On the Normal bundle of an Exceptional curve in a higher dimensional algebraic manifold

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§1. Introduction.

An algebraic curve C in an algebraic manifold X is called *exceptional* or *contractible*, if there exists a birational projective morphism $\varphi: X \to Y$ whose exceptional set is C, here Y may be an algebraic space or an analytic space. The morphism φ is called a *small contraction*, and the singularity $(\varphi(X), \varphi(C))$ is called a *small singularity*, if $\dim X \geqq 3$.

When X is an algebraic surface, C is exceptional if and only if its normal bundle $N_{C/X}$ is negative. We consider the case $\dim X \geqq 3$. Grauert has shown that if $N_{C/X}$ is negative, then C is exceptional. But we know many exceptional curves whose normal bundles are not negative. In the case $\dim X = 3$, the author and Nakayama have proved the following: (See [1], [2] and [14])

Theorem 1.1. *If C is an exceptional rational curve, then its normal bundle $N_{C/X} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ (assume $a \leqq b$) satisfies $a + 2b < 0$ and $(a, b) \neq (-1, 0)$. Conversely, for every pair of integers (a, b) which satisfies the above conditions, we can construct such exceptional rational curves.*

In this paper, we aim to generalize the above theorem to the higher dimensional case. Our results are the following:

Theorem 1.2. *Let C be a smooth exceptional curve of genus g, and let $\mathcal{M}$ be a subbundle of $N_{C/X}$ of the maximal degree. Put $b = \deg \mathcal{M}$ and $a = \deg N_{C/X} - \deg \mathcal{M}$. Then $a + 2b < 0$ and $a + b < 0$. Moreover if $g = 0$, then $a + b \leqq -n + 1$.*

Theorem 1.3. *If $(d_1, \cdots, d_{n-1})$ are integers such that $d_1 \leqq \cdots \leqq d_h < 0 \leqq d_{h+1} \leqq \cdots \leqq d_{n-1}$ and that $(d_1 + \cdots + d_h) + 2(d_{h+1} + \cdots + d_{n-1}) \leqq -n + 1$. Then we can construct an* *exceptional smooth rational curve C whose normal bundle is $N_{C/X} \cong \mathfrak{O}_C(d_1)\oplus\cdots\oplus\mathfrak{O}_C(d_{n-1})$.*

In the case of surfaces, the singularity obtained by contracting a rational curve is at worst a rational singularity. This is no longer true if $\dim X \geqq 3$. Related to this, the following facts are known.

(1) (Laufer[10]) A small singularity obtained by contracting a smooth rational curve whose normal bundle is semi-negative (i.e. $d_i \leqq 0$ for all i) is a rational singularity.

(2) There exists the natural surjection $\mathbf{R}^1\varphi_*\mathfrak{O}_X \to \mathrm{H}^1(C, I_C/I_C^2)$, here I_C is the defining ideal of $C \cong \mathbf{P}^1$ in $\mathfrak{O}_X$. Note that $I_C/I_C^2 = N^*_{C/X}$. Therefore if $d_i \geqq 2$ for some i, then $\mathbf{R}^1\varphi_*\mathfrak{O}_X \neq 0$.

(3) We can constract (X, C) with $\mathrm{H}^1(C, I_C/I_C^2) = 0$ and $\mathbf{R}^1\varphi_*\mathfrak{O}_X \neq 0$. (See §3.)

(4) A small singularity is rational if and only if it is Cohen-Macauley. But a rational small singularity need not be Gorenstien. (Proposition 3 in p.363 of [16]is not true.)

§2. Proof of Theorem 1.2.

We summarize some results which have been proved in some preceding papers. Some results are quoted in slightly extended form. So, we give proofs of them, though they are based on the original papers.

Let I_C be the defining ideal of a non-singular curve C in $\mathfrak{O}_X$.

Definition. (Saturation) An ideal $\mathfrak{I}$ of $\mathfrak{O}_X$ is called *saturated* if $\mathfrak{O}_X/\mathfrak{I}$ has no embeded primes. Under the condition $\mathrm{Supp}\,\mathfrak{O}_X/\mathfrak{I} = C$, $\mathfrak{I}$ is saturated if and only if there exists a filtration $\mathfrak{O}_X = \mathfrak{F}_0 \supset \mathfrak{F}_1 \supset \cdots \supset \mathfrak{F}_r = \mathfrak{I}$ such that every $\mathfrak{F}_{i-1}/\mathfrak{F}_i$ is an invertible $\mathfrak{O}_C$-module. For any ideal $\mathfrak{J} \subset \mathfrak{O}_X$ and $\mathrm{Supp}\,\mathfrak{O}_X/\mathfrak{J} = C$, there exists the unique saturated ideal $\mathfrak{I}$ such that $\mathfrak{I} \supset \mathfrak{J}$ and that $\mathfrak{I}/\mathfrak{J}$ is a zero-dimensional sheaf or $\mathfrak{I} = \mathfrak{J}$. We call this ideal $\mathfrak{I}$ to be *the saturation of* $\mathfrak{J}$, and represent as $\mathrm{Sat}(\mathfrak{J})$. A filtration of ideals included in I_C is called saturated if every successive quotient is a locally free $\mathfrak{O}_C$-module.

Definition. (Degree and length) For two ideals $\mathfrak{I}$ and $\mathfrak{J}$ with $I_C \supset \mathfrak{I} \supset \mathfrak{J}$ and $\mathrm{Supp}\,\mathfrak{O}_X/\mathfrak{J} = C$, there exists a filtration $\mathfrak{I} = \mathfrak{F}_0 \supset \mathfrak{F}_1 \supset \cdots \supset \mathfrak{F}_r \supset \mathfrak{J}$ such that every $\mathfrak{F}_{i-1}/\mathfrak{F}_i$ is a locally free $\mathfrak{O}_C$-module and that $\mathfrak{F}_r/\mathfrak{J}$ is a zero-dimensional sheaf. Then we define $\deg \mathfrak{I}/\mathfrak{J} = \sum_{i=1}^r \deg \mathfrak{F}_{i-1}/\mathfrak{F}_i + \deg \mathfrak{F}_r/\mathfrak{J}$, and $\mathrm{length}\,\mathfrak{I}/\mathfrak{J} = \sum_{i=1}^r \mathrm{rank}\,\mathfrak{F}_{i-1}/\mathfrak{F}_i$. Clearly these are independent of the choice of a filtration.

Definition. (f-base) Assume that $\mathfrak{I}$ and $\mathfrak{J}$ are ideals with $I_C \supset \mathfrak{I} \supset \mathfrak{J}$, and that $\mathfrak{I}/\mathfrak{J}$ is a locally free $\mathfrak{O}_C$-module. Let $\mathbb{F}$: $\mathfrak{I} = \mathfrak{F}_0 \supset \mathfrak{F}_1 \supset \cdots \supset \mathfrak{F}_r = \mathfrak{J}$ be a filtration such that $\mathfrak{F}_{i-1}/\mathfrak{F}_i$

are invertible $\mathcal{O}_C$-modules. A set of functions $g_1, \cdots, g_r$ on a open set U is said to be an *f-base* (filtration base) corresponding to $\mathbb{F}$, if $g_i \in \Gamma(U, \mathcal{F}_{i-1})$ and if the class of g_i modulo $\mathcal{F}_i$ is a local base of the invertible sheaf $\mathcal{F}_{i-1}/\mathcal{F}_i$. We say $h_1, \cdots, h_r$ is an f-base, if it is an f-base corresponding to a suitable filtration.

Definition. (Numerical equivalence) Two vector bundles $\mathcal{F}$ and $\mathcal{G}$ over a curve C is called to be numerical equivalent (denoted $\mathcal{F} \approx \mathcal{G}$), if $\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{G}$ and $\deg \mathcal{F} = \deg \mathcal{G}$.

Theorem 2.1. (Vanishing Theorem, Ando[2].) *Assume that the ground field is an algebraically closed field of any characteristic. Let $\varphi\colon X \to Y$ is a surjective morphism such that the general fibers of φ are at most one-dimensional. (Hence, $\dim X = \dim Y$ or $\dim X = \dim Y + 1$.) If C is an irreducible component of an one-dimensional fiber of φ, then there exists a divisor A (independent of the choice of Γ) such that $\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma(A)) = 0$ for any subscheme Γ whose support is C.*

Proof. Since the argument is local, we can assume that Y is an affine neighborhood of $y = \varphi(C)$. Put $m = \dim Y = n - \dim \varphi$. Let $B_1, \cdots, B_m \subset Y$ be divisors such that $y \in B_1 \cap \cdots \cap B_m$ and that $B_1 \cap \cdots \cap B_m$ is zero-dimensional. Let $H_i = \varphi^* B_i$. Then we choose $r_1, \cdots, r_m \in \mathbb{N}$ so that $\Gamma \subset r_1 H_1 \cap \cdots \cap r_m H_m$ as a scheme. Put $q_j = 0$ or r_j. Since φ is projective morphism, we can choose a sufficiently φ-ample divisor A such that $\mathbf{R}^i \varphi_* \mathcal{O}_X(A) = 0$ for $i > 0$. Then $\mathbf{R}^i \varphi_* \mathcal{O}_X(A - q_1 H_1 - \cdots - q_m H_m) \cong \mathbf{R}^i \varphi_* \mathcal{O}_X(A) \otimes \mathcal{O}_Y(-q_1 B_1 - \cdots - q_m B_m) = 0$ for $i > 0$. Since Y is affine, $\mathrm{H}^i(Y, \varphi_* A - q_1 B_1 - \cdots - q_m B_m) = 0$ for $i > 0$. Thus $\mathrm{H}^i(X, A - q_1 H_1 - \cdots - q_m H_m) = 0$ for $i > 0$. Therefore $\mathrm{H}^i(r_1 H_1 \cap \cdots \cap r_m H_m, A) = 0$ for $i \geqq 1$. Since $\dim(\operatorname{Supp}(\operatorname{Ker}(\mathcal{O}_{r_1 H_1 \cap \cdots \cap r_m H_m} \to \mathcal{O}_\Gamma))) \leqq 1$, we have $\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma(A)) = 0$. $\square$

Theorem 2.2. (Existence of an ample ideal, Nakayama[14].) *If C is an exceptional curve, then there exists an ideal $\mathcal{A}$ included in I_C such that C is a component of $\operatorname{Supp}(\mathcal{O}_X/\mathcal{A})$ and that $\mathcal{A}/I_C\mathcal{A}$ is an ample $\mathcal{O}_C$-module. Conversely, if there exists such an ideal $\mathcal{A}$, then C is exceptional.*

Proof. Since the argument is local, we can assume that Y is a (formal or an analytic) neighborhood of y, and that X is a (formal or an analytic) neighborhood of C. Since φ is projective, we can find effective Cartier divisors $D_1, \cdots, D_{n-1}$ such that $(D_i \cdot C)_X < 0$ for $1 \leqq i \leqq n-1$, and that $\dim(D_1 \cap \cdots \cap D_{n-1}) = 1$. Let $\mathcal{A}$ be the ideal $\mathcal{O}_X(-D_1) + \cdots + \mathcal{O}_X(-D_{n-1})$. Since $(D_i \cdot C)_X < 0$, $C \subset D_i$. Hence $\mathcal{A} \subset I_C$. Since $\mathcal{A}/I_C\mathcal{A} \cong \mathcal{A} \otimes \mathcal{O}_C \cong \mathcal{O}_C(-D_1) \oplus \cdots \oplus \mathcal{O}_C(-D_{n-1})$, $\mathcal{A}/I_C\mathcal{A}$ is ample.

