.17(4), and we have (2).
If $I_{xz} = I_{yx} = I_{zy} = 2$, then f is an element of

$$\mathfrak{P}_3 := \left\{g \in \mathfrak{P}^+_{3,3} \;\middle|\; g(P) = g(P_x) = g_y(P_x) = g(P_y) = g_z(P_y) = g(P_z) = g_x(P_z) = 0\right\}.$$

Then, $\mathfrak{P}_3 = \mathbb{R}_+ \cdot \mathfrak{f}_{000}(x, z, y)$, and we have (3).

(ii) Consider the case $I_x + I_y + I_z \leq 5$ and $I_{xy} > I_{xz}$.
Then $I_y \leq 2$, $I_z \leq 1$ and $I_{xz} = 1$. Let $G(x, y, z) = y(x - z)^2$. Then $G_x = 2y(x - z)$, $G_y(x - z)^2$, $G_z = -2y(x - z)$, $G_{xx} = G_{zz} = 2y$ and $G_{yy} = 0$. Note that $G_y(P_x) \neq 0$ and $G_y(P_z) \neq 0$.
Since $G(P) = G_x(P) = G_y(P) = G_z(P) = 0$, we have $G \in \mathfrak{L}_P$. Since $G(P_x) = G_z(P_x) = G_{zz}(P_x) = 0$, we have $G \in \mathfrak{L}_{P_x}$. Since $G(P_y) = G_x(P_y) = G_z(P_y) = 0$, we have $G \in \mathfrak{L}_{P_y}$. Since $G(P_z) = 0$, we have $G \in \mathfrak{L}_{P_z}$. Thus $G \in \mathfrak{L}_f = \mathbb{R}_+ \cdot f$. A contradiction.

(iii) Consider the case $I_x + I_y + I_z \leq 5$ and $I_{xy} < I_{xz}$.
As the above argument, we have $G(x, z, y) = z(x - y)^2 \in \mathfrak{L}_f = \mathbb{R}_+ \cdot f$. A contradiction.

(iv) Consider the case $I_x + I_y + I_z \leq 5$ and $I_{xy} = I_{xz}$.
Then $I_x = I_y = I_z = 1$, and Thus $G \in \mathfrak{L}_f = \mathbb{R}_+ \cdot f$.
Thus, we complete the proof. ∎

3.6. Proofs of Theorems in §1.

Theorem 1.1 follows from Theorem 3.20(2) and Lemma 3.17(1). Theorem 1.2 follows from Theorem 3.15(2), (3) and Lemma 3.14(1). Theorem 1.4 follows frrom Theorem 3.13 and Lemma 3.12.

Proof of Theorem 1.5. (I) If $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ is reducible, then f is (4) or (5) in Theorem 1.5, by Theorem 1.1.

Assume that $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ is irreducible, Then, $V_{\mathbb{C}}(f)$ is a rational curve on $\mathbb{P}^2_{\mathbb{C}}$ whose unique singular point lies on $\mathbb{P}^2_+$.

(II) Consider the case that $V_{\mathbb{C}}(f)$ has a cusp. Then $P \in L_x \cup L_y \cup L_z$ by Theorem 3.13. If $P = (0\colon a\colon 1) \in L_x$ $(a > 0)$, put $f_1(x, y, z) = f(x, ay, z)$. Then, $V_{\mathbb{C}}(f_1)$ is a rational curve whose cusp is at $(0\colon 1\colon 1)$. Thus $f_1 = \alpha\mathfrak{h}_{pq}$ $(\exists\alpha > 0)$by Theorem 1.4. So, $f(x, y, z) = \alpha\mathfrak{h}_{pq}(x, y/a, z)$.

If $P = (1\colon 0\colon a) \in L_y$ $(a > 0)$, put $f_1(x, y, z) = f(z, x, ay)$. Then, $V_{\mathbb{C}}(f_1)$ is a rational curve whose cusp is at $(0\colon 1\colon 1)$. Thus $f_1(x, y, z) = \alpha\mathfrak{h}_{pq}(x, y, z)$ and $f(x, y, z) = \alpha\mathfrak{h}_{pq}(y, z/a, x)$.

Similarly, if $P = (a\colon 1\colon 0) \in L_z$ $(a > 0)$, then $f(x, y, z) = \alpha\mathfrak{h}_{pq}(z, x/a, y)$.

