# On extremal rays of the higher dimensional varieties

Tetsuya Ando
Department of Mathematics, Tokyo university, Hongo, Tokyo, 113 Japan.

**0. Introduction.**
The purpose of this paper is to study the structure of contraction of extremal rays on non-singular projective varieties. The notion of extremal ray was first introduced by Mori[8], that is a half line representing an edge of a numerical cone generated by effective 1-cycles in a half space (see Sect. 1). Extremal ray consists of curves that are contracted by a projective morphism, which was proved by Shokurov[13] in the general situation. This projective morphism is called a contraction of the extremal ray. In the case of dimension two and three, the structure of extremal rays was completely determined by Mori[8]. We shall study the structure of extremal rays and their contractions in the higher dimensional cases. Contractions of extremal rays may be roughly classified into two types. One is the type giving a birational morphism and another type gives a fiber space. Our main result is summarized as follows.

Let $X$ be a non-singular projective variety over an algebraically closed field of characteristic zero. Assuming that the canonical divisor $K_X$ is not nef, we have an extremal ray $\mathbb{R}_+[C]$ (Mori[8]). Let $f\colon X \to Y$ be a contraction of $\mathbb{R}_+[C]$. Our main results are stated as follows:
(i) If $f$ is birational and if the exceptional set of $f$ is a divisor $D$, then the general fiber $F$ of $f_D\colon D \to f(D)$ is a Gorenstein Fano variety with index greater than 1. (see Theorem 2.1 for details)
(ii) In addition to the above, if $\dim f(D) = \dim D - 1$ and if $f_D$ is equi-dimensional, then $Y$ and $f(D)$ are non-singular and $f$ is a blowing up of $Y$ along the center $f(D)$. (see Theorem 2.3)
(iii) If $\dim Y < \dim X$, then the general fiber of $f$ is a Fano manifold.
(iv) Moreover if $\dim Y = \dim X - 1$ and if $f$ is equi-dimensional, $f$ induces a conic bundle structure. (Theorem 3.1)

The morphism $f$ in (i) is called a good contraction (see Kawamata[6]). In this case, our classification is rather satisfactory to understand the properties of extremal curves. But in the case when $f$ is birational and its exceptional set has codimension bigger than one, our result is insufficient; for example the exceptional sets in this case may be reducible. But the author does not have such an example when $X$ is non-singular. In case (iii), we have only informations on general fibers, as $f$ might not be equi-dimensional.

In Sect. 1, we shall explain two important theorems on which this paper is based. One is called the cone theorem, which asserts the existence of extremal rays, that was proved by Mori [8] and generalized by Kawamata[6]. The other is called the base point free theorem, which claims the existence of contraction, that was established by Shokurov[13] and Kawamata[6].

In Sect. 2, we shall treat the case where the contraction is birational, and in Sect. 3, we study the contraction giving rise to a fiber space.

**1. Preliminaries.**
We assume that the ground field $k$ is an algebraically closed field of characteristic zero. Let $X$ be a projective variety over $k$ with only canonical singularities. This means that the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier and that there exists a desingularization $f\colon Y \to X$ such that

$$K_Y = f^*K_X + \sum a_i F_i,$$

$F_i$ being prime divisors, where $a_i \geqq 0$. We fix the following notation.

$N^1(X) := (\{$ Cartier divisors on $X$ $\}/ \approx) \otimes_{\mathbb{Z}} \mathbb{R}$

$N_1(X) := (\{$ 1-cycles on $X$ $\}/ \approx) \otimes_{\mathbb{Z}} \mathbb{R}$

$\overline{NE}(X) :=$ the closure of the convex cone generated by effective 1-cycles in $N_1(X)$.

Here the symbol $\approx$ means the numerical equivalence. Further the symbol $\sim$ denotes the linear equivalence.

A Cartier divisor $D$ is called *numerically effective* or *nef* if $(D \cdot C)_X \geqq 0$ for all curves $C$ on $X$.

Similarly 1-cycle $Z$ is also said to *be numerically effective* or *nef* if $(D \cdot Z)_X \geqq 0$ for any effective Cartier divisor $D$.

The *numerical Kodaira dimension* of a nef Cartier divisor $D$ is defined to be $\kappa_{\mathrm{num}}(D) := \max\{\ d \mid D^d \not\approx 0\ \}$ (see Reid[11]). Then $\kappa(D) \leqq \kappa_{\mathrm{num}}(D) \leqq n = \dim X$, where $\kappa(D)$ denotes the $D$-dimension of $X$, i.e. $\kappa(D) := \kappa(D, X)$.

A Cartier divisor $D$ is called *big* if $\kappa(D) = n$.

A linear system is called *free* if it has neither fixed components nor base points.

*Definition.* A curve $C$ on $X$ is called *extremal* if
(i) $(K_X \cdot C) < 0$,
(ii) given $A$, $B \in \overline{NE}(X)$, $A$, $B \in \mathbb{R}_+[C]$ if $A + B \in \mathbb{R}_+[C]$.

*Definition.* Let $C$ be an extremal curve. A Cartier divisor $H$ is called a *good supporting divisor* with respect to $C$, if
(i) $H$ is nef,
(ii) for $Z \in \overline{NE}(X)$, $(H \cdot Z)_X = 0$ if and only if $Z \in \mathbb{R}_+[C]$.

The following two theorems are fundamental.

