Some operator monotone functions related to Petz-Hasegawa's functions

Masato Kawasaki and Masaru Nagisa

Abstract

Let f be an operator monotone function on $[0,\infty)$ with $f(t) \geq 0$ and $f(1) = 1$. If $f(t)$ is neither the constant function 1 nor the identity function t, then

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))} \qquad t \geq 0$$

is also operator monotone on $[0,\infty)$, where $a, b \geq 0$ and

$$f^\sharp(t) = \frac{t}{f(t)} \qquad t \geq 0.$$

1 Introduction

We call a real continuous function $f(t)$ on an interval I operator monotone on I (in short, $f \in \mathbb{P}(I)$), if $A \leq B$ implies $f(A) \leq f(B)$ for any self-adjoint matrices A, B with their spectrum containd in I. In this paper, we consider only the case $I = [0,\infty)$ or $I = (0,\infty)$. We denote $f \in \mathbb{P}_+(I)$ if $f \in \mathbb{P}(I)$ satisfies $f(t) \geq 0$ for any $t \in I$.

Let $\mathbb{H}_+$ be the upper half-plane of $\mathbb{C}$, that is,

$$\mathbb{H}_+ = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\} = \{z \in \mathbb{C} \mid |z| > 0,\ 0 < \arg z < \pi\},$$

where $\mathrm{Im} z$ (resp. $\arg z$) means the imaginary part (resp. the argument) of z. As Loewner's theorem, it is known that f is operator monotone on I if and only if f has an analytic continuation to that maps $\mathbb{H}_+$ into itself and also has an analytic continuation ot the lower half-plane $\mathbb{H}_-(= -\mathbb{H}_+)$, obtained by the reflection across I (see [1],[2]).

D. Petz [5] proved that an operator monotone function $f : [0,\infty) \longrightarrow [0,\infty)$ satisfying the functional equation

$$f(t) = tf(t^{-1}) \qquad t \geq 0$$

is related to a Morozova-Chentsov function which gives a monotone metric on the manifold of $n \times n$ density matrices. In the work [6], the concrete functions (Petz-Hasegawa's functions)

$$f_a(t) = a(1-a)\frac{(t-1)^2}{(t^a-1)(t^{1-a}-1)} \qquad (-1 < a < 2)$$

appeared and their operator monotonicity was proved. V.E.S. Szabo introduced an interesting idea for checking their operator monotonicity in [7]. We use a similar idea as Szabo's in our argument. M. Uchiyama [8] proved the operator monotonicity of the following extended functions:

$$\frac{(t-a)(t-b)}{(t^p-a^p)(t^{1-p}-b^{1-p})}$$

for $0 < p < 1$ and $a, b > 0$. It is well known that the function t^p $(0 \le p \le 1)$ is operator monotone as Loewner-Heinz's inequality. The main result of this paper is as follows:

Theorem 1. *Let a and b be non-negative real. If $f \in \mathbb{P}_+[0,\infty)$ and both f and $f^\sharp$ are not constant, then*

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))}$$

is operator monotone on $[0,\infty)$, where

$$f^\sharp(t) = \frac{t}{f(t)} \qquad t \ge 0.$$

2 Proof of Main result

The following statement was proved by M. Uchiyama [8]. Here we prove it based on the fact that any operator monotone function is a Pick function, but this is essentially same as his proof.

Proposition 2. *Let $f \in \mathbb{P}(0,\infty)$ be not constant and a be positive real. Then we have that*

$$g(t) = \frac{t-a}{f(t)-f(a)}$$

is operator monotone on $[0,\infty)$.

Proof. Since $f(\mathbb{H}_+) \subset \mathbb{H}_+$, $\mathrm{Im}(f(z)-f(a)) = \mathrm{Im}(f(z)) > 0$ for all $z \in \mathbb{H}_+$. Therefore $g(z) = \dfrac{z-a}{f(z)-f(a)}$ is holomorphic on $\mathbb{H}_+$.

Since f is not constant, $f'(a) \neq 0$. We also have

$$\lim_{t\to 0+} g(t) = \begin{cases} \dfrac{a}{f(a)-f(0)} \ (>0) & \text{if } f(0) \text{ exists} \\ 0 & \text{otherwise} \end{cases}.$$

This means $g(z)$ is continuous on $\mathbb{H}_+ \cup [0,\infty)$ and we have $g([0,\infty)) \subset [0,\infty)$.