Conversely, assume that there exists such $\mathcal{A}$. We can suppose that $\mathcal{A}$ is saturated and $\operatorname{Supp}\operatorname{Spec}\mathcal{O}_X/\mathcal{A} = C$. We can apply the contractible criterion of Artin[3], Fujiki[5] to Spec($\mathcal{O}_X$ /$\mathcal{A}$). □

Corollary 2.3. *Assume that there exists a birational projective morphism $\varphi: X \to Y$ whose exceptional set is a union of curves $\cup_i C_i$, and if C is an irreducible component of $\cup_i C_i$. then C is contractible.*

Theorem 2.4. (Jiménez[6] Theorem 1.) *If C is an exceptional curve of genus g, then there exists a line bundle $\mathcal{L}$ on a neighborhood of C in X such that $\deg(\mathcal{L}|_C) \geqq g-2$ and $\dim \mathrm{H}^1(\hat{X}, \hat{\mathcal{L}}) < +\infty$. Here $\hat{X}$ is the formal completion of X along C, and $\hat{\mathcal{L}}$ is the formal completion of $\mathcal{L}$ along C. Conversely, if there exists such $\mathcal{L}$, and if the ground field is $\mathbb{C}$, then C is contractible.*

Let $I_C = \mathcal{J}_0 \supset \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_{n-1} = I_C^2$ be a filtration such that all $\mathcal{J}_{i-1}/\mathcal{J}_i$ are invertible sheaves. We can find such a filtration, by Atiyah[4]. For some $1 \leqq h \leqq n-2$, we put $J = J_h$. Let $a_i = \deg \mathcal{J}_{i-1}/\mathcal{J}_i$, $a = -(a_{h+1} + \cdots + a_{n-1})$ and $b = -(a_1 + \cdots + a_h)$. We can find a f-base $x_1, \cdots, x_{n-1}$ corresponding to the above filtration on a sufficiently small neighborhood U of $p \in C$. Clearly,

$$I_C|_U = (x_1, \cdots, x_{n-1}),$$
$$J|_U = (x_1, \cdots, x_h)^2 + (x_{h+1}, \cdots, x_{n-1}).$$

We can easily check the following:

Proposition 2.5.

$$(J^r/I_C J^r)|_U = \langle x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \mid (m_1 + \cdots + m_h) + 2(m_{h+1} + \cdots + m_{n-1}) = 2r \rangle$$
$$(I_C J^r/J^{r+1})|_U = \langle x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \mid (m_1 + \cdots + m_h) + 2(m_{h+1} + \cdots + m_{n-1}) = 2r+1 \rangle$$

Here $\langle g_1, \cdots, g_r \rangle$ implies the $\mathcal{O}_{C \cap U}$-module whose base is $g_1, \cdots, g_r$.

By the above proposition, we recognize that J^r and $I_C J^r$ are saturated. If we read carefully the proof of Theorem 3.2 in Ando [2], we obtain the following results.

Theorem 2.6.

$$\deg J^r/I_C J^r = -\, b\sum_{i=1}^{[h/2]}\binom{h-1}{2i-1}\binom{n+r-i-2}{n-2} - (a+2b)\sum_{i=0}^{[h/2]}\binom{h}{2i}\binom{n+r-i-2}{n-1},$$

$$\operatorname{rank} J^r/I_C J^r = \sum_{i=0}^{[h/2]}\binom{h}{2i}\binom{n+r-i-2}{n-2},$$

$$\deg I_C J^r/J^{r+1} = -\, b\sum_{i=0}^{[(h-1)/2]}\binom{h-1}{2i}\binom{n+r-i-2}{n-2} - (a+2b)\sum_{i=0}^{[h/2]}\binom{h}{2i+1}\binom{n+r-i-2}{n-1},$$

$$\operatorname{rank} I_C J^r/J^{r+1} = \sum_{i=0}^{[(h-1)/2]}\binom{h}{2i+1}\binom{n+r-i-2}{n-2},$$

$$\deg \mathbf{O}_X/J^r = -\, b\sum_{i=1}^{h}\binom{h-1}{i-1}\binom{r-[i/2]+n-2}{n-1} - (a+2b)\sum_{i=0}^{h}\binom{h}{i}\binom{r-[i/2]+n-2}{n},$$

$$\operatorname{length} \mathbf{O}_X/J^r = \sum_{i=0}^{h}\binom{h}{i}\binom{r-[i/2]+n-2}{n-1}.$$

Here, $[x]$ implies the round down of real number x.

Proof. We start by proving the following:

Claim 2.6.1. $\mathfrak{J}_p^r/(I_C\mathfrak{J}_q\mathfrak{J}_p^{r-1}) \approx \bigoplus_{k=0}^r \mathrm{S}^{2k}(I_C/\mathfrak{J}_q)\otimes \mathrm{S}^{r-k}(\mathfrak{J}_p/I_C^2)$ for $1 \leqq q \leqq p < n-1$.

Proof. Put $G_k = I_C^{2k-3}\mathfrak{J}_q\mathfrak{J}_p^{r-k+1} + I_C^{2k}\mathfrak{J}_p^{r-k}$, and $H_k = I_C^{2k-2}\mathfrak{J}_p^{r-k+1}\cap I_C\mathfrak{J}_q\mathfrak{J}_p^{r-1}$, here we formally put $J^0 = J^{-1} = J^{-2} = \mathbf{O}_X$. First we prove $G_k \supset H_k$. It is enough to show this on U. Let x^α be a monomial in $x_1,\cdots,x_q$ of degree α, y^β be a monomial in $x_{q+1},\cdots,x_p$ of degree β, and z^γ be a monomial in $x_{p+1},\cdots,x_{n-1}$ of degree γ. Since G_k and H_k are generated by the monomials in the form $x^\alpha y^\beta z^\gamma$, it is enough to show that if $x^\alpha y^\beta z^\gamma \in H_k$ then $x^\alpha y^\beta z^\gamma\in G_k$. Note that $x^\alpha y^\beta z^\gamma \in I_C^i\mathfrak{J}_p^j$ if and only if $\alpha+\beta+\gamma \geqq i+j$ and $\alpha+\beta+2\gamma\geqq i+2j$. Assume $x^\alpha y^\beta z^\gamma\in H_k$ and $x^\alpha y^\beta z^\gamma \notin G_k$. Since $x^\alpha y^\beta z^\gamma \notin I_C^{2k}\mathfrak{J}_p^{r-k}$ and since $x^\alpha y^\beta z^\gamma \in I_C^{2k-2}\mathfrak{J}_p^{r-k+1}$, we have $\alpha+\beta+\gamma = r+k-1$ and $\alpha+\beta+2\gamma \geqq 2r$. Since $G_k \supset I_C^{2k-3}\mathfrak{J}_q\mathfrak{J}_p^{r-k+1}\supset I_C^{2k-3}\mathfrak{J}_p^{r-k+2}$, $x^\alpha y^\beta z^\gamma \notin I_C^{2k-3}\mathfrak{J}_p^{r-k+2}$. Thus $\alpha+\beta+2\gamma < 2r+1$. Therefore $\gamma = r-k+1$, $\alpha+\beta = 2k-2$. If $\beta\geqq 1$ and $\alpha+\beta = 2k-2$, then $x^\alpha y^\beta \in I_C^{2k-3}\mathfrak{J}_q$. Hence $x^\alpha y^\beta z^\gamma\in I_C^{2k-3}\mathfrak{J}_q\mathfrak{J}_p^{r-k+1}\subset G_k$. Thus $\beta = 0$. But $x^{2k-2}z^{r-k+1}\notin I_C\mathfrak{J}_q\mathfrak{J}_p^{r-1}$. Thus we have $G_k\supset H_k$.

Since $I_C^2 \subset \mathfrak{J}_p\subset\mathfrak{J}_q\subset I_C$, we have $I_C^{2k-3}\mathfrak{J}_q\mathfrak{J}_p^{r-k+1}\subset H_k$ and $H_{k+1} = I_C^{2k}\mathfrak{J}_p^{r-k}\cap H_k$. Thus

$G_k = I_C^{2k}\mathfrak{J}_p^{r-k} + H_k$. Put $F_k = G_k/H_k$. By the isomorphism theorem, $F_k \cong I_C^{2k}\mathfrak{J}_p^{r-k}/H_{k+1}$. On the other hand

$$\mathrm{S}^{2k}(I_C/\mathfrak{J}_q) \otimes \mathrm{S}^{r-k}(\mathfrak{J}_p/I_C^2) \cong I_C^{2k}\mathfrak{J}_p^{r-k}/G_{k+1}.$$

Thus

$$0 \to F_{k+1} \to F_k \to \mathrm{S}^{2k}(I_C/\mathfrak{J}_q) \otimes \mathrm{S}^{r-k}(\mathfrak{J}_p/I_C^2) \to 0$$

is exact for $0 \leqq k \leqq r$. By similar arguments, we have $F_0 \cong J_p^r/I_C\mathfrak{J}_q J_p^{r-1}$ and $F_{r+1} \cong I_C^{2r-1}\mathfrak{J}_q/(I_C^{2r} \cap I_C\mathfrak{J}_q J_p^{r-1}) = 0$. □

Claim 2.6.2. $(\mathfrak{J}_q/\mathfrak{J}_{q+1}) \otimes (\mathfrak{J}_p^r/I_C\mathfrak{J}_{q+1}\mathfrak{J}_p^{r-1}) \cong \mathfrak{J}_q\mathfrak{J}_p^r/\mathfrak{J}_{q+1}\mathfrak{J}_p^r$ for $0 \leqq q < p \leqq n-1$.