(III) We consider the case that $V_{\mathbb{C}}(f)$ has an acnode P. Then $P \in \mathrm{Int}(\mathbb{P}^2_+)$ by Lemma 3.11. Let $P = (1\colon a\colon b)$ $(a > 0,\ b > 0)$, and $f_1(x, y, z) := f(x, ay, bz)$. Then $f_1 \in \mathcal{E}(\mathcal{P}^+_{3,3})$ and $V_{\mathbb{C}}(f_1)$ has an acnode at $(1\colon 1\colon 1)$. After a suitable permutation σ of x, y, z, $f_1(\sigma(x), \sigma(y), \sigma(z))$ is equal to one of $\alpha\mathfrak{f}_{pqr}(x, y, z)$, $\alpha\mathfrak{g}_{\frac{1}{p},\frac{1}{q}}(x, y, z)$, $\alpha\mathfrak{f}_{\frac{1}{p},0,q}(x, z, y)$, $\alpha\mathfrak{f}_{p,\frac{1}{q},0}(x, y, z)$, $\alpha\mathfrak{g}_{\frac{1}{p},0}(x, y, z)$, $\alpha\mathfrak{f}_{p,0,0}(x, y, z)$, $\alpha\mathfrak{g}_{0,0}(x, y, z)$, $\alpha\mathfrak{f}_{0,0,0}(x, y, z)$ and $\alpha\mathfrak{f}_{0,0,0}(x, z, y)$ by results in §3.5. On the other hand

$$\begin{aligned}
\mathfrak{g}_{pq}(y, x, z) &= \mathfrak{g}_{qp}(x, y, z)\\
\mathfrak{f}_{pqr}(y, x, z) &= \mathfrak{f}_{qrp}(x, z, y),\\
\mathfrak{f}_{pqr}(y, z, x) &= \mathfrak{f}_{rpq}(x, y, z),\\
\mathfrak{f}_{pqr}(z, x, y) &= \mathfrak{f}_{qrp}(x, y, z),\\
\mathfrak{f}_{pqr}(z, y, x) &= \mathfrak{f}_{rpq}(x, z, y),\\
\mathfrak{f}_{pqr}(y, x, z) &= p^4q^4r^4\mathfrak{f}_{\frac{1}{q},\frac{1}{p},\frac{1}{r}}(x, y, z),
\end{aligned}$$

Thus $f(x, y, z)$ can be represented as one of the forms in (1), (2) and (3). □

Corollary 3.25. (1) $\mathcal{E}(\mathcal{P}^e_{3,6}) \subset \mathcal{E}(\mathcal{P}_{3,6})$.
(2) (Theorem 1.7) *If* $f(x, y, z) \in \mathcal{E}(\mathcal{P}^+_{3,3})$, *then* $f(x^2, y^2, z^2) \in \mathcal{E}(\mathcal{P}_{3,6})$.

Proof. (1) and (2) are equivalent, since $\mathcal{P}^+_{3,3} \cong \mathcal{P}^e_{3,6}$ by the correspondence $f(x, y, z) \to f(x^2, y^2, z^2)$.

Take $g \in \mathcal{E}(\mathcal{P}^e_{3,6})$. There exists a $f \in \mathcal{E}(\mathcal{P}^+_{3,3})$ such that $g(x, y, z) = f(x^2,\ y^2,\ z^2)$. If $f(x, y, z) = \mathfrak{f}_{pqr}(x,\ y/a,\ z/b)$, then $g \in \mathcal{E}(\mathcal{P}_{3,6})$ by Theorem 3.20(1). Similarly, we obtain $g \in \mathcal{E}(\mathcal{P}_{3,6})$ in the cases (2), (3), (4) of Theorem 1.5, by Lemma 3.12 and Theorem 3.15.

Consider the case $f(x, y, z) = x(ax+bx+cx)^2$ with $\dim\bigl(V_{\mathbb{R}}(ax+by+cz)\cap\mathbb{P}^2_+\bigr) = 1$. Then $V_{\mathbb{R}}(ax^2+by^+cz^2)$ is an irreducible real quadric curve. Thus $g(x, y, z) = x^2(ax^2+bx^2+cx^2)^2 \in \mathcal{E}(\mathcal{P}_{3,6})$.

It is easy to see that $x^2y^2z^2 \in \mathcal{E}(\mathcal{P}_{3,6})$. Thus we have the conclusion. □

Proposition 3.26. *Let* $\mathcal{P}^{c+}_{3,3} := \bigl\{ f \in \mathcal{P}^+_{3,3} \bigm| f(y, z, x) = f(x, y, z) \text{ (i.e. } f \text{ is cyclic)} \bigr\}$. *Then,* $\mathcal{E}(\mathcal{P}^{c+}_{3,3}) \subset \mathcal{E}(\mathcal{P}^+_{3,3})$.