**Theorem 1.1.** Base point free theorem (Shokurov[13], Kawamata [6]). *Let $X$ be a projective variety with only canonical singularities. Assume that a Cartier divisor $H$ is nef and that $aH - K_X$ is nef and big for some $a \in \mathbb{N}$. Then $|mH|$ is free, for $m \gg 0$.*

**Theorem 1.2.** Cone theorem (Kawamata[6], Mori[8], Kollár[5]). *Assume that $X$ has only canonical singularities. Fix an ample divisor L. Then for any $\epsilon > 0$, there exist extremal curves $C_1, \ldots, C_r$ such that*

$$\overline{NE}(X) = \sum_{i=1}^{r} \mathbb{R}_+[C_i] + \overline{NE}_\epsilon(X).$$

*Here $\overline{NE}_\epsilon(X) := \{\ Z \in \overline{NE}(X) \mid (K_X \cdot Z) > -\epsilon(L \cdot Z)\ \}$.*

**Corollary 1.3.** *For any extremal curve $C$, there exists a good supporting divisor $H$ such that*
(i) *$|mH|$ is free for any $m \gg 0$,*
(ii) *if $E$ is a Cartier divisor such that $(E \cdot C)_X > 0$, then for $m \gg 0$, $mH + E$ is ample. Especially, $mH - K_X$ is ample for $m \gg 0$. Moreover if $X$ is non-singular, then $H^i(X, mH + E) = 0$ and $H^i(X, mH) = 0$ for $i > 0$ and $m \gg 0$.*

The morphism $f = \Phi_{|mH|}\colon X \to Y$ are isomorphic to each other if $m \gg 0$, and so $f$ is called a *contraction associated with $C$*.

**Lemma 1.4.** *Let $C$ be an extremal curve and $H$ be a good supporting divisor with respect to $C$. $H$ is big if and only if $C$ is not nef.*

*Proof.* If $H$ is big, then $H^i(X, mH + K_X) = 0$ for $m \gg 0$, and $i > 0$ by Corollary 1.3. Since $\chi(mH + K_X) = \frac{1}{n!}(H^n)_X \cdot m^n +$ (lower terms in $m$) $\gg 0$ for $m \gg 0$, $|mH + K_X| \neq \phi$ for $m \gg 0$. Moreover, since $(mH + K_X \cdot C)_X = (K_X \cdot C)_X < 0$, it follows that $C$ is not nef.

Conversely if $C$ is not nef, there exists an effective divisor $D$ such that $(D \cdot C)_X < 0$. By Corollary 1.3, $rH - D$ is ample for $r \gg 0$. Since $\frac{1}{n!} r^n (H^n)_X \cdot m^n +$(lower terms in $m$) $= h^0(X, mrH) \geqq h^0(X, m(rH - D)) = \frac{1}{n!}((rH - D)^n)_X \cdot m^n +$ (lower terms in $m$) for $m \gg 0$, we have $(H^n)_X > 0$. ∎

The following lemma is related to a conic bundle structure.

**Lemma 1.5.** *Let $X$ be a non-singular projective variety of dimension $n$. Further let $C$ be an irreducible curve on $X$ such that*
(i) $(K_X \cdot C) < 0$,
(ii) *$\chi(\mathcal{O}_{C'}) \geqq 0$ for any subscheme $C'$ in $X$ with $(C')_{\mathrm{red}} = C$. Then $C = \mathbb{P}^1$ and $N_{C/X}$ is one of the following four cases:*

$$N_{C/X} \cong \mathcal{O}_C^{\oplus(n-1)}, \tag{1}$$
$$N_{C/X} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C^{\oplus(n-2)}, \tag{2}$$
$$N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(-2) \oplus \mathcal{O}_C^{\oplus(n-3)}, \tag{3}$$
$$N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(-1)^{\oplus 2} \oplus \mathcal{O}_C^{\oplus(n-4)}. \tag{4}$$

*Moreover, in the cases of* (3) *and* (4)*, letting* $J \subset \mathcal{O}_X$ *be the ideal such that* $I_C \supset J \supset I_C^2$ *and* $I_C/J \cong \mathcal{O}_C(-1)$*, we have* $J/J^2 \cong (\mathcal{O}_X/J)^{\oplus(n-1)}$*, where* $I_C$ *is the ideal of* $\mathcal{O}_X$ *defining* $C$ *in* $X$*.*

*Proof.* Since $h^1(C, \mathcal{O}_C) \leqq 1$, $C$ is isomorphic to $\mathbb{P}^1$ or a plane cubic curve. If $C$ is a plane cubic curve, then $\chi(\Omega_C^1) = 0$. By the exact sequence

$$0 \to I_C/I_C^2 \to \Omega_X^1 \otimes \mathcal{O}_C \to \Omega_C^1 \to 0$$

$\chi(\mathcal{O}_X/I_C^2) = \chi(\mathcal{O}_C) + \chi(I_C/I_C^2) = \chi(I_C/I_C^2) = \chi(\Omega_X^1 \otimes \mathcal{O}_C) - \chi(\Omega_C^1) = (K_X \cdot C) < 0$, which contradicts hypothesis (ii). Thus $C \cong \mathbb{P}^1$.

Hence, letting $I := I_C$, the sheaf $I/I^2$ is locally free and so $I/I^2 \cong \mathcal{O}_C(p_1) \oplus \cdots \oplus \mathcal{O}_C(p_{n-1})$ for some $p_1,\ldots, p_{n-1}$ with $p_1 \leqq p_2 \leqq \cdots \leqq p_{n-1}$. Since $C$ is a locally complete intersection,

$$0 \to I/I^2 \to \Omega_X^1 \otimes \mathcal{O}_C \to \Omega_C^1 \to 0$$

is exact. Thus $p_1 + p_2 + \cdots + p_{n-1} = (K_X \cdot C) + 2 \leqq 1$. Especially $p_1 \leqq 0$.