Proof. There exists the canonical surjection from the left hand side to the right hand side. Thus it is enough to show that the both have the same rank at a general point of C. We use the same notations as above. $\mathfrak{J}_q\mathfrak{J}_p^r$ is generated by the monomials $x^\alpha y^\beta z^\gamma$ with (α, β, γ) which satisfy $\alpha + \beta + 2\gamma \geqq 1 + 2r$ and $\alpha + 2\beta + 2\gamma \geqq 2 + 2r$. Thus $\mathfrak{J}_q\mathfrak{J}_p^r/\mathfrak{J}_{q+1}\mathfrak{J}_p^r$ is generated by $x^\alpha x_{q+1}^\beta z^\gamma$ which satisfy $\alpha + \beta + 2\gamma = 1 + 2r$ and $\beta \geqq 1$. On the other hand, $J_p^r/I_C\mathfrak{J}_{q+1}\mathfrak{J}_p^{r-1}$ is generated by $x^\alpha x_{q+1}^\beta z^\gamma$ with $\alpha + \beta + 2\gamma = 2r$. Thus $(\mathfrak{J}_q/\mathfrak{J}_{q+1}) \otimes (J_p^r/I_C\mathfrak{J}_{q+1}\mathfrak{J}_p^{r-1})$ is generated by $x^\alpha x_{q+1}^\beta z^\gamma$ which satisfy $\alpha + \beta + 2\gamma = 1 + 2r$ and $\beta \geqq 1$. □

Put

$$\mathbf{Z}(r) = \left\{ (m_1, \cdots, m_{n-1}) \in \mathbb{Z}^{n-1} \;\middle|\; \begin{array}{l} (m_1 + \cdots + m_h) + 2(m_{h+1} + \cdots + m_{n-1}) = r, \\ \text{and } m_i \geqq 0 \text{ for } 0 \leqq i \leqq n-1 \end{array} \right\}.$$

If we apply Claim 2.6.1 with $p = q = h$, then we have

$$\begin{aligned} J^r/I_C J^r &\approx \bigoplus_{k=0}^{r} \mathrm{S}^{2k}(I_C/J) \otimes \mathrm{S}^{r-k}(J/I_C^2) \\ &\approx \bigoplus_{(m_1, \cdots, m_{n-1}) \in \mathbf{Z}(2r)} \mathcal{O}_C(m_1 a_1 + \cdots m_{n-1} a_{n-1}). \end{aligned}$$

Here $\mathcal{O}_C(d)$ means an invertible $\mathcal{O}_C$-sheaf of degree d. ($\mathcal{O}_C(d)$ is unique up to numerial equivalence, though it need not be unique up to isomorphisms.) Consider the coarse filtration

$$I_C J^r \supset \mathfrak{J}_1 J^r \supset \mathfrak{J}_2 J^r \supset \cdots \supset \mathfrak{J}_{h-1} J^r \supset J^{r+1}.$$

By Claim 2.6.2, we have

$$I_C J^r/J^{r+1} \approx \bigoplus_{q=1}^{h} \mathcal{O}_C(a_q) \otimes (J^r/I_C\mathfrak{J}_q J^{r-1}).$$

By Claim 2.6.1, we have

$$\begin{aligned} I_C J^r/J^{r+1} &\approx \bigoplus_{q=1}^{h} \bigoplus_{k=0}^{h} \mathcal{O}_C(a_q) \otimes \mathrm{S}^{2k}(I_C/\mathfrak{J}_q) \otimes \mathrm{S}^{r-k}(J/I_C^2) \\ &\approx \bigoplus_{(m_1, \cdots, m_{n-1}) \in \mathbf{Z}(2r+1)} \mathcal{O}_C(m_1 a_1 + \cdots m_{n-1} a_{n-1}). \end{aligned}$$

For each element $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_h) \in (\mathbb{Z}/2\mathbb{Z})^h$, we put

$$\mathbf{Z}(r, \varepsilon) = \{(m_1, \cdots, m_{n-1}) \in \mathbf{Z}(r) \mid m_i \equiv \varepsilon_i (\mathrm{mod}\ 2) \text{ for } 1 \leqq i \leqq p\}.$$

Let $rk(r, \varepsilon)$ be the number of elements in $\mathbf{Z}(r, \varepsilon)$, and let

$$d(r, \varepsilon) = \sum_{(m_1, \cdots, m_{n-1}) \in \mathbf{Z}(r, \varepsilon)} (m_1 a_1 + \cdots + m_{n-1} a_{n-1}).$$

Note that

$$\begin{aligned}
\operatorname{rank} J^r / I_C J^r &= \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^h} rk(2r, \varepsilon), \\
\operatorname{rank} I_C J^r / J^{r+1} &= \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^h} rk(2r+1, \varepsilon), \\
\deg J^r / I_C J^r &= \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^h} d(2r, \varepsilon), \\
\deg I_C J^r / J^{r+1} &= \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^h} d(2r+1, \varepsilon).
\end{aligned}$$

When ε contains p pieces of 1 and $(h-p)$ pieces of 0, we denote $v(\varepsilon) = p$. We consider the case: $\varepsilon = (1, \cdots, 1, 0, \cdots, 0)$, $v(\varepsilon) = p$. If we put $m_1 = 2l_1 + 1$, $\cdots$, $m_p = 2l_p + 1$; $m_{p+1} = 2l_{p+1}$, $\cdots$, $m_h = 2l_h$; $m_{h+1} = l_{h+1}$, $\cdots$, $m_{n-1} = l_{n-1}$, then we can denote as

$$\mathbf{Z}(r, \varepsilon) = \{(m_1, \cdots, m_{n-1}) \in \mathbb{Z}^{n-1} \mid 2(l_1 + \cdots + l_{n-1}) = r - p\}.$$

Thus, if $r - p$ is an odd number, then $\mathbf{Z}(r, \varepsilon) = \phi$. If $r - p$ is an even number: $r - p = 2k$, then

$$rk(r, \varepsilon) = \binom{n+k-2}{k} = \binom{k+n-2}{n-2}.$$

Let's calculate $d(r, \varepsilon)$ in this case.

$$\begin{aligned}
d(r, \varepsilon) &= \sum_{(m_1, \cdots, m_{n-1}) \in \mathbf{Z}(r, \varepsilon)} ((2l_1 + 1)a_1 + \cdots + (2l_p + 1)a_p \\
&\qquad + 2l_{p+1}a_{p+1} + \cdots + 2l_h a_h + l_{h+1}a_{h+1} + \cdots + l_{n-1}a_{n-1}) \\
&= \sum_{l_1 + \cdots l_{n-1} = k} ((l_1(2a_1) + \cdots + l_h(2a_h) + l_{h+1}a_{h+1} + \cdots + l_{n-1}a_{n-1}) + (a_1 + \cdots + a_p)) \\
&= \binom{k+n-2}{n-1}(2(a_1 + \cdots + a_h) + (a_{h+1} + \cdots + a_{n-1})) + \binom{k+n-2}{n-2}(a_1 + \cdots + a_p) \\
&= \binom{k+n-2}{n-1}(-a - 2b) + \binom{k+n-2}{n-2}(a_1 + \cdots + a_p).
\end{aligned}$$

Now we vary ε under the condition $v(\varepsilon) = p = r - 2k$. The number of such ε is $\binom{h}{p}$.

$$\begin{aligned}
\sum_{v(\varepsilon) = p} d(r, \varepsilon) &= \binom{h}{p}\binom{k+n-2}{n-1}(-a - 2b) + \frac{p}{h}\binom{h}{p}\binom{k+n-2}{n-2}(a_1 + \cdots + a_h) \\
&= \binom{h}{p}\binom{k+n-2}{n-1}(-a - 2b) + \binom{h-1}{p-1}\binom{k+n-2}{n-2}(a_1 + \cdots + a_h), \\
\sum_{v(\varepsilon) = p} rk(r, \varepsilon) &= \binom{h}{p}\binom{k+n-2}{n-2}.
\end{aligned}$$

Varying $0 \leqq p \leqq h$, we have the conclusions. □

Proof of Theorem 1.2. Let $\mathcal{M}$ be a subbundle of $N_{C/X}$ of the maximal degree. Since I_C/I_C^2 is the dual sheaf of $N_{C/X}$, we can find the ideal $I_C \supset J \supset I_C^2$ such that $(I_C/J)^* \cong \mathcal{M}$. Let $h = \operatorname{rank} I_C/J$. Note that I_C/J and J/I_C^2 are locally free $\mathcal{O}_C$-modules, $\deg I_C/J = -\deg \mathcal{M} = -b$ and that $\deg J/I_C^2 = -a$. By Atiyah[4], we can find a filtration $I_C = \mathcal{J}_0 \supset \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_h = J \supset \mathcal{J}_{h+1} \supset \cdots \supset \mathcal{J}_{n-1} = I_C^2$ such that all $\mathcal{J}_{i-1}/\mathcal{J}_i$ are invertible sheaves.