Proof. By Theorem 3.1 of [1], $f \in \mathcal{E}(\mathcal{P}^{c+}_{3,3})$ is equal to $cxyz$ or $c\mathfrak{f}_s$ ($\exists c > 0$, $s \in [0, +\infty]$) where

$$\mathfrak{f}_s(x,y,z) := s^2(x^3+x^3+z^3) - (2s^3-1)(x^2y+y^2z+z^2x) \\ + (s^4-2s)(xy^2+yz^2+zx^2) - 3(s^4-2s^3+s^2-2s+1)xyz,$$
$$\mathfrak{f}_\infty(x,y,z) := xy^2+yz^2+zx^2-3xyz.$$

Note that $\mathfrak{f}_{sss}(x,y,z) = (s^2+s+1)(s^2-s+1)^2\mathfrak{f}_s(x,y,z)$. Thus $\mathfrak{f}_s(x,y,z) \in \mathcal{E}(\mathcal{P}^{+}_{3,3})$ for $s \geq 0$. Since $xyz \in \mathcal{E}(\mathcal{P}^{+}_{3,3})$, we have $\mathcal{E}(\mathcal{P}^{c+}_{3,3}) \subset \mathcal{E}(\mathcal{P}^{+}_{3,3})$. □

Corollary 3.27. *All the cyclic elements of $\mathcal{E}(\mathcal{P}^{c+}_{3,3})$ are $c\mathfrak{f}_s$ ($s \in [0, +\infty]$) and $cxyz$ ($c > 0$). All the symmetric elements of $\mathcal{E}(\mathcal{P}^{c+}_{3,3})$ are $c\mathfrak{f}_1$ and $cxyz$ ($c > 0$).*

Note that $\mathfrak{f}_{t^2}(x^2, y^2, z^2)$ appeared as $S_t(x, y, z)$ in [15, (1.9), (6.20)].

§4. Appendix.

Hilbert proved $P_{3,4} = \Sigma_{3,4}$. We shall give an alternative proof of this theorem. The proof of [4, Proposition 6.3.4] is one of classical type proofs oh this Hilbelt's theorem.

Theorem 4.1. (Hilbert) *If $f \in \mathcal{E}(\mathcal{P}_{3,4})$, then f is the square of a quadratic polynomial.*

Proof. Assume that $f \in \mathcal{E}(\mathcal{P}_{3,4})$ is not a square of a quadratic polynomial. It is easy to see that this implies that f is irreducible in $\mathbb{R}[x,y,z]$.

If f is not exposed, then there exists $f_n \in \mathcal{E}(\mathcal{P}_{3,4})$ ($n \in \mathbb{N}$) such that $\lim_{n\to\infty} f_n = f$ (with respcet to the Euclidean topology of $\mathcal{H}_{3,4}$), and that all f_n are exposed (see [16, Theorem 2.1.7]). If f is irreducible in $\mathbb{R}[x,y,z]$, we can take f_n so that f_n are irreducible in $\mathbb{R}[x,y,z]$. So we may assume that f is exposed.

Consider $V_{\mathbb{R}}(f)$. It is easy to see that if $\dim_{\mathbb{R}} V_{\mathbb{R}}(f) = 1$ as a topological space, then f is not irreducible in $\mathbb{R}[x,y,z]$, since f is PSD. If $V_{\mathbb{R}}(f) = \emptyset$, then f cannot be extremal. Thus $V_{\mathbb{R}}(f)$ is a set of isolated points. Since f is exposed, $V_{\mathbb{R}}(f)$ does not contain infinitely near points. Since $\dim_{\mathbb{R}} \mathcal{H}_{3,4} = 15$, $V_{\mathbb{R}}(f)$ must contain at least $5 = 15/3$ points.

If $V_{\mathbb{C}}(f)$ is irreducible, this is impossible because any curves on $\mathbb{P}^2_{\mathbb{C}}$ whose arithmetic genus is equal to 2, can have at most 4 singular points.