By Loewner's theorem, we have the following integral representation of f: for $z \in \mathbb{H}_+ \cup (0,\infty)$

$$f(z) = \alpha + \beta z + \int_0^\infty \left(-\frac{1}{x+z} + \frac{x}{x^2+1}\right) d\nu(x) \quad (\alpha \in \mathbb{R},\ \beta \ge 0),$$

where ν is a positive measure on $(0,\infty)$ such that

$$\int_0^\infty \frac{1}{x^2+1} d\nu(x) < \infty.$$

Using this relation, we have

$$g(z) = \frac{1}{\beta + \int_0^\infty \frac{1}{(x+z)(x+a)} d\nu(x)}.$$

If we show that $\mathrm{Im}(g(z)) \geq 0$ for any $z \in \mathbb{H}_+$, then $g \in \mathbb{P}_+[0,\infty)$. We remark that

$$\begin{aligned} z \in \mathbb{H}_+ \Rightarrow 0 < \arg z < \pi &\Rightarrow 0 < \arg(x+z) < \pi \quad (x \in [0,\infty)) \\ &\Rightarrow \pi < \arg \frac{1}{(x+z)(x+a)} < 2\pi \quad (x \in [0,\infty)) \\ &\Rightarrow \mathrm{Im} \frac{1}{(x+z)(x+a)} < 0 \quad (x \in [0,\infty)). \end{aligned}$$

So we have

$$\mathrm{Im}(\beta + \int_0^\infty \frac{1}{(x+z)(x+a)} d\nu(x)) = \int_0^\infty \mathrm{Im} \frac{1}{(x+z)(x+a)} d\nu(x) < 0.$$

This shows that $\mathrm{Im}(g(z)) \geq 0$. □

For $f \in \mathbb{P}[0,\infty)$, we have the following integral representation:

$$f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{z+\lambda} dw(\lambda) \qquad (z \in \mathbb{H}_+ \cup [0,\infty)),$$

where $\beta \geq 0$ and

$$\int_0^\infty \frac{\lambda}{1+\lambda} dw(\lambda) < \infty.$$

When $f(0) \geq 0$ (i.e., $f \in \mathbb{P}_+[0,\infty)$), $f(z)$ can be approximated by

$$\sum_{i=1}^n f_i(z),$$

where each $f_i(z)$ satisfies that

$$0 < \arg f_i(z) \leq \arg z \text{ whenever } 0 < \arg z < \pi.$$

So we have $0 < \arg f(z) \leq \arg z$ whenever $0 < \arg z < \pi$.

By using elementary geometry, it easily holds that

$$\arg(z - |z|) = \frac{\pi + \arg z}{2}$$

for any $z \in \mathbb{H}_+$. So we can get the following statement:

Lemma 3. *For any $z \in \mathbb{H}_+$ and $l > 0$, we have*

$$\arg z < \arg(z - l) < \frac{\pi + \arg z}{2} \quad \textit{if} \quad |z| > l.$$

Now we can prove the following theorem and remark that Theorem 1 easily follows from this:

Theorem 4. *Let* f*,* $g \in \mathbb{P}_+[0,\infty)$ *and both* f *and* g *be non-constant. If* $\dfrac{f(t)g(t)}{t}$ *is operator monotone on* $[0,\infty)$*, then*

$$h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(g(t)-g(b))}$$

is also operator monotone on $[0,\infty)$ *for any* $a, b \geq 0$.

Proof. By the assumption we can consider the function

$$h(z) = \frac{(z-a)(z-b)}{(f(z)-f(a))(g(z)-g(b))} \quad z \in \mathbb{H}_+.$$

It is clear that $h(z)$ is holomorphic on $\mathbb{H}_+$. We may consider values, by taking the limit,

$$h(a) = \frac{a-b}{f'(a)(g(a)-g(b))} \text{ and } h(b) = \frac{b-a}{g'(b)(f(b)-f(a))}.$$

So we have $h([0,\infty)) \subset [0,\infty)$.

We assume that $f(z)$ and $g(z)$ are continuous on the closure $\overline{\mathbb{H}_+}$ of $\mathbb{H}_+$ and

$$f(t) - f(a) \neq 0 \text{ and } g(t) - g(b) \neq 0 \text{ for any } t \in (-\infty, 0).$$

Then $h(z)$ is continuous on $\overline{\mathbb{H}_+}$.

Since $\dfrac{z-a}{f(z)-f(a)}$ and $\dfrac{z-b}{g(z)-g(b)}$ belong to $\mathbb{P}_+[0.\infty)$ by Proposition 2, it is clear that $\arg h(z) \geq 0$ if $0 \leq \arg z \leq \pi$.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\begin{aligned}
&\arg h(z) \\
=& \arg(z-a) - \arg(f(z)-f(a)) + \arg(z-b) - \arg(g(z)-g(b)) \\
\leq& \pi - \arg f(z) + \pi - \arg g(z) \\
\leq& 2\pi - \arg z = \pi \qquad (\text{since } \arg f(z) + \arg g(z) - \arg z \geq 0).
\end{aligned}$$

So $0 \leq \arg h(z) \leq \pi$.