Let $\mathcal{L}$ be the line bundle in Theorem 2.4. By Theorem 2.6, we have

$$\chi(\mathcal{O}_X/J^r \otimes \mathcal{L}) = -\,b\sum_{i=1}^{h}\binom{h-1}{i-1}\binom{r-[i/2]+n-2}{n-1} - (a+2b)\sum_{i=0}^{h}\binom{h}{i}\binom{r-[i/2]+n-2}{n}$$
$$+ (\deg \mathcal{L}|_C + \chi(\mathcal{O}_C))\sum_{i=0}^{h}\binom{h}{i}\binom{r-[i/2]+n-2}{n-1}.$$

Since $\deg \mathcal{L}|_C \leqq g-2$, we have $\deg \mathcal{L}|_C + \chi(\mathcal{O}_C) < 0$. Moreover $-b \leqq 0$ by assumption. Since

$$\dim \mathrm{H}^1(X, \mathcal{O}_X/J^r \otimes \mathcal{L}) \leqq \dim \mathrm{H}^1(\hat{X}, \hat{\mathcal{L}}),$$

we have

$$\chi(\mathcal{O}_X/J^r \otimes \mathcal{L}) \geqq -\dim \mathrm{H}^1(\hat{X}, \hat{\mathcal{L}})$$

for any $r \gg 0$. Thus $-(a+2b)$ must be positive. Therefore $a+2b<0$.

Similarly, we have $a+b<0$, by

$$\chi(\mathcal{O}_X/I_C^r \otimes \mathcal{L}) = -(a+b)\binom{r+n-2}{n} + (\deg \mathcal{L}|_C + \chi(\mathcal{O}_C))\binom{r+n-2}{n-1}.$$

Assume that $C \cong \mathbf{P}^1$. The dimension of the Hilbert scheme $Hilb_{C/X}$ at $[C]$ is not less than $h^0(N_{C/X}) - h^1(N_{C/X}) = \deg N_{C/X} + (n-1)$. Since C is exceptional, the dimension of the Hilbert scheme at $[C]$ must be zero. Thus $a+b = \deg N_{C/X} \leqq -n+1$. □

Remark. Jiménetz[6] Proposition 4.1 insists the similar inequality $a+(h+1)b \leqq 1$ instead of $a+2b \leqq 1$. But this is incorrect. Theorem 1.3 gives a counter example for it. He have made an error in calculation of $\chi(J^r/J^{r+1} \otimes \mathcal{L})$.

§3. Examples of exceptional rational curves.

Proof of Theorem 1.3. It is convenient to fix better notations. We shall thereby prove the following:

Let q and r are non-negative integers with $q+r=n-1$, and let $n_1, \cdots, n_q$ and $p_1, \cdots, p_r$ be any integers such that $-n_i<0$, $p_j \geqq 0$ for $1 \leqq i \leqq q$, $1 \leqq j \leqq r$, and that

(3.1) $\quad -(n_1+\cdots+n_q) + 2(p_1+\cdots+p_r) \leqq -n+1.$

Then, there exist an n-dimensional algebraic variety X and an exceptional curve C in X whose normal bundle is

$$N_{C/X} = \mathcal{O}_C(-n_1) \oplus \cdots \oplus \mathcal{O}_C(-n_q) \oplus \mathcal{O}_C(p_1) \oplus \cdots \oplus \mathcal{O}_C(p_r). \tag{3.2}$$

Let U and V are $\mathbb{C}^n$ with coordinates $(t_1, \ldots, t_q, s_1, \ldots, s_r, w)$ and $(z_1, \ldots, z_q, y_1, \ldots, y_r, x)$ respectively. Let $P_j = 2(p_1 + \cdots p_j) + j$ $(0 \leqq j \leqq r,\ P_0 = 0)$, $N_i = n_1 + \cdots + n_i - i$ $(0 \leqq i \leqq q,\ N_0 = 0)$, and

$$\sigma = y_1^2 + x^{P_1} y_2^2 + \cdots + x^{P_{r-1}} y_r^2$$

(if $r = 0$ then $\sigma = 0$). Note that $N_i \geqq 0$. We construct X by patching U and V by the following transition functions:

$$\begin{cases} t_i = x^{n_i} z_i + x^{1-N_{i-1}} \sigma & (1 \leqq i \leqq q) \\ s_j = x^{-p_j} y_j & (1 \leqq j \leqq r) \\ w = x^{-1}. & \end{cases}$$

Let C be the curve in X defined by $t_1 = \cdots = t_q = s_1 = \cdots = s_r = 0$ in U, and $z_1 = \cdots = z_q = y_1 = \cdots = y_r = 0$ in V. Clearly C is a non-singular rational curve which satisfies (3.2). We shall show that C is an exceptional. We shall directly construct the contraction morphism of C. For that purpose, let's find some holomorphic functions on X which vanish on C. Let

$$u_i = \begin{cases} t_1 = x^{n_1} z_1 + x\sigma & \text{if } i = 1, \\ t_i - w^{n_{i-1}-1} t_{i-1} = x^{n_i} z_i - x z_{i-1} & \text{if } 2 \leqq i \leqq q. \end{cases}$$

$u_1, \ldots, u_q$ are such functions. Let

$$\sigma_j = \sum_{k=j}^{r} x^{P_{k-1} - P_{j-1}} y_k^2$$

for $1 \leqq j \leqq r$. Formally, put $\sigma_{r+1} = 0$. Note that $\sigma_1 = \sigma$. For $1 \leqq j \leqq r$, let $I(j)$ be an integer such that $N_{I(j)} + 1 > P_j \geqq N_{I(j)-1}$ ($I(j)$ is not always unique). Since $N_q \geqq P_r$ by (3.1), we have $1 \leqq I(1) \leqq \cdots \leqq I(r) \leqq q$. Let

$$v_j = w^{P_j - N_{I(j)-1}} t_{I(j)} - \sum_{k=1}^{j} w^{P_j - P_k} s_k^2 = x^{1+N_{I(j)} - P_j} z_{I(j)} + x\sigma_{j+1}.$$

Then $v_1, \ldots, v_r$ are holomorphic functions on X which vanish on C. Moreover, u_i and v_j can be divided by x on V. Thus wu_i, wv_j, $s_k u_i^{p_k}$ and $s_k v_j^{p_k}$ are also such functions $(1 \leqq i \leqq q$, $1 \leqq j \leqq r$, $1 \leqq k \leqq r)$.

Now we have $(q+r)(2+r)$ holomorphic functions u_i, v_j, wu_i, wv_j, $s_k u_i^{p_k}$ and $s_k v_j^{p_k}$. Aligning these functions, we have the holomorphic map $h\colon X \longrightarrow \mathbb{C}^{(n-1)(2+r)}$. Let $X \xrightarrow{\varphi} Y \xrightarrow{g} \mathbb{C}^{(n-1)(2+r)}$ be the Stein factorization of h. Clearly, $\varphi(C)$ is a point. We shall show that $X - C \cong Y - \varphi(C)$. It is enough to show that h is finite map on $X - C$. It is elementary (and somewhat tiresome) work to check it. These details are similar to the last part of the proof of Theorem 3.1. Thus we omit it. $\square$

Theorem 3.1. *For any positive integer m, there exists an 3-dimensional algebraic variety X and an exceptional rational curve C in X with*

$$N_{C/X} = \mathcal{O}_C(-2m-1) \oplus \mathcal{O}_C(m).$$

Proof. Let U and V be $\mathbb{C}^3$ with coordinates $(t,\ s,\ w)$ and $(z,\ y,\ x)$. We construct X by patching U and V by the following transition functions:

$$\begin{cases} t = x^{2m+1}z + y^2 + x^{2m}y^3 \\ s = x^{-m}y \\ w = x^{-1} \end{cases}$$

It is clear that the curve $C \subset X$ defined by $t = s = 0$ in U and $z = y = 0$ in V is a rational curve with $I_C/I_C^2 \cong \mathcal{O}_C(2m+1) \oplus \mathcal{O}_C(-m)$ where I_C is a defining ideal of C in $\mathcal{O}_X$. We shall show that this $(-2m-1,\ m)$-curve C is exceptional. The first step is to find five holomorphic functions $v_1,\ \cdots,\ v_5$ on X. Let $v_1 = t = x^{2m+1}z + y^2 + x^{2m}y^3$ and $v_2 = w^{2m}t - s^2 = xz + y^3$. v_1 and v_2 are holomorphic functions on X. Since $u = v_2^2 - v_1^3 = 2xy^3z + x^2z^2 + x^{2m}u_0(x,\ z,\ y)$ can be divided by x, $v_3 = su^m$ and $v_4 = wu$ are also holomorphic functions on X. For $r \geqq 0$, let

$$f_{0,r} = \begin{cases} w^m t^{r/2} & \text{if } r \text{ is even} \\ s t^{(r-1)/2} & \text{if } r \text{ is odd.} \end{cases}$$

Inductively, for $q \geqq 1$, let $f_{q,r} = f_{0,r}v_2^q - \sum_{i=0}^{q-1} \binom{q}{i} f_{i,3q-3i+r}$.

Claim. $\displaystyle\sum_{i=a}^{b} (-1)^{i-a} \binom{b}{i}\binom{i}{a} = 0.$

Proof. Let $r = b - a$. Then

$$\sum_{i=a}^{b} (-1)^{i-a} \binom{b}{i}\binom{i}{a} = \frac{b!}{a!}\sum_{i=a}^{b} \frac{(-1)^{i-a}}{(b-i)!(i-a)!} = \frac{b!}{a!}\sum_{j=0}^{r} \frac{(-1)^j}{(r-j)!j!} = \frac{b!}{a!r!}\sum_{j=0}^{r} (-1)^j \binom{r}{j} = 0.$$

$\square$

Let

$$\sigma_i^{(q,r)} = \begin{cases} w^m t^{(3q+r-3i)/2} v_2^i & \text{if } q+r+i \text{ is even} \\ s t^{(3q+r-3i-1)/2} v_2^i & \text{if } q+r+i \text{ is odd.} \end{cases}$$

Claim. $\displaystyle f_{q,r} = \sum_{i=0}^{q} (-1)^{q-i} \binom{q}{i} \sigma_i^{(q,r)}.$

Proof. Note that $\sigma_i^{(k,3q-3k+r)} = \sigma_i^{(q,r)}$. Let $\sigma_i = \sigma_i^{(q,r)}$. We shall show that

$$f_{k,3q-3k+r} = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sigma_i.$$

Note that when we put $k = q$, we have the claim. If $k = 0$, the above equality trivially holds. If $k > 0$, we shall prove this by the induction on k. Since $f_{0,3q-3k+r}v_2^k = \sigma_k^{(q,r)}$,

$$\begin{aligned} f_{k,3q-3k+r} &= f_{0,3q-3k+r}v_2^k - \textstyle\sum_{i=0}^{k-1}\binom{k}{i} f_{i,3k-3i+r} \\ &= \sigma_k - \textstyle\sum_{i=0}^{k-1}\binom{k}{i}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\sigma_j \\ &= \sigma_k - \textstyle\sum_{j=0}^{k-1}\left(\sum_{i=j}^{k-1}(-1)^{i-j}\binom{k}{i}\binom{i}{j}\right)\sigma_j \\ &= \sigma_k + \textstyle\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}\sigma_j \\ &= \textstyle\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sigma_j. \end{aligned}$$

□

Trivially, $f_{q,r}$ is a polynomial with respect to w, t, s. On the other hand, by construction, $f_{q,r}(x, z, y) = x^{q-m}z^q y^r + x^m g(x, z, y)$, where g is a suitable polynomial. Especially, $f_{m,0}$ is also a polynomial with respect to x, z, y. Thus, $v_5 = f_{m,0}$ is a holomorphic function on X.