Thus $f = g\overline{g}$ where $g \in \mathbb{C}[x,y,z] - \mathbb{R}[x,y,z]$ is a quadratic. Note that $V_{\mathbb{R}}(f) \subset V_{\mathbb{C}}(g) \cap V_{\mathbb{C}}(\overline{g})$. Thus $\#V_{\mathbb{R}}(f) \leq 4$. A contradiction. □

If $f \in \mathcal{E}(\mathcal{P}_{3,6}) - \Sigma_{3,6}$, then $x^{2d}f \in \mathcal{E}(\mathcal{P}_{3,6+2d}) - \Sigma_{3,6+2d}$. Here we shall give examples of irreducible $f \in \mathcal{E}(\mathcal{P}_{3,d}) - \Sigma_{3,d}$ for $d = 8$ and 10. Such f will be more interesting than reducible ones.

Theorem 4.2. *There exists $f(x,y,z) \in \mathcal{E}(\mathcal{P}^{+}_{3,4})$ such that $f(x^2, y^2, z^2) \in \mathcal{E}(\mathcal{P}_{3,8})$ and $f(x^2, y^2, z^2)$ is irreducible in $\mathbb{C}[x,y,z]$.*

Proof. In $\mathcal{E}(\mathcal{P}^{+}_{3,4})$, the equality conditions $f(1,1,1) = f(2,3,1) = f(1,2,3) = f(0,4,3) = f(6,0,5) = f(0,1,0) = 0$ determine the polynomial

$$f(x,y,z) := 591900050x^4 + 437205100x^3y - 766414561x^2y^2 + 217365672xy^3 \\ - 1650610670x^3z - 102695021x^2yz + 248518503xy^2z + 549666y^3z \\ + 1531736792x^2z^2 + 118221267xyz^2 + 101630538y^2z^2 \\ - 636743352xz^3 - 273946320yz^3 + 183282336z^4$$

up to a constant multiplication. We can prove that $f \in \mathcal{P}^+_{3,4}$ if we observe $f(1+x,\, 1+tx,\, 1)/x^2$ carefully. It is easy to prove that $f(x^2,\, y^2,\, z^2)$ is irreducible in $\mathbb{C}[x,y,z]$ (see Theorem 4.3 below). $f(x^2,\, y^2,\, z^2)$ has the following 17 isolated zeros: $(1{:}\,1{:}\,1)$, $(1{:}\,1{:}\,-1)$, $(1{:}\,-1{:}\,1)$, $(-1{:}\,1{:}\,1)$, $(\sqrt{2}{:}\,\sqrt{3}{:}\,1)$, $(\sqrt{2}{:}\,\sqrt{3}{:}\,-1)$, $(\sqrt{2}{:}\,-\sqrt{3}{:}\,1)$, $(-\sqrt{2}{:}\,\sqrt{3}{:}\,1)$, $(1{:}\,\sqrt{2}{:}\,\sqrt{3})$, $(1{:}\,\sqrt{2}{:}\,-\sqrt{3})$, $(1{:}\,-\sqrt{2}{:}\,\sqrt{3})$, $(-1{:}\,\sqrt{2}{:}\,\sqrt{3})$, $(0{:}\,2{:}\,\sqrt{3})$, $(0{:}\,2{:}\,-\sqrt{3})$, $(\sqrt{6}{:}\,0{:}\,\sqrt{5})$, $(\sqrt{6}{:}\,0{:}\,-\sqrt{5})$ and $(0{:}\,1{:}\,0)$. Solve the differential equations for $F \in \mathcal{H}_{3,8}$ such that $F(P) = F_x(P) = F_y(P) = 0$ for the above 17 points P. The solution space of this equation is $\mathbb{R} \cdot f(x^2, y^2, z^2)$. Thus $f(x^2,\, y^2,\, z^2) \in \mathcal{E}(\mathcal{P}_{3,8})$. □

Theorem 4.3. *There exists $f(x,y,z) \in \mathcal{E}(\mathcal{P}^+_{3,5})$ such that $f(x^2,\, y^2,\, z^2) \in \mathcal{E}(\mathcal{P}_{3,10})$ and $f(x^2,\, y^2,\, z^2)$ is irreducible in $\mathbb{C}[x,y,z]$.*

Proof. The equality conditions $f(4,1,1) = f(1,4,1) = f(1,1,4) = f(1,9,9) = f(9,1,9) = f(9,9,1) = f(1,0,0) = f(0,1,0) = 0$ determine the polynomial

$$\begin{aligned} f(x,y,z) = 837x^4y - 645x^3y^2 - 645x^2y^3 + 837xy^4 + 1755x^4z \\ - 17181x^3yz + 23876x^2y^2z - 17181xy^3z + 1755y^4z - 3486x^3z^2 \\ + 19594x^2yz^2 + 19594xy^2z^2 - 3486y^3z^2 + 3287x^2z^3 - 11030xyz^3 \\ + 3287y^2z^3 - 1692xz^4 - 1692yz^4 + 648z^5. \end{aligned}$$