*Case I.* $p_1 = 0$. Then $(p_1,\ldots, p_{n-1}) = (0,\ldots, 0)$ or $(0,\ldots,0, 1)$, corresponding to (1) or (2) respectively.

*Case II.* $p_1 \leqq -1$. Let $J$ and $K$ be the ideals of $\mathcal{O}_X$ defined by $I \supset J \supset K \supset I^2$, $I/J \cong \mathcal{O}_C(p_1)$, $J/K \cong \mathcal{O}_C(p_2)$ and $K/I^2 \cong \mathcal{O}_C(p_3) \oplus \cdots \oplus \mathcal{O}_C(p_{n-1})$. We claim:

$$\chi(\mathcal{O}_X/J^r) = 2 \cdot {}_nH_{r-1} + ({}_nH_{r-1} + 4 \cdot {}_{n+1}H_{r-2}) \cdot p_1 + 2{}_{n+1}H_{r-2}(p_2 + \cdots + p_{n-1}), \tag{$*$}$$

$$\begin{aligned}\chi(\mathcal{O}_X/K^r) =& (3 \cdot {}_nH_{r-1} + {}_nH_{r-2}) \\ &+ (6 \cdot {}_{n+1}H_{r-2} + 2 \cdot {}_{n+1}H_{r-3} + {}_nH_{r-1} + {}_nH_{r-2})(p_1 + p_2) \\ &+ (3 \cdot {}_{n+1}H_{r-2} + {}_{n+1}H_{r-3})(p_3 + \cdots + p_{n-1})\end{aligned} \tag{$**$}$$

where ${}_nH_r = \binom{n+r-1}{r}$.

Since $I$, $J$ and $K$ are locally free, we have $I^r/I^{r+1} \cong S^r(I/I^2)$, $J^r/IJ^r \cong S^r(J/IJ)$, and $IJ^r/J^{r+1} \cong (J^r/IJ^r) \otimes (I/J)$. We can describe $J/IJ$ as $\mathcal{O}_C(q_1) \oplus \cdots \oplus \mathcal{O}_C(q_{n-1})$ ($q_1 \leqq \cdots \leqq q_{n-1}$). Since

$$0 \to I^2/IJ \to J/IJ \to J/I^2 \to 0$$

is exact, $I^2/IJ \cong S^2(I/J) \cong \mathcal{O}_C(2p_1)$, and since $J/I^2 \cong \mathcal{O}_C(p_2) \oplus \cdots \oplus \mathcal{O}_C(p_{n-1})$, we have $q_1 + \cdots + q_{n-1} = \deg J/IJ = \deg I^2/IJ + \deg J/I^2 = 2p_1 + p_2 + \cdots p_{n-1}$. Therefore

$$\begin{aligned}\chi(\mathcal{O}_X/J^r) &= \sum_{i=0}^{r-1} \chi(S^i(\mathcal{O}_C(q_1) \oplus \cdots \oplus \mathcal{O}_C(q_{n-1}))) + \sum_{i=0}^{r-1} \chi(S^i(\mathcal{O}_C(q_1) \oplus \cdots \oplus \mathcal{O}_C(q_{n-1})) \otimes \mathcal{O}_C(p_1)) \\ &= \sum_{r_1+\cdots+r_{n-1}<r} \{\chi(\mathcal{O}_C) + \chi(\mathcal{O}_C(p_1)) + 2(r_1q_1 + \cdots + r_{n-1}q_{n-1})\} \\ &= {}_nH_{r-1}(2 + p_1) + 2{}_{n+1}H_{r-2}(q_1 + \cdots + q_{n-1})\end{aligned}$$

Thus we have ($*$). Now we shall prove ($**$). Since $\operatorname{rank} K/IK = n$ we can express $K/IK$ as $\mathcal{O}_C(s_1) \oplus \cdots \oplus \mathcal{O}_C(s_n)$ ($s_1 \leqq \cdots \leqq s_n$), where $s_1 + \cdots + s_n = \deg K/IK = \deg I^2/IK + \deg K/IJ - \deg I^2/IJ = 3(p_1 + p_2) + (2p_1 + p_3 + \cdots + p_{n-1}) - 2p_1 = 3p_1 + 3p_2 + p_3 + \cdots + p_{n-1}$. Since $0 \to \mathcal{O}_C(2p_1 + 2p_2) \otimes S^{r-2}(K/IK) \to S^r(K/IK) \to K^r/IK^r \to 0$ is exact and since $IK^r/JK^r \cong I/J \otimes K^r/IJK^{r-1} \cong I/J \otimes S^r(K/IJ)$, $JK^r/K^{r+1} \cong J/K \otimes K^r/IK^r$, we have ($**$).

We regard $\chi(\mathcal{O}_X/J^r)$ and $\chi(\mathcal{O}_X/K^r)$ as polynomials in $r$. By ($*$) and ($**$), we have

$$\chi(\mathcal{O}_X/J^r) = \frac{r^n}{n!}(4p_1 + 2(p_2 + \cdots + p_{n-1})) + \text{lower terms in } r, \tag{\#}$$

$$\chi(\mathcal{O}_X/K^r) = \frac{r^n}{n!}(8(p_1 + p_2) + 4(p_3 + \cdots + p_{n-1})) + \text{lower terms in } r. \tag{\#\#}$$

By the assumption that $\chi(\mathcal{O}_X/J^r) \geqq 0$ and $\chi(\mathcal{O}_X/K^r) \geqq 0$ for any $r > 0$, it follows that $2p_1+p_2+\cdots+p_{n-1} \geqq 0$, and $2p_1 + 2p_2 + p_3 + \cdots + p_{n-1} \geqq 0$. Thus we have $(p_1,\ldots, p_{n-1}) = (-1,0,\ldots,0,2)$ or $(-1, 0,\ldots, 0, 1, 1)$.