Now we have a holomorphic mapping $h = (v_1, \cdots, v_5)\colon X \longrightarrow \mathbb{C}^5$. Let $X \xrightarrow{\varphi} Y \xrightarrow{g} \mathbb{C}^5$ be the Stein factorization of h. Since $C = h^{-1}(0)$, $\varphi(C)$ is a point. We shall show that $X - C \cong Y - \varphi(C)$. Since g is a finite map, and since every fiber of φ is connected, it is enough to show that h is finite except the origin 0.

Let $v = (v_1, \cdots, v_5) \in h(U) - \{0\}$. If $v_2^2 - v_1^3 \neq 0$, then $t = v_1$, $s = v_3/(v_2^2 - v_1^3)^m$ and $w = v_4/(v_2^2 - v_1^3)$. Thus $h^{-1}(v)$ is just a point.

Assume $v_2^2 - v_1^3 = 0$. If $v_2 = 0$, then we can derive $v_1 = v_2 = v_3 = v_4 = v_5 = 0$. Thus $v_2 \neq 0$. Since $f_{m,0} = v_5$, we have $\alpha w^m - \beta s = v_5$, where

$$\alpha = \begin{cases} \displaystyle\sum_{j=0}^{m/2}\binom{m}{2j} v_1^{(3m-6j)/2} v_2^{2j} & \text{if } m \text{ is even} \\ \displaystyle\sum_{j=0}^{(m-1)/2}\binom{m}{2j+1} v_1^{(3m-6j-3)/2} v_2^{2j+1} & \text{if } m \text{ is odd} \end{cases}$$

and

$$\beta = \begin{cases} \displaystyle\sum_{j=0}^{(m/2)-1}\binom{m}{2j+1} v_1^{(3m-6j-4)/2} v_2^{2j+1} & \text{if } m \text{ is even} \\ \displaystyle\sum_{j=0}^{(m-1)/2}\binom{m}{2j} v_1^{(3m-6j-1)/2} v_2^{2j} & \text{if } m \text{ is odd.} \end{cases}$$

Since $v_2^2 = v_1^3$ and since $v_2 \neq 0$, we have $(\alpha, \beta) \neq (0, 0)$. Thus the equations on w and s

$$\begin{cases} \alpha w^m - \beta s = v_5 \\ v_1 w^{2m} - s^2 = v_2 \end{cases}$$

have at most $2m$ common solutions. Thus $h_{|U-C}$ is finite.

Let $v = (v_1, \cdots, v_5) \in h(V-U) - \{0\}$. Then $x = 0$. Thus $v_1 = y^2$, $v_2 = y^3$, $v_3 = 2^m z^m y^{3m+1}$, $v_4 = 2zy^3$, and $v_5 = z^m$. Therefore $h_{|V-U-C}$ is finite.

Thus we have $h_{|X-C}$ is finite. Therefore C is exceptional. □

Lemma 3.2. Let $\varphi\colon (X, C) \to (Y, y)$ be a small contraction of $C \cong \mathbf{P}^1$, and $\mathfrak{m}_y$ be the maximal ideal at y in $\mathcal{O}_Y$. If $I_C^2 \subset \varphi^*\mathfrak{m}_y$ and $\mathrm{H}^1(C, I_C/I_C^2) = 0$ then $\mathbf{R}^1\varphi_*\mathcal{O}_X = 0$.

Proof. Let $\mathfrak{m}_y = (y_1, \ldots, y_r)$, $J = \varphi^*\mathfrak{m}_y$ and $z_i = \varphi^* y_i$ $(1 \leqq i \leqq r)$. Define $g\colon \mathcal{O}_X^{\oplus r} \to J$ by $g(a_1, \ldots, a_r) = a_1 z_1 + \cdots + a_r z_r$. Since $\mathbf{R}^2\varphi_*(\operatorname{Ker} g) = 0$, $\overline{g}\colon \mathbf{R}^1\varphi_*\mathcal{O}_X^{\oplus r} \to \mathbf{R}^1\varphi_* J$ is surjective. Since $\mathrm{H}^1(C, I_C/I_C^2) = 0$, $\overline{\iota}\colon \mathbf{R}^1\varphi_* I_C^2 \to \mathbf{R}^1\varphi_* I_C$ is surjective. Since $I_C^2 \subset J$, $\overline{\iota}$ factor through the surjection $\mathbf{R}^1\varphi_* J \to \mathbf{R}^1\varphi_* I_C$. Since $\mathrm{H}^1(C, \mathcal{O}_C) = 0$, $\mathbf{R}^1\varphi_* I_C \cong \mathbf{R}^1\varphi_*\mathcal{O}_X$. Thus we have a surjection $h\colon \mathbf{R}^1\varphi_*\mathcal{O}_X^{\oplus r} \to \mathbf{R}^1\varphi_*\mathcal{O}_X$.

Assume that $\mathbf{R}^1\varphi_*\mathcal{O}_X \neq 0$. Let $\xi_1, \ldots, \xi_d$ be basis of $\mathbf{R}^1\varphi_*\mathcal{O}_X$. Since h factors through $\overline{g}$, ξ_i can be written as

$$\xi_i = \sum_{k=1}^{r} z_k \sum_{j=1}^{d} a_{ijk}\xi_j,$$

here $a_{ijk} \in \mathbb{C}$. The matrix $\left(\sum_{k=1}^{r} a_{ijk} z_k\right)_{i,j}$ must be invertible. But values of z_k are zero on C. This is a contradiction. Thus $\mathbf{R}^1\varphi_*\mathcal{O}_X = 0$. □

By the above Lemma, the small singularities obtained as in the proof of Theorem 1.3 are rational, if $\mathrm{H}^1(C, I_C/I_C^2) = 0$. But the following are examples with $\mathrm{H}^1(C, I_C/I_C^2) = 0$ but $\mathbf{R}^1\varphi_*\mathcal{O}_X \neq 0$.

Example 3.3. Let U and V be $\mathbb{C}^3$ with coordinates (t, s, w) and (z, y, x). We construct X and C by the following transition functions as the proof of Theorem 3.1.

$$\begin{cases} t = x^5 z + x y^3 \\ s = x^{-1} y \\ w = x^{-1} \end{cases}$$

C is contracted by

$$\begin{cases} v_1 = t = x^5 z + x y^3 \\ v_2 = wt = x^4 z + y^3 \\ v_3 = w^4 t - s^3 = xz \\ v_4 = w^5 t - w s^3 = z \end{cases}$$

Since $I_C/I_C^2 \cong \mathcal{O}_C(5) \oplus \mathcal{O}_C(-1)$, $\mathrm{H}^1(C, I_C/I_C^2) = 0$.

We shall show that $\mathbf{R}^1\varphi_*\mathcal{O}_X \neq 0$. Note that $\mathrm{H}^1(X, \mathcal{O}_X) \cong \mathbf{R}^1\varphi_*\mathcal{O}_X$ An elementary calculation of the Čech cohomology show us that the class of $x^{-1}y^2$ is not zero in $\mathrm{H}^1(X, \mathcal{O}_X)$. Thus $\mathbf{R}^1\varphi_*\mathcal{O}_X \neq 0$. (A careful calculation lead us to $\dim \mathbf{R}^1\varphi_*\mathcal{O}_X = 1$.)

Example 3.4. Let $n \geqq 3$. Let U and V be $\mathbb{C}^3$ with coordinates $(t,\ s,\ w)$ and $(z,\ y,\ x)$. We construct X and C by the following transition functions as above.

$$\begin{cases} t = x^{n+1}z + y^n + x^n y^{n+1} \\ s = x^{-1}y \\ w = x^{-1} \end{cases}$$

C is contracted by

$$\begin{cases} v_1 = t = x^{n+1}z + y^n + x^n y^{n+1} \\ v_2 = w^n t - s^n = xz + y^{n+1} \\ v_3 = w^{n+1}t - ws^n - st = z - x^n yz - x^{n-1}y^{n+2} \\ v_4 = sv_2^{n-1} - wt^n \end{cases}$$

Note that v_4 is a polynomial with $(x,\ y,\ z)$. Since $I_C/I_C^2 \cong \mathcal{O}_C(n+1) \oplus \mathcal{O}_C(-1)$, $\mathrm{H}^1(C, I_C/I_C^2) = 0$. Onthe other hand,

$$\mathbf{R}^1\varphi_*\mathcal{O}_X = \bigoplus_{i=1}^{n-1}\bigoplus_{j=1}^{i} \mathbb{C}\cdot x^{-j}y^{i+1}$$

. Thus $\dim \mathbf{R}^1\varphi_*\mathcal{O}_X = (n-1)(n-2)/2$.

Remark. By the above exsamples, we conclude the following: A three dismensional small singularity obtained by contracting $(a, 1)$-curve is a rational singularity if $a = -3$. But if $a \leqq -4$, we can construct both examples that are rational and that are irrational.