Elementary but somewhat long calculation shows that $f \in \mathcal{P}^+_{3,5}$. Let $g(x,\, y,\, z) := f(x^2,\, y^2,\, z^2)$. Then $V_{\mathbb{R}}(g)$ has the following 26 acnodes: $V_{26} := \big\{(2{:}\,1{:}\,1),\ (2{:}\,1{:}\,-1),\ (2{:}\,-1{:}\,1),\ (-2{:}\,1{:}\,1),\ (1{:}\,2{:}\,1),\ (1{:}\,2{:}\,-1),\ (1{:}\,-2{:}\,1),\ (-1{:}\,2{:}\,1),\ (1{:}\,1{:}\,2),\ (1{:}\,1{:}\,-2),\ (1{:}\,-1{:}\,2),\ (-1{:}\,1{:}\,2),\ (1{:}\,3{:}\,3),\ (1{:}\,3{:}\,-3),\ (1{:}\,-3{:}\,3),\ (-1{:}\,3{:}\,3),\ (3{:}\,1{:}\,3),\ (3{:}\,1{:}\,-3),\ (3{:}\,-1{:}\,3),\ (-3{:}\,1{:}\,3),\ (3{:}\,3{:}\,1),\ (3{:}\,3{:}\,-1),\ (3{:}\,-3{:}\,1),\ (-3{:}\,3{:}\,1),\ (1{:}\,0{:}\,0),\ (0{:}\,1{:}\,0)\big\}$. The solution space of the system of equalities $g(P) = g_x(P) = g_y(P) = g_z(P) = 0$ for all $P \in V_{26}$, is equal to $\mathbb{R} \cdot f(x^2, y^2, z^2)$. Thus $g \in \mathcal{E}(\mathcal{P}_{3,10})$. So, $f \in \mathcal{E}(\mathcal{P}^+_{3,5})$. We shall show that g is irreducible in $\mathbb{C}[x,y,z]$.

(1) To begin with, we prove that g is irreducible in $\mathbb{R}[x,y,z]$. Assume that $g = h_1h_2$ where $h_1 \in \mathcal{H}_{3,d}$, $h_2 \in \mathcal{H}_{3,e}$ with $d+e = 10$, $d \le e$. Since $g \in \mathcal{E}(\mathcal{P}_{3,10})$, we have $h_1 \in \mathcal{E}(\mathcal{P}_{3,d})$ and $h_2 \in \mathcal{E}(\mathcal{P}_{3,e})$. If d is odd, then $\mathcal{P}_{3,d} = 0$. Thus d is even. If $d = 2$, then $h_1 = h_3^2$ $(\exists h_3 \in \mathcal{H}_{3,1})$. Then V_{26} must contain a line $V_{\mathbb{R}}(h_3)$. If $d = 4$, then $h_1 = h_4^2$ $(\exists h_4 \in \mathcal{H}_{3,2})$. Since $\mathrm{Sing}(V_{\mathbb{R}}(h_4)) \subset V_{26}$, we have $V_{\mathbb{R}}(h_4) = \emptyset$ or $h_4 = h_5^2 + h_6^2$ $(\exists h_5,\, h_6 \in \mathcal{H}_{3,1})$. Is is easy to see that these are impossible.

(2) Thus, if g is reducible, there exists an imaginary $h_7 \in \mathbb{C}[x,y,z]$ such that $g = h_7\overline{h_7}$ where $\overline{h_7}$ is the complex conjugate of h_7. If $P \in \mathrm{Sing}(V_{\mathbb{C}}(h_7)) \cap \mathbb{P}^2_{\mathbb{R}} \neq \emptyset$, then $P \in \mathrm{Sing}(V_{\mathbb{C}}(\overline{h_7})) \cap \mathbb{P}^2_{\mathbb{R}}$. This is impossible, since $P \in V_{26}$ is an acnode. Thus $\mathrm{Sing}(V_{\mathbb{C}}(h_7)) \cap \mathbb{P}^2_{\mathbb{R}} = \emptyset$. This implies $V_{26} \subset V_{\mathbb{C}}(h_7) \cap V_{\mathbb{C}}(\overline{h_7})$. But $\#\big(V_{\mathbb{C}}(h_7) \cap V_{\mathbb{C}}(\overline{h_7})\big) \le 5^2 = 25$. □

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