Finally we shall prove that $J/J^2 \cong (\mathcal{O}_X/J)^{\oplus(n-1)}$.

*Case II.1.* $q_1 = 0$. Then $q_1 = \cdots = q_{n-1} = 0$, i.e. $J/IJ \cong \mathcal{O}_C^{\oplus(n-1)}$. Noting that $IJ/J^2 \cong I/J \otimes J/IJ \cong \mathcal{O}_C(-1) \otimes \mathcal{O}_C^{\oplus(n-1)} \cong \mathcal{O}_C(-1)^{\oplus(n-1)}$, we have the following diagram.

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_C(-1)^{\oplus(n-1)} & \to & (\mathcal{O}_X/J)^{\oplus(n-1)} & \to & \mathcal{O}_C^{\oplus(n-1)} & \to & 0 \\ & & \| & & & & \| & & \\ 0 & \to & IJ/J^2 & \to & J/J^2 & \to & J/IJ & \to & 0 \end{array}$$

Since $H^0(IJ/J^2) = 0$ and since $H^1(IJ/J^2) = 0$, we have $H^0(J/J^2) \cong H^0(J/IJ) \cong k^{n-1}$. By this isomorphism we get the surjection $(\mathcal{O}_X/J)^{\oplus(n-1)} \twoheadrightarrow J/J^2$ making the above diagram commutative. Thus we have $(\mathcal{O}_X/J)^{\oplus(n-1)} \cong J/J^2$.

*Case II.2.* $q_1 \leqq -1$. Let $L \subset \mathcal{O}_X$ be an ideal such that $J \supset L \supset IJ$, $J/L \cong \mathcal{O}_C(q_1)$, and that $L/IJ \cong \mathcal{O}_C(q_2) \oplus \cdots \oplus \mathcal{O}_C(q_{n-1})$. Now we shall calculate $\chi(\mathcal{O}_X/L^r)$. If $L \subset I^2$, then we have

$$\begin{aligned} \chi(\mathcal{O}_X/K^r) = &(3 \cdot {}_nH_{r-2} + {}_nH_{r-2}) + ({}_nH_{r-1} + {}_nH_{r-2})p_1 \\ &+ (6 \cdot {}_{n+1}H_{r-2} + 2 \cdot {}_{n+1}H_{r-3} + {}_nH_{r-1} + {}_nH_{r-2})q_1 \\ &+ (3 \cdot {}_{n+1}H_{r-2} + {}_{n+1}H_{r-3})(q_2 + \cdots + q_{n-1}) \end{aligned}$$

by the same argument as in the proof of $(**)$. We assume $L \not\subset I^2$. Since $(L^r/IL^r)/\mathrm{tor} \cong S^r((L/IL)/\mathrm{tor})$, $\#(L^r/IL^r)_{\mathrm{tor}} \leqq \#(L/IL)_{\mathrm{tor}} \cdot {}_{n-1}H_{r-2}$, $JL^r/L^{r+1} \cong L^r/IL^r \otimes J/L$, and since $0 \to T \to L^r/IL^r \otimes I/J \to IL^r/JL^r \to 0$ is exact for some torsion $\mathcal{O}_C$-module $T$, we have

$$\chi(\mathcal{O}_X/L^r) \leqq 3 \cdot {}_{n+1}H_{r-2}(p_1 + q_1 + \cdots + q_{n-1}) + {}_nH_{r-1}(3 + p_1 + q_1) + 3 \cdot {}_nH_{r-2}\chi((L/IL)_{\mathrm{tor}}).$$

Thus $\chi(\mathcal{O}_X/L^r) < 0$ for $r \gg 0$. This is a contradiction. □

*Remark.* If $N_{C/X}$ is one of (1), (3) or (4), then $C$ is nef.

**2. Birational case.**

Throughout this section, we assume that $X$ is non-singular. We fix an extremal curve $C$ and assume that $C$ is not nef. Let $H$ be a good supporting divisor with respect to $C$. Since $C$ is not nef, the contraction $f = \Phi_{|mH|}: X \to Y$ for $m \gg 0$ is a birational morphism. Let $E$ be the exceptional set of $f$. $C$ being not nef, there is a prime divisor $D$ such that $(D \cdot C) < 0$. If $\dim E = n-1$ ($n = \dim X$), then such a $D$ is unique and $D = E$. But in the case $\dim E < n-1$, there are infinitely many such $D$. In fact, (i) $\dim |mH + K_X| \gg 0$ for $m \gg 0$, because $H$ is big and $H^i(X, mH + K_X) = 0$ for $i > 0$ and $m \gg 0$. (ii) Any general member of $|mH + K_X|$ is an irreducible divisor, since $\mathrm{Bs}\,|mH + K_X| \subset E$. Actually this follows from the fact that $mH + K_X$ is ample outside $E$. Thus any irreducible member of $|mH + K_X|$ satisfies the condition for $D$.

We treat the case $\dim E = n-1$.

*Definition.* (See Fujita[1, 2]) A Gorenstein projective variety $X$ with $\dim X \geqq 3$ is called *Del Pezzo variety* if

(i) there exists an ample Cartier divisor $L$ such that $-K_X \sim (n-2)L$, and
(ii) $H^i(X, tL) = 0$ for all $t \in \mathbb{Z}$, $0 < i < n$.

*Definition.* (See Mukai[9]) A Gorenstein projective variety $X$ with $\dim X \geqq 4$ is called *Mukai variety* if

(i) there exists an ample Cartier divisor $L$ such that $-K_X \sim (n-3)L$, and
(ii) $H^i(X, tL) = 0$ for all $t \in \mathbb{Z}$, $0 < i < n$.