§4. Moving Fiber.

Consider the case that $\varphi\colon X \to Y$ is a morphism such that the general fibers of φ are at most one-dimensional. We choose a and b as in Theorem 1.2. Then we can prove that $a+2b \leqq 0$ holds, similarly to the proof of Theorem 1.2 using Theorem 2.1 and 2.6. We shall consider the case $a + 2b = 0$. Let C be an irreducible component of an one-dimensional fiber of φ. We expect that some multiple of C is movable. The following theorem gives a partial answer to this. This argument play one of the important roles in the classification theory of the extremal ray (Mori[11]).

We choose the ideal J of $\mathcal{O}_X$ as in the proof of Theorem 1.2. That is, J is the ideal such that $I_C \supset J \supset I_C^2$, and that I_C/J gives the quotient bundle of I_C/I_C^2 of the minimal degree.

Theorem 4.1. *Assume that $a + 2b = 0$ and that $C \cong \mathbb{P}^1$, then $J^e/I_C J^e$ and $I_C J^e/J^{e+1}$ are semi-negative for any positive integer e.*

Proof. We shall show that $J^e/I_C J^e$ is semi-negative for any positive integer e.

Assume that $J^e/I_C J^e$ is not semi-negative. Let's derive a contradiction. Since $J^e/I_C J^e$ is not semi-negative, we can find an ideal L such that $J^e \supset L \supset I_C J^e$ and that $L/I_C J^e$ is an ample invertible $\mathcal{O}_C$-module. Let $f = 2e+1$ and let $L_r = \sum_{i+2j+fl=r} I_C^i J^j L^l$. We take a divisor A of X as in Theorem 2.1. We want to derive the contradiction $h^0(X, \mathcal{O}_X/L_r \otimes_{\mathcal{O}_X} \mathcal{O}_X(A)) = \chi(\mathcal{O}_X/L_r \otimes_{\mathcal{O}_X} \mathcal{O}_X(A)) < 0$ for $r \gg 0$. We have to calculate $\deg \mathcal{O}_X/L_r$ and length $\mathcal{O}_X/L_r$. Let's construct a saturated filtration between L_r and L_{r+1}.

For an integer i with $0 \leqq i \leqq [r/f]$, we define numeric functions $\gamma(r,i) = [r/f] - i$, $\beta'(r,i) = (r\%f + fi)$, $\beta(r,i) = [\beta'(r,i)/2]$ and $\alpha(r,i) = \beta'(r,i)\%2$, where [ ] is the Gaussian symbol, and % is the remainder operator (adopted in the programming language $\mathcal{C}$). Note that $\beta'(r,i) + f\gamma(r,i) = \alpha(r,i) + 2\beta(r,i) + f\gamma(r,i) = r$. Using these symbols, we define $L_{r,i}$ as the following:

$$J_l = J^{[l/2]} I_C^{l\%2},$$

$$P_{r,i} = I_C^{\alpha(r,i)} J^{\beta(r,i)} L^{\gamma(r,i)} = J_{\beta'(r,i)} L^{\gamma(r,i)},$$

$$L_{r,i} = L_{r+1} + \sum_{k=i}^{[r/f]} P_{r,k}.$$

Note that $L_r = \sum_{i+2j+fl \geqq r} I_C^i J^j L^l = \sum_{k=0}^{[r/f]} P_{r,k}$. We have the filtration $L_r = L_{r,0} \supset L_{r,1} \supset \cdots \supset L_{r,[r/f]} \supset L_{r+1}$. Put $L_{r,[r/f]+1} = L_{r+1}$ for convenience' sake. We shall introduce an another filtration. Let

$$U_{l,i}^r = L_{l,i} + J_r.$$

This definition is valid if $0 \leqq i \leqq [l/f]+1$. But $U_{l,i}^r$ have significance, only if $r+1 \leqq l < [(fr-1)/(2e)]$ and if $0 \leqq i < r - l + [l/f]$. Consider the coarse filtration $L_{r,[r/f]} = U_{r+1,0}^r \supset U_{r+2,0}^r \supset U_{r+3,0}^r \supset \cdots \supset J_r$, and the fine filtration $U_{l,0}^r \supset U_{l,1}^r \supset U_{l,2}^r \supset \cdots \supset U_{l+1,0}^r$. By the following claim, we consent that the conditions $r+1 \leqq l < [(fr-1)/(2e)]$ and $0 \leqq i < r-l+[l/f]$ are reasonable.

Claim 4.1.1 If $l \geqq [(fr-1)/(2e)]$, then $U_{l,i}^r \subset J_r$. If $i \geqq r - l + [l/f]$, then $U_{l,i}^r \subset U_{l+1,0}^r$.

Proof. Note that $P_{l,i} \subset J_r$ if and only if $\beta'(l,i) + 2e\gamma(l,i) \geqq r$. Hence $U_{l,i}^r \subset U_{l+1,0}^r$ if and only if $\beta'(l,i) + 2e\gamma(l,i) \geqq r$. Note that $\beta'(l,i) + 2e\gamma(l,i) = (l\%f + fi) + (f-1)([l/f] - i) = (l\%f + f[l/f]) - ([l/f] - i) = l - [l/f] + i$. Thus $U_{l,i}^r \subset U_{l+1,0}^r$ if and only if $i \geqq r - l + [l/f]$.

Assume $l \geqq [(fr-1)/(2e)]$. Since $l > [(fr-1)/(2e)] - 1 = [f(r-1)/(2e)]$, $l > f(r-1)/(f-1)$. Hence $l - (l/f) > r - 1$. Thus $\beta'(l,i) + 2e\gamma(l,i) \geqq l - [l/f] \geqq r$ for $i \geqq 0$. Therefore $U_{l,i}^r \subset J_r$.

With the symbol $\mathbb{U}^r$, we denote the filtration $L_{r,[r/f]} = U^r_{r+1,0} \supset U^r_{r+1,1} \supset \cdots \supset U^r_{r+2,0} \supset \cdots \supset J_r$. For $0 \leqq i \leqq [r/(2e)]$, let $J_{r,i} = J_{r-2ei}L^i + J_{r+1}$, and let $J_{r,[r/(2e)]+1} = J_{r+1}$. Then we have a filtration $J_r \supset J_{r,1} \supset \cdots \supset J_{r,[r/(2e)]} \supset J_{r+1}$. Consider the natural surjective map $\mathrm{S}^i(L/I_CJ^e) \otimes J_{r-2ei}/J_{r-2ei,1} \to J_{r,i}/J_{r,i+1}$.

Claim 4.1.2 $J_{r,i}/J_{r,i+1}$ is a locally free $\mathcal{O}_C$-module for $0 \leqq i \leqq [r/(2e)]$, and

$$J_{r,i}/J_{r,i+1} \cong \mathrm{S}^i(L/I_CJ^e) \otimes J_{r-2ei}/J_{r-2ei,1}.$$

Proof. To begin with, we shall show that $J_{r,i}/J_{r,i+1}$ are locally free $\mathcal{O}_C$-modules. We show this Subclaim by an induction on r. Put $r = 2ep + q$ $(0 \leqq q < 2e)$. Since $J^e/I_CJ^e \cong J^e/L \oplus L/I_CJ^e$, we obtain an injection $f_0\colon J_q/J_{q+1} \otimes \mathrm{S}^p(J^e/L) \to J_q/J_{q+1} \otimes \mathrm{S}^p(J^e/I_CJ^e)$. Let $g_0\colon J_q/J_{q+1} \otimes \mathrm{S}^p(J^e/I_CJ^e) \to J_r/J_{r+1}$ be the natural surjection, and let $M_{r,0} = \mathrm{Image}(g_0 \circ f_0)$. $M_{r,0}$ is a locally free $\mathcal{O}_C$-module, and it have a canonical surjection $\pi_0\colon M_{r,0} \to J_r/J_{r,1}$. We assume that $M_{k,0} \cong J_k/J_{k,1}$ for $k < r$ as an induction hypothesis. By this assumption, we obtain the canonical injection, $f_i\colon M_{r-2ei,0} \otimes \mathrm{S}^i(L/I_CJ^e) \to J_{r-2ei}/J_{r-2ei+1} \otimes \mathrm{S}^i(J^e/I_CJ^e)$ for $1 \leqq i \leqq [r/(2e)]$. There also exists the canonical surjection $g_i\colon J_{r-2ei}/J_{r-2ei+1} \otimes \mathrm{S}^i(J^e/I_CJ^e) \to J_r/J_{r+1}$. Let $M_{r,i} = \mathrm{Image}(g_i \circ f_i)$. $M_{r,i}$ is a locally free $\mathcal{O}_C$-module, and it have a canonical surjection $\pi_i\colon M_{r,i} \to J_{r,i}/J_{r,i+1}$. Since $M_{r,i} \cap (\sum_{j \neq i} M_{r,j}) = 0$, we have $\sum_{i=0}^{[r/(2e)]} \mathrm{rank}\, M_{r,i} \leqq \mathrm{rank}\, J_r/J_{r+1}$. Thus $\mathrm{rank}\, J_r/J_{r+1} = \sum_{i=0}^{[r/(2e)]} \mathrm{rank}\, J_{r,i}/J_{r,i+1} \leqq \sum_{i=0}^{[r/(2e)]} \mathrm{rank}\, M_{r,i} \leqq \mathrm{rank}\, J_r/J_{r+1}$. Therefore we have $M_{r,i} \cong J_{r,i}/J_{r,i+1}$ for $0 \leqq i \leqq [r/(2e)]$. Thus we have proved that $J_{r,i}/J_{r,i+1}$ are locally free $\mathcal{O}_C$-modules. Since the natural injection $\mathrm{S}^i(L/I_CJ^e) \otimes J_{r-2ei}/J_{r-2ei,1} \to J_r/J_{r+1}$ can be factored as $\mathrm{S}^i(L/I_CJ^e) \otimes J_{r-2ei}/J_{r-2ei,1} \to M_{r,i} \to J_r/J_{r+1}$, we have that $\mathrm{S}^i(L/I_CJ^e) \otimes J_{r-2ei}/J_{r-2ei,1} \to J_{r,i}/J_{r,i+1}$ is injective. Since $J_{r-2ei}/J_{r-2ei,1} \cong M_{r-2ei,0}$, this map is surjective, therefore isomorphic.

Since $L_{r,[r/f]} = J_r + L_{r+1}$, we have the natural surjection $J_r/J_{r+1} \to L_{r,[r/f]}/L_{r+1}$. Since $J_{r,1} \subset L_{r+1}$, the above map can be reduce to the surjection $J_r/J_{r,1} \to L_{r,[r/f]}/L_{r+1}$. On the other hand, we obtain the natural surjections

(LU-r) $\quad L_{l,i}/L_{l,i+1} \to U^r_{l,i}/U^r_{l,i+1} \quad (r+1 \leqq l < [(fr-1)/(2e)]$ and $0 \leqq i < r - l + [l/f])$.

Note that, if $r < f$, these are isomorphisms. By induction on r, we will afterward show that these surjections are all isomorphisms. We proceed our argument assuming that (LU-k) are isomorphisms for $k \leqq r$. Moreover we assume that

(LF-r) "$L_{l,i}/L_{l,i+1}$ are locally free $\mathcal{O}_C$-modules if ($l \leqq r$ and $0 \leqq i \leqq [l/f]$) or ($r+1 \leqq l < [(fr-1)/(2e)]$ and $0 \leqq i < r - l + [l/f]$)."

Note that (LF-$(f-1)$) holds trivially.

We consider two filtrations between $L_{r,[r/f]}$ and J_{r+1}:

$(\mathbb{F}_1): L_{r,[r/f]} = U^r_{r+1,0} \supset \cdots$ (filtration $\mathbb{U}^r$) $\cdots \supset J_r \supset J_{r,1} \supset \cdots \supset J_{r+1}$,

$(\mathbb{F}_2): L_{r,[r/f]} \supset L_{r+1} \supset L_{r+1,1} \supset \cdots \supset L_{r+1,[(r+1)/f]} \supset \cdots$ (filtration $\mathbb{U}^{r+1}$) $\cdots \supset J_{r+1}$.

We can couple the successive quotients of both filtrations as the following. Any quotients $U^r_{r+1,i}/U^r_{r+1,i+1}$ in $(\mathbb{F}_1)$ can be coupled with $L_{r+1,i}/L_{r+1,i+1}$ in $(\mathbb{F}_2)$ — (CPL-1). For $l \geqq r+2$, any quotients $U^r_{l,i}/U^r_{l,i+1}$ in $(\mathbb{F}_1)$ can be coupled with $U^{r+1}_{l,i}/U^{r+1}_{l,i+1}$ in $(\mathbb{F}_2)$ — (CPL-2). The quotient $J_r/J_{r,1}$ in $(\mathbb{F}_1)$ can be coupled with $L_{r,[r/f]}/L_{r+1}$ in $(\mathbb{F}_2)$ — (CPL-3). $J_{r,1}/J_{r,2}$ in $(\mathbb{F}_1)$ can be coupled with $L_{r+1,[(r+1)/f]-1}/L_{r+1,[(r+1)/f]}$ in $(\mathbb{F}_2)$ — (CPL-4). For $2 \leqq i \leqq [r/(2e)]$, $J_{r,i}/J_{r,i+1}$ in $(\mathbb{F}_1)$ can be coupled with $U^{r+1}_{r+i,[(r+i)/f]-i}/U^{r+1}_{r+i,[(r+i)/f]-i+1}$ in $(\mathbb{F}_2)$ — (CPL-5). By the above couplings, we have an one-to-one corresponding between the successive quotients of $(\mathbb{F}_1)$ and $(\mathbb{F}_2)$. We shall show that these couples are isomorphic to each other. About (CPL-1), since (LU-r) is an isomorphism, we have $U^r_{r+1,i}/U^r_{r+1,i+1} \cong L_{r+1,i}/L_{r+1,i+1}$. About (CPL-2), we have the natural surjection,

(SJ-2.r) $\quad U^r_{l,i}/U^r_{l,i+1} \cong L_{l,i}/L_{l,i+1} \to U^{r+1}_{l,i}/U^{r+1}_{l,i+1}$

About (CPL-3), we already have the natural surjection

(SJ-3.r) $\quad J_r/J_{r,1} \to L_{r,[r/f]}/L_{r+1}$.

About (CPL-4), since $P_{r+1,[(r+1)/f]-1} = J_{r-2e}L$, we have the natural surjection

(SJ-4.r) $\quad J_{r,1}/J_{r,2} \cong L/I_C J^e \otimes J_{r-2e}/J_{r-2e+1,1} \to L_{r+1,[(r+1)/f]-1}/L_{r+1,[(r+1)/f]}$.

About (CPL-5), since $P_{r+i,[(r+i)/f]-i} = J_{r-2ei}L^i$, we have the natural surjection

(SJ-5.r)
$$\begin{aligned} J_{r,i}/J_{r,i+1} &\cong S^i(L/I_C J^e) \otimes J_{r-2ei}/J_{r-2ei,1} \\ &\to L_{r+i,[(r+i)/f]-i}/L_{r+i,[(r+i)/f]-i+1} \\ &\to U^{r+1}_{r+i,[(r+i)/f]-i}/U^{r+1}_{r+i,[(r+i)/f]-i+1}. \end{aligned}$$

Claim 4.1.3. The surjections (LU-$(r+1)$), (SJ-2.r), (SJ-3.r), (SJ-4.r) and (SJ-5.r) are isomorphisms. Moreover (LF-$(r+1)$) holds.

Proof. We are now proving the above claim by an induction on r. Since (SJ-2.r), (SJ-3.r), (SJ-4.r) and (SJ-5.r) are obtained as the one to one coupling between the successive quotients of the filtrations $(\mathbb{F}_1)$ and $(\mathbb{F}_2)$, we should mention that if each left hand sides of these surjections are locally free $\mathcal{O}_C$-modules, these surjections are all isomorphisms. Since (SJ-3.r), (SJ-4.r) and (SJ-5.r) clearly satisfy this condition, it is enough to check that $L_{l,i}/L_{l,i+1}$ are locally free for $l \geqq r+2$ and $0 \leqq i < r-l+[l/f]$. This condition is satisfied by the induction hypothesis

(LF-r). Thus (SJ-2.r), (SJ-3.r), (SJ-4.r) and (SJ-5.r) are isomorphisms. Since (SJ-2.r) and (SJ-5.r) are isomorphisms, we know that (LU-$r+1$) are isomorphisms. Therefore (LF-$(r+1)$) holds. Thus we complete the proof of Claim 4.1.3.

Claim 4.1.4. The natural morphism $L_{\beta'(r,i),i}/L_{\beta'(r,i),i+1} \otimes \mathrm{S}^{\gamma(r,i)}(L/I_C J^e) \to L_{r,i}/L_{r,i+1}$ *is an isomorphism for* $0 \leqq i < [r/f]$.

Proof. Since

$$L/I_C J^e \otimes J_{r-2e}/J_{r-2e,1} \cong L_{r+1,[(r+1)/f]-1}/L_{r+1,[(r+1)/f]},$$

and

$$J_{r-2e}/J_{r-2e,1} \cong L_{r-2e,[(r-2e)/f]}/L_{r-2e,[(r-2e)/f]+1},$$

we have the following isomorphism:

$$L/I_C J^e \otimes L_{r-2e,[(r-2e)/f]}/L_{r-2e,[(r-2e)/f]+1} \to L_{r+1,[(r+1)/f]-1}/L_{r+1,[(r+1)/f]}.$$

Similarly, for $2 \leqq i \leqq [r/(2e)]$, since

$$\mathrm{S}^i(L/I_C J^e) \otimes J_{r-2ei}/J_{r-2ei,1} \cong L_{r+i,[(r+i)/f]-i}/L_{r+i,[(r+i)/f]-i+1},$$

and since

$$J_{r-2ei}/J_{r-2ei,1} \cong L_{r+i,[(r+i)/f]-i}/L_{r+i,[(r+i)/f]-i+1},$$

we have the following isomorphism:

$$\mathrm{S}^i(L/I_C J^e) \otimes L_{r-2ei,[(r-2ei)/f]}/L_{r-2ei,[(r-2ei)/f]+1} \to L_{r+i,[(r+i)/f]-i}/L_{r+i,[(r+i)/f]-i+1}.$$

Varying r and i, we have Claim 4.1.4.

Put $rk(r) = \operatorname{rank} L_{r,[r/f]}/L_{r+1}$ and put $d(r) = \deg L_{r,[r/f]}/L_{r+1}$. Since $L_{r,[r/f]}/L_{r+1} \cong J_r/J_{r,1}$, and since $\mathrm{S}^i(L/I_C J^e) \otimes J_{r-2ei}/J_{r-2ei,1} \cong J_{r,i}/J_{r,i+1}$, we have $\sum_{i=0}^{[r/(2e)]} rk(r-2ei) = \operatorname{rank} J_r/J_{r+1}$, and $\sum_{i=0}^{[r/(2e)]}(d(r-2ei) + rk(r-2ei) \cdot i \deg L/I_C J^e) = \deg J_r/J_{r+1}$.

Claim 4.1.5.

$$\text{(4.1.5.a)} \quad rk(2r) = \sum_{k=0}^{[h/2]} \binom{h}{2k} \sum_{j=0}^{e-1} \binom{r-j-k+n-3}{n-3},$$

$$\text{(4.1.5.b)} \quad rk(2r+1) = \sum_{k=0}^{[(h-1)/2]} \binom{h}{2k+1} \sum_{j=0}^{e-1} \binom{r-j-k+n-3}{n-3}.$$

Proof. We shall prove (4.1.5.a). By Theorem 2.6, we have $\sum_{i=0}^{[2r/(2e)]} rk(2r-2ei) = \sum_{k=0}^{[h/2]} \binom{h}{2k} \binom{r-k+n-2}{n-2}$. Assume that $rk(2r-2ei) = \sum_{k=0}^{[h/2]} \binom{h}{2k} \sum_{j=0}^{e-1} \binom{r-ei-j-k+n-3}{n-3}$ holds for $i \geqq 1$. Then it is enough to show $\sum_{i=0}^{[r/e]} \sum_{k=0}^{[h/2]} \binom{h}{2k} \sum_{j=0}^{e-1} \binom{r-ei-j-k+n-3}{n-3} = \sum_{k=0}^{[h/2]} \binom{h}{2k} \binom{r-k+n-2}{n-2}$. Note that $\sum_{j=0}^{m} \binom{j+q}{q} = \binom{m+q+1}{q+1}$. Put $p = 2ei + j$. Then $\sum_{i=0}^{[r/e]} \sum_{j=0}^{e-1} \binom{r-2ei-j-k+n-3}{n-3} =$

$\sum_{p=0}^{e[r/e]+e-1}\binom{r-k-p+n-3}{n-3} = \sum_{p=0}^{r-k}\binom{r-k-p+n-3}{n-3} = \binom{r-k+n-2}{n-2}$. Thus (4.1.5.a) hold. Similarly we can prove (4.1.5.b).