**Theorem 2.1.** *Assume that* $\dim E = n-1$. *Let* $F$ *be a general fiber of* $f_D: D \to f(D)$ *(Note that if* $\dim f(D) = 0$, *then* $F = D$*). Then there exists a Cartier divisor* $L$ *on* $X$ *such that*

*(i)* $\mathrm{Im}(\mathrm{Pic}\, X \to \mathrm{Pic}\, F) = \mathbb{Z}[L|_F]$ *and* $L|_F$ *is ample on* $F$,
*(ii)* $\mathcal{O}_F(-K_X) \cong \mathcal{O}_F(pL)$ *and* $O_F(-D) \cong \mathcal{O}_F(qL)$ *for some* $p, q \in \mathbb{N}$,
*(iii)* $H^i(F, tL) = 0$ *for* $0 < i < \dim F$, *unless* $-q < t < -p$.
*Especially* $H^i(F, tL) = 0$ *for all* $t \in \mathbb{Z}$ *and* $0 < i < \dim F$, *if* $\dim F \leqq 4$.

By these properties, we can classify $F$ into the lower dimensional cases in the following way:

(a) If $\dim F = 1$, $F \cong \mathbb{P}^1$.
(b) If $\dim F = 2$, $F$ is $\mathbb{P}^2$ or $Q^2$.
(c) If $\dim F = 3$, $F$ is $\mathbb{P}^3$, $Q^3$, or a Del Pezzo 3-fold.
(d) If $\dim F = 4$, $F$ is $\mathbb{P}^4$, $Q^4$, a Del Pezzo 4-fold, or a Mukai 4-fold.

**Lemma 2.2.** *Let $P$ be a Cartier divisor in $X$ such that $mH + P - K_X$ is nef and big for any $m \gg 0$. Then $H^i(F, \mathcal{O}_F(P)) = 0$ for $i > 0$. Moreover if $F'$ is a subscheme of $X$ with $(F')_{\mathrm{red}} \subset F$, then $H^r(F', \mathcal{O}_{F'}(P)) = 0$, where $r = \dim F$.*

*Proof.* Let $A_1, \dots, A_b$ ($b = \dim f(D)$) be very ample divisors in $Y$ such that $f(D) \cap A_1 \cap \cdots \cap A_b \ni f(F)$, $H_j := f^* A_j$ ($1 \leqq j \leqq b$), and let $s$ be a fixed natural number. Noting that $H_j$ are good supporting divisors with respect to $C$, $mH + P - K_X - t_1 H_1 - \cdots - t_b H_b$ ($t_j = 0$ or $s$) is nef and big for $m \gg 0$. Since $mH - D$ is ample for $m \gg 0$ by Corollary 1.3, it follows that $mH + P - t_1 H_1 - \cdots - t_b H_b - K_X$ and $mH + P - t_1 H_1 - \cdots - t_b H_b - sD - K_X$ are nef and big for $m \gg 0$. Thus

$$H^i(X, mH + P - t_1 H_1 - \cdots - t_b H_b) = H^i(X, mH + P - t_1 H_1 - \cdots - t_b H_b - sD) = 0$$

for $i > 0$, by Kawamata-Viehweg vanishing Theorem([7, 14]). Using the exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_X(mH + P - t_1 H_1 - \cdots - t_b H_b - sD) \to \mathcal{O}_X(mH + P - t_1 H_1 - \cdots - t_b H_b) \\ &\to \mathcal{O}_{sD}(mH + P - t_1 H_1 - \cdots - t_b H_b) \to 0, \end{aligned}$$

we have $H^i(\mathcal{O}_{sD}(mH + P - t_1 H_1 - \cdots - t_b H_b)) = 0$ for any $i > 0$. Again using the exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_{sD}(mH + P - sH_1 - t_2 H_2 - \cdots - t_b H_b) \to \mathcal{O}_{sD}(mH + P - t_2 H_2 - \cdots - t_b H_b) \\ &\to \mathcal{O}_{sD \cap sH_1}(mH + P - t_2 H_2 - \cdots - t_b H_b) \to 0, \end{aligned}$$

we have $H^i(\mathcal{O}_{sD \cap sH_1}(mH + P - t_2 H_2 - \cdots - t_b H_b) = 0$ for any $i > 0$. Repeating this process we have $H^i(\mathcal{O}_{sD \cap sH_1 \cap \cdots \cap sH_b}(mH + P)) = 0$ for $i > 0$.

On the other hand, $F$ is a connected component of $D \cap H_1 \cap \cdots \cap H_b$. Therefore

$$H^i(F, \mathcal{O}_F(mH + P)) \subset H^i(\mathcal{O}_{D \cap H_1 \cap \cdots \cap H_b}(mH + P)) = 0$$

for $i > 0$. Since $|mH|$ is free and $H|_F \approx 0$, it follows that $\mathcal{O}_F(H) \cong \mathcal{O}_F$. Thus, we have $H^i(\mathcal{O}_F(P)) = 0$ for $i > 0$.