We shall represent $\sum_{l=0}^{r} d(l)$ as a function on r. Let $T_0(h) = \sum_{k=0}^{[h/2]}\binom{h}{2k}$, and $T_1(h) = \sum_{k=0}^{[(h-1)/2]}\binom{h}{2k+1}$. Note that $T_0(h)+T_1(h) = \sum_{k=0}^{h}\binom{h}{k} = 2^h$. If h is an odd number, then $T_0(h) = T_1(h) = 2^{h-1}$. If h is an even number, then $T_0(h) = 2^{h-1} + \binom{h}{h/2}/2$ and $T_1(h) = 2^{h-1} - \binom{h}{h/2}/2$. Since $\binom{r-j-k+n-3}{n-3} \sim r^{n-3}/(n-3)!$, we have

$$rk(2r) \sim T_0(h)\frac{er^{n-3}}{(n-3)!},$$

and

$$rk(2r+1) \sim T_1(h)\frac{er^{n-3}}{(n-3)!}.$$

Thus we have

$$\sum_{i=0}^{[r/e]}(rk(2r-2ei)\cdot i) \sim T_0(h)\frac{r^{n-1}}{e(n-1)!},$$

and

$$\sum_{i=0}^{[r/e]}(rk(2r+1-2ei)\cdot i) \sim T_1(h)\frac{r^{n-1}}{e(n-1)!}.$$

Put $c = \deg L/I_CJ^e$. Recall that $-b = \deg I_C/J$ and $-a = \deg J/I_C^2$. Since $\deg J^r/I_CJ^r \sim T_0(h)(-a-2b)r^{n-1}/(n-1)!$ and $\deg I_CJ^r/J^{r+1} \sim T_1(h)(-a-2b)r^{n-1}/(n-1)!$, we have

$$\begin{aligned}\sum_{i=0}^{[r/e]} d(2r-2ei) &= \deg J^r/I_CJ^r - \sum_{i=0}^{[r/e]}(rk(2r-2ei)\cdot i\deg L/I_CJ^e)\\ &\sim T_0(h)\frac{((-a-2b)-c/e)r^{n-1}}{(n-1)!},\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{[r/e]} d(2r+1-2ei) &= \deg I_CJ^r/J^{r+1} - \sum_{i=0}^{[r/e]}(rk(2r+1-2ei)\cdot i\deg L/I_CJ^e)\\ &\sim T_1(h)\frac{((-a-2b)-c/e)r^{n-1}}{(n-1)!}.\end{aligned}$$

Hence

$$\sum_{i=0}^{r} d(l) \sim \frac{((-a-2b)e-c)r^{n-1}}{2^{n-1-h}(n-1)!}.$$

Let's calculate $\deg L_r/L_{r+1}$. Put $r = pf+q$ $(0 \leqq q < f)$. Since

$$L_{pf+q,i}/L_{pf+q,i+1} \cong L_{if+q,i}/L_{if+q,i+1} \otimes \mathrm{S}^{p-i}(L/I_CJ^e),$$

we have

$$\begin{aligned}\deg L_{pf+q}/L_{pf+q+1} &= \sum_{i=0}^{p}\deg L_{if+q,i}/L_{if+q,i+1} \otimes \mathrm{S}^{p-i}(L/I_CJ^e)\\ &= \sum_{i=0}^{p}(d(if+q)+(p-i)crk(if+q))\\ &\sim \sum_{i=0}^{p} d(if+q) + \frac{p^{n-1}f^{n-3}c}{2^{n-h-2}(n-1)!}.\end{aligned}$$

We sum up the above equalities on $q = 0, \cdots, f-1$, thereby we modify $\sum_{i=0}^{p} d(if+q)$ to $\sum_{i=0}^{f(p+1)-1} d(i)$, and we consequently have

$$\deg L_{pf}/L_{(p+1)f} \sim \frac{(pf)^{n-1}}{2^{n-h-1}(n-1)!}(((-a-2b)e-c)+\frac{2c}{f}) \sim \frac{(ef(-a-2b)-(f-2)c)(pf)^{n-1}}{2^{n-h-1}(n-1)!f}.$$

Finally we have

$$\deg \mathcal{O}_X/L_r \sim \frac{r^n}{2^{n-h-1}f^2 n!}(ef(-a-2b)-(f-2)c).$$

By Theorem 2.1, $0 < \chi(\mathcal{O}_X/L_r \otimes_{\mathcal{O}_X} \mathcal{O}_X(A)) \sim \deg \mathcal{O}_X/L_r$. Therefore $(ef(-a-2b) - (f-2)c) \geqq 0$. On the other hand, by the given condition, $a+2b=0$, $c = \deg L/I_C J^e > 0$, $f = 2e+1 \geqq 3$. This implies $(ef(-a-2b)-(f-2)c) < 0$. A contradiction. Thus we conclude that $J^e/I_C J^e$ is semi-negative for any positive integer e.

We shall show that $I_C J^e/J^{e+1}$ is semi-negative for any positive integer e.

Assume that $I_C J^e/J^{e+1}$ is not semi-negative. Take an ample invertible sub module M of $I_C J^e/J^{e+1}$. Let $f = 2e+1$ and $\psi\colon \mathrm{S}^2(I_C J^e/J^{e+1}) \to J^f/I_C J^f$. Clearly $\psi(\mathrm{S}^2(M)) \neq 0$. But this contradict to the fact that $J^f/I_C J^f$ is semi-negative. □

Corollary 4.2. Under the assumption in Theorem 4.1, we assume $I_C/J \cong \mathcal{O}_C(-1)$. Then $\mathrm{Spec}(\mathcal{O}_X/J)$ is movable.

Proof. Since $\deg J/I_C J = -a-2b = 0$ and $J/I_C J$ is semi-negative, we have $J/I_C J \cong \mathcal{O}_C^{\oplus(n-1)}$ and $I_C J/J^2 \cong I_C/J \otimes J/I_C J \cong \mathcal{O}_C(-1)^{\oplus(n-1)}$. The isomorphism $\mathrm{H}^0(X, J/J^2) \cong \mathrm{H}^0(X, (\mathcal{O}_X/J)^{\oplus(n-1)}) \cong k^{n-1}$ derives the isomorphism $J/J^2 \cong (\mathcal{O}_X/J)^{\oplus(n-1)}$. Thus $\mathrm{Spec}(\mathcal{O}_X/J)$ is movable. □

This result will be used in Kachi[7].

References.

[1] T. Ando, *An example of exceptional* $(-2n-1,\ n)$*-curve in algebraic 3-fold,* Proc. Japan Acad., **66** Ser.A, 269-271 (1990)

[2] T. Ando, *On the normal bundle of* $\mathbf{P}^1$ *in the higher dimensional projective variety,* Amer. J. Math. **113**, 949-961 (1991)

[3] M. Artin, *Algebraization of formal moduli II: Existence of the formal modifications,* Ann. of Math., bf 91, 88-135 (1970)

[4] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc., **3**, VII, 414-452 (1958)

[5] A.Fujiki, *On the blowing down of analytic spaces,* Publ. RIMS, Kyoto Univ., **10**, 473-507 (1975)

[6] J. Jiménez, *Contraction of Nonsingular Curves,* Duke Math. J., **65**, 313-332 (1992)

[7] Kachi, Extremal Contractions from 4-dimensinal Manifolds to 3-folds, Ann. Scuola. norm. sup. Pisa, Cl. Sci. **24**, 63-131 (1997).

[8] Y. Kawamata, *A generalization of Kodaira-Ramanujam's Vanishing Theorem,* Math. Ann., **261**, 43-46 (1982)

[9] S. Katz and D. Morrison, *Gorenstein Threefold Singularities with Small Resolutions via Invariant Theory for Weyl Groups,* Preprint.

[10] H. Laufer, *On* $\mathbf{CP}^1$ *as an exceptional set, Recent developments in several complex variables,* Ann. of Math. Stud., Princeton University Press **100**, 261-275 (1981)

[11] S. Mori, *Threefolds whose canonical bundles are not numerically effective,* Ann. of Math., **116**, 133-176 (1982)

[12] S. Mori, *Flip theorem and the existence of minimal models for 3-fold,* J. Am. Math. Soc., **1**, No.1, 117-253 (1988)

[13] D. Morrison, *The Birational Geometry of Surfaces with Rational Double Points,* Math. Ann., **271**, 415-438 (1985)

[14] N. Nakayama, *On smooth exceptional curves in threefolds,* J. Fac. Sci. Univ. Tokyo. Sect. IA, Math., **37**, 511-525 (1990)

[15] T. Ohsawa, *On analytic families of submanifolds and holomorphical convexity of neighbourhoods of* $\mathbf{P}^1$*,* J. Math. Soc. Japan, **33**, 711-731 (1981)

[16] H. Pinkham. *Factorization of birational maps in dimension 3,* Proc. of A.M.S. Summer Inst. on Singularities, Arcata, 1981. Proc. Symposia in Pure Math., A.M.S. **40**, Part 2, 343-371 (1983)

[17] M. Reid. *Minimal models of canonical 3-folds,* Advanced Studies in Pure Math., **1**, Algebraic Varieties and Analytic Varieties, Kinokuniya, 131-180 (1983)