Let $s$ be an integer such that $sD \cap sH_1 \cap \cdots \cap sH_b \supset F'$. Then $\mathcal{O}_{sD \cap sH_1 \cap \cdots \cap sH_b}(mH + P) \to \mathcal{O}_{F'}(mH + P)$ is surjective. Thus

$$H^r(F', \mathcal{O}_{F'}(P)) = H^r(F', \mathcal{O}_{F'}(mH + P)) = 0.$$

□

*Proof of Theorem 2.1.* We take $A_j$ and $H_j$ ($1 \leqq j \leqq b$) as in proof of Lemma 2.2. Since $|H_j|_D|$ are free, every connected component of $|D \cap H_1 \cap \cdots \cap H_b|$ is irreducible and reduced for general $A_1, \dots, A_b$ by a theorem of Bertini. Thus $F$ is irreducible and reduced Gorenstein variety. Take any curve $Z$ in $F$. Then $(H \cdot Z)_X = 0$. Therefore $Z \in \mathbb{R}_+[C]$. This means that $\mathrm{rank}(\mathrm{Im}(N_1(F) \to N_1(X))) = 1$. We shall show that $\mathrm{Im}(\mathrm{Pic}\, X \to \mathrm{Pic}\, F) \cong \mathbb{Z}$. Take any element $M \in \mathrm{Pic}\, X$ such that $M|_F \approx 0$. Since $\mathrm{rank}(\mathrm{Im}(N^1(X) \to N^1(F))) = 1$, it suffices to show that $M|_F \sim 0$. $mH + M - K_X$ is ample for $m \gg 0$, we have $H^i(\mathcal{O}_F(M)) = 0$ for $i > 0$, by Lemma 2.2. Therefore $h^0(\mathcal{O}_F(M)) = \chi(\mathcal{O}_F(M)) = \chi(\mathcal{O}_F) = h^0(\mathcal{O}_F) = 1$. So $\mathcal{O}_F(M) \cong \mathcal{O}_F$. This implies that $\mathrm{Im}(\mathrm{Pic}\, X \to \mathrm{Pic}\, F) \cong \mathbb{Z}$.

Let $L \in \mathrm{Pic}\, X$ be a generator of $\mathrm{Im}(\mathrm{Pic}\, X \to \mathrm{Pic}\, F)$ such that $L|_F$ is ample. Then $\mathcal{O}_F(-K_X) \cong \mathcal{O}_F(pL)$ and $\mathcal{O}_F(-D) \cong \mathcal{O}_F(qL)$ for some $p$, $q \in \mathbb{N}$. By the adjunction formula, $\omega_F \cong \mathcal{O}_F((-p-q)L)$. By Lemma 2.2, we conclude $H^i(F, tL) = 0$ for $i > 0$, $t \geqq -p$. By Serre duality $H^i(F, tL) = 0$ for $i < \dim F$, $t \leqq -q$.

Let $r := \dim F$ and $d := (L^r)_F$. Now we classify $F$ in the case $r \leqq 4$. Let

$$P(t) := \chi(\mathcal{O}_F(tL)) = \frac{d}{r!} t^r + \frac{(p+q)d}{2(r-1)!} t^{r-1} + \text{lower terms in } t.$$

By Serre duality, $P(-t) = (-1)^r P(t-p-q)$. $P(0) = \chi(\mathcal{O}_F) = 1$. By (iii), $P(t) = 0$ for any $t$ such that $-p \leqq t < 0$. By these properties we have

(a) If $r = 1$, $P(t) = dt + 1$.

(b) If $r = 2$, $P(t) = \frac{d}{2}t(t+p+q)$.

(c) If $r = 3$, $P(t) = \frac{d}{12}t(t+p+q)(2t+p+q) + \frac{2t}{p+q} + 1$.

(d) If $r = 4$, $P(t) = \frac{1}{24}\{t^2(t+p+q)^2 d + t(t+p+q)(pqd + \frac{24}{pq})\} + 1$.

We can compute $\Delta$-genera of $(F, L)$ defined by $\Delta(F, L) = (n-1) + d - P(1)$ in the sense of Fujita [1] as follows.

First we treat the case $\dim F = 3$. If $p + q > 4$, then $\Delta(F, L) < 0$. This contradicts Fujita's result to the effect that $\Delta(F, L) \geqq 0$ (see [1]). Thus $2 \leqq \Delta(F, L) \leqq 4$.

Case (p, $q$) = (3, 1), (2, 2), (1, 3). Then $\Delta = \frac{3}{2}(1-d) \in \mathbb{Z}$. Therefore $\Delta = 0$, $d = 1$ and so $F \cong \mathbb{P}^3$ by Fujita[1].

Case $(p, q) = (2, 1), (1, 2)$. Then $\Delta = \frac{2}{3}(2-d) \in \mathbb{Z}$. Therefore $\Delta = 0$, $d = 2$ and so $F \cong Q^3$, again by [1].

Case $(p, q) = (1, 1)$. Then $\Delta = 0$ and so $F$ is a Del Pezzo variety which allows hypersurface singularities.

In the case of $\dim F = 4$, by the same discussion as above we have $2 \leqq p + q \leqq 5$ and finally we obtain the following table.

Table 1

| | | | | |
|---|---|---|---|---|
| $p+q$ | 5 | 4 | 3 | 2 |
| $\Delta(F, L)$ | 0 | 0 | 0 | $2d$ |
| $d$ | 1 | 2 | $d \geqq 1$ | $d \geqq 1$ |
| $F$ | $\mathbb{P}^4$ | $Q^4$ | Del Pezzo varieties | Mukai varieties |

□

**Theorem 2.3.** *If $\dim f(D) = \dim D - 1 = n - 2$ and $f_D$ is equi-dimensional then both $Y$ and $f(D)$ are non-singular, and moreover $f: X \to Y$ is the blowing up along the smooth center $f(D)$.*

*Proof.* Take any irreducible curve $Z \in \mathbb{R}_+[C]$. Let $Z'$ be an arbitrary subscheme of $X$ with $(Z')_{\mathrm{red}} = Z$. Since every fibre of $f_D: D \to f(D)$ is one-dimensional, it follows that $H^1(\mathcal{O}_{Z'}) = 0$ and $H^1(\mathcal{O}_{Z'}(K_X)) = 0$ by Lemma 2.2. Thus $Z$ satisfies the condition of Lemma 1.5. Since $Z$ is not nef, we have $N_{Z/X} \cong \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z^{\oplus n-2}$. Therefore $(K_X \cdot Z)_X = -1$. This implies that any fiber of $f_D$ is irreducble and isomorphic to $\mathbb{P}^1$.

Let $T$ be a connected component of the Hilbert scheme $Hilb_{X/k}$ containing a fiber $C$ of $f_D$. Choose any closed point $t \in T$, and let $Z$ be a closed subscheme of $X$ representated by $t$. We regard $Z$ as a 1-cycle and denote it by $\Gamma$ which is written as $\operatorname{Spec} \mathcal{O}_Z/K$. Since $(Z \cdot -K_X)_X = 1$ and since any irreducible component of $\Gamma$ belongs to $\mathbb{R}_+[C]$, $\Gamma$ is irreducible as a 1-cycle. By a property of Hilbert scheme, $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_C) = 1$. Since $\Gamma \approx C$, $\chi(\mathcal{O}_\Gamma) = \chi(\mathcal{O}_C) = 0$. Therefore $\chi(K) = 0$. Since $K$ is a skyscraper sheaf, it follows that $K = 0$. Thus $Z$ is an irreducible and reduced curve, and $Z \in \mathbb{R}_+[C]$. This implies that $Z \cong \mathbb{P}^1$ and $N_{Z/X} \cong \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z^{\oplus n-2}$. Since $H^0(N_{Z/X}) = k^{n-1}$ and $H^1(N_{Z/X}) = 0$, $T$ is smooth and $(n-1)$-dimensional at $t$. Thus $T$ is an irreducible non-singular $(n-1)$-dimensional projective variety. Let $U \subset T \times X$ be a universal family containing all fibers of $f_D$, and $\pi: U \to X$, $\varphi: U \to T$ be natural morphisms. Since $U$ is a Zariski $\mathbb{P}^1$-bundle, $U$ is non-singular. Moreover since any fiber of $\varphi$ belongs to $\mathbb{R}_+[C]$ and since for any point of $D$ there is a fiber of $f_D$ passing through the point, we have $\pi(U) = D$. Take any point $x \in D$. Since only one fiber of $f_D$ passes through $x$, $\varphi(\pi^{-1}(x))$ is one point. Therefore $\pi$ is an isomorphism on any fiber of $\varphi$. Thus $\pi$ is bijective. We claim that $\pi$ is the isomorphism. Actually, $\pi^*\omega_D \cong \mathcal{O}_U(K_U + \Delta)$ where $\Delta$ is an effective divisor. For any fiber $\Gamma$ of $\varphi$, $(K_U \cdot \Gamma)_U = (\pi^*\omega_D \cdot \Gamma)_U = -1$. Therefore $(\Delta \cdot \Gamma)_U = 0$. Since $\Delta$ doesn't contain any fiber of $\varphi$, $\Delta = 0$. Thus $\pi$ is an isomorphism. By contraction theorem of Nakano [10], we see that $f$ is the blowing up along $f(D)$. □

*Remark.* If $\dim X = 3$, $f_D$ is always equi-dimensional. In fact, if a fibre of $f_D$ contains a surface, it must be $D$.

Now we consider the case $\dim E < n-1$. It is not difficult to see that $\dim E \geqq 2$. In fact:

Assume $\dim E = 1$. Take the reduced curve $C$ of an irreducible component of $E$. Let $H_1, \ldots, H_{n-1} \in |mH + K_X|$ be hypersurfaces of $X$ which intersect transversally. By the same argument as in Lemma 2.2, we have

$H^i(sH_1 \cap \cdots \cap sH_{n-1}, \mathcal{O}_{sH_1 \cap \cdots \cap sH_{n-1}}(mH)) = 0$ for $i \geqq 1$, $m \gg 0$,

repeating cutting process by $H_1, \ldots, H_{n-1}$. Thus $H^1(C', \mathcal{O}_{C'}) = H^1(C', \mathcal{O}_{C'}(mH)) = 0$ for any subscheme $C'$ of $X$ with $(C')_{\mathrm{red}} = C$. Since $C$ is not nef, it follows that $C \cong \mathbb{P}^1$ and $N_{C/X} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C^{\oplus(n-2)}$, by Lemma 1.5. Thus $C$ is movable, which contradicts the fact that $C$ is a component of the exceptional locus. Therefor $\dim E \geqq 2$ fas been established.

Note that if $X$ and $Y$ are toric varieties, then $\dim E \geqq \frac{1}{2} \dim X$. (See Reid[12].) It seems to be true that this holds in general for smooth varieties. Of cource if $X$ has singularities, there are many examples in which $\dim E = 1$. (See Reid[12].) In such case, $E$ is a rational tree, since $H^1(E, \mathcal{O}_E) = 0$. (See Kawamata[6].)

## 3. Fiber case.

Throughout this section, assume that $X$ is non-singular and that the extremal curve $C$ is nef. Let $H$ be a good supporting divisor with respect to $C$, and $f := \Phi_{|mH|} \colon X \to Y$ for $m \gg 0$, which is a contraction of $C$. Let $b$ denote $\dim Y = \kappa(H) = \kappa_{\mathrm{num}}(H)$. Since $C$ is nef, it follows that $b < n = \dim X$.

**Theorem 3.1.** (i) *A general fiber of $f$ is a Fano $(n-b)$-fold.*

(ii) *If $b = n-1$ and $f$ is equi-dimensional, then $Y$ is non-singular and $f$ induces a conic bundle structure on $X$.*

*Proof.* (i) Let $p$ be a general point of $Y$. Take ample divisors $A_1, \ldots, A_b$ in $Y$ such that $p \in A_1 \cap \cdots \cap A_b$ and $\dim(A_1 \cap \cdots \cap A_b) = 0$. Let $H_j := f^* A_j$. Then $F := f^{-1}(p)$ is an irreducible component of $H_1 \cap \cdots \cap H_b$.

Since $F$ is a complete intersection and $\mathcal{O}_F(H_j) = \mathcal{O}_F$, we have $\omega_F = \mathcal{O}_F(K_X)$. On the other hand, any curve $Z$ on $F$ belongs to $\mathbb{R}_+[C]$. Hence $-K_X$ is ample on $F$. By definition, $F$ is a Fano variety.

(ii) Let $\Gamma$ be a one-dimensional subscheme of $X$ whose support is included in a fiber of $f \colon X \to Y$. By the same argument as in the proof of Lemma 2.2, we have $H^1(\mathcal{O}_\Gamma) = 0$. In particular any irreducible reduced component $C$ of a fiber of $f \colon X \to Y$ satisfies the assertions of Lemma 1.5. Especially $C \cong \mathbb{P}^1$. We claim that any fiber $F := f^{-1}(y)$ of $f$ is isomorphic to a plane conic. Since $(K_X \cdot F) = -2$, $F$ has at most two irreducible components.

*Case 1.* $F$ is irreducible. Then $F \cong \mathbb{P}^1$ and $N_{F/X} \cong \mathcal{O}_F^{\oplus(n-1)}$.

*Case 2.* $F = C_1 \cup C_2$. We claim that $C_1 \cap C_2$ consists of only one point.

Since $C_1 \cong C_2 \cong \mathbb{P}^1$ and

$$0 \to \mathcal{O}_F \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0$$

is exact, we have

$$\chi(\mathcal{O}_{C_1 \cap C_2}) = \chi(\mathcal{O}_{C_1}) + \chi(\mathcal{O}_{C_2}) - \chi(\mathcal{O}_F) = 2 - \chi(\mathcal{O}_F).$$

On the other hand, noting that $H^1(\mathcal{O}_F) = 0$, we have $\chi(\mathcal{O}_F) = 1$. Thus $\chi(\mathcal{O}_{C_1 \cap C_2}) = 1$, i.e. $C_1 \cap C_2$ is one point.

*Case 3.* $F = 2C$ as a 1-cycle. Let $I := I_C$ and $J := I_F$. Considering on each local ring, we have $I \supset J \supset I^2$. Since $1 = \chi(\mathcal{O}_F) = \chi(\mathcal{O}_C) + \chi(I/J) = 1 + \chi(I/J)$, $0 = \chi(I/J) = \deg I/J + \operatorname{rank} I/J$. By Lemma 1.5, $\deg I/J \geqq -1$. Thus $\deg I/J = -1$, $\operatorname{rank} I/J = 1$. This implies that $N_{C/X}$ is of type (3) or (4) in Lemma 1.5, and that $J/J^2 \cong \mathcal{O}_F^{\oplus(n-1)}$.

Since any fiber of $f$ is isomorphic to a plane conic, $f$ is flat. Therefore $Y$ is non-singular. □

*Remark.* If $\dim X = 3$ and $\dim Y = 2$, $f$ is equi-dimensional. In fact, if some fibre $f^{-1}(p)$ has dimension two, then $(f^{-1}(p) \cdot C)_X < 0$ for any curve $C \subset f^{-1}(p)$. But since $C$ is an extremal curve, $C$ must be nef.

## References


[1] Fujita, T.: On the structure of polarized manifolds with total deficiency one I, II, III. J. Math. Soc. Japan **32**, 709-725 (1980); **33**, 415-434 (1981); **36**, 75-89 (1984)

[2] Fujita, T.: Classification of projective varieties of Δ-genus one. Proc. Japan Acad., Ser.A **58**, 113-116 (1982)

[3] Hidaka, F. and Watanabe, K.: Normal Gorenstein surface with ample anti-canonical divisor. Tokyo J. Math. **4**, 319-330 (1981)
[4] Iskovskih, V. A.: Fano 3-folds I, II. Math. USSR, Izv. **11**, 485-527 (1977); **12**, 469-506 (1978)
[5] Kollár, J.: The cone theorem. Ann. Math., **120**, 1-5 (1984)
[6] Kawamata, Y.: The cone of curves of algebraic varieties. Ann. Math. **119**, 603-633 (1984)
[7] Kawamata, Y.: A Generalization of Kodaira-Ramanujam's Vanish ing Theorem. Math. Ann. **261**, 43-46 (1982)
[8] Mori, S.: Threefolds whose canonical bundles are not numeri cally effective. Ann. Math. **116**, 133-176 (1982)
[9] Mukai, S.: On Fano manifolds of coindex 3. (Preprint 1983)
[10] Nakano, S.: On the inverse of monoidal transformations. Publ. Res. Inst. Math. Sci. **6**, 483-502 (1971)
[11] Reid, M.: Projective morphisms according to Kawamata. University of Warwick (Preprint 1983)
[12] Reid, M.: Decomposition of toric morphisms. In: Arithmetic and Geometry, II (M. Artin, J. Tata, eds) Prog. Math. **36**, 395-418. Boston-Basel-Stuttgart: Birkhauser 1983
[13] Shokurov, V. V.: Non vanishing theorem (in Russian). Izv. Akad. Nauk SSSR (To appear)
[14] Viehveg, E.: Vanishing theorems. J. Reine Angew. Math. **335**, 1-8 (1982)