In the case $z \in \mathbb{H}_+$ satisifying $|z| > \max\{a, b\}$, it holds that

$$\arg z < \arg(z-a), \arg(z-b) < \frac{\pi + \arg z}{2}$$

by above lemma. Since

$$\begin{aligned}
\arg h(z) &= \arg(z-a) - \arg(f(z)-f(a)) + \arg(z-b) - \arg(g(z)-g(b)) \\
&\leq \frac{\pi + \arg z}{2} - \arg f(z) + \frac{\pi + \arg z}{2} - \arg g(z) \\
&= \pi + \arg z - \arg f(z) - \arg g(z) \leq \pi,
\end{aligned}$$

we have $0 < \arg h(z) < \pi$.

For $r > 0$, we define $H(r) = \{z \in \mathbb{C} \mid |z| \leq r, \mathrm{Im} z \geq 0\}$. Whenever $r > l = \max\{a, b\}$, we can get

$$0 \leq \arg h(z) \leq \pi$$

on the boundary of $H(r)$. Since $h(z)$ is holomorphic on $H(r)$, $\mathrm{Im}h(z)$ is harmonic on $H(r)$. Because $\mathrm{Im}h(z) \geq 0$ on the boundary of $H(r)$, we have $h(H(r)) \subset \overline{\mathbb{H}_+}$ by the minimum principle of the harmonic function. This implies

$$h(\overline{\mathbb{H}_+}) = h(\bigcup_{r>l} H(r)) \subset \overline{\bigcup_{r>l} h(H(r))} \subset \overline{\mathbb{H}_+},$$

and $h \in \mathbb{P}_+[0, \infty)$.

In general case, we set

$$\frac{f(t)g(t)}{t} = F(t) \text{ and } \tilde{f}(t) = \frac{f(t)}{F(t)} \qquad (t \geq 0).$$

By the relation $\tilde{f}(t)g(t) = t$, we have $\tilde{f} \in \mathbb{P}_+[0, \infty)$. We define the function f_p, $\tilde{f}_p$ and g_p $(0 < p < 1)$ as follows:

$$f_p(z) = f(z^p), \quad \tilde{f}_p(z) = \tilde{f}(z^p),$$
$$\text{and } g_p(z) = (\tilde{f}_p)^\sharp(z) = \frac{z}{\tilde{f}_p(z)} = \frac{zF(z^p)}{f(z^p)} = z^{1-p}g(z^p)$$

for $z \in \overline{\mathbb{H}_+}$. Then we have f_p, $g_p \in \mathbb{P}_+[0, \infty)$ and

$$h_p(z) = \frac{(z-a)(z-b)}{(f_p(z) - f_p(a))(g_p(z) - g_p(b))}$$

is holomorphic on $\mathbb{H}_+$ and continuous on $\overline{\mathbb{H}_+}$. By the fact $\dfrac{f_p(t)g_p(t)}{t} = F(t^p)$ is operator monotone on $[0, \infty)$, $h_p(t)$ becomes operator monotone on $[0, \infty)$. Since

$$\begin{aligned} h_p(t) &= \frac{(t-a)(t-b)}{(f_p(t) - f_p(a))(g_p(t) - g_p(b))} \\ &= \frac{(t-a)(t-b)}{(f(t^p) - f(a^p))(t^{1-p}g(t^p) - b^{1-p}g(b^p))} \qquad \text{for } t \geq 0, \end{aligned}$$

we have

$$\lim_{p \to 1-0} h_p(t) = h(t).$$

So we can get the operator monotonicity of $h(t)$. □

We can generalize this result for many operator monotone functions under some assumption([4]).

We remark that, for $f \in \mathbb{P}[0, \infty)$, f_0 belongs to $\mathbb{P}_+[0, \infty)$, where we put $f_0(t) = f(t) - f(0)$. Let $g \in \mathbb{P}_+[0, \infty)$ and $\dfrac{f_0(t)g(t)}{t}$ be operator monotone on

$[0,\infty)$. Under the assumption that f and g are not constant, we have

$$\begin{aligned}&\frac{(t-a)(t-b)}{(f_0(t)-f_0(a))(g(t)-g(b))}\\=&\frac{(t-a)(t-b)}{(f(t)-f(a))(g(t)-g(b))}\in\mathbb{P}_+[0,\infty)\end{aligned}$$

for any $a,b\geq 0$.

Corollary 5. *Let $f\in\mathbb{P}_+(0,\infty)$ and both f and $f^\sharp$ be not constant. For any $a>0$, we define*

$$h_a(t)=\frac{(t-a)(t-a^{-1})}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(a^{-1}))}\qquad t\in(0,\infty).$$

Then we have

(1) h_a is operator monotone on $(0,\infty)$.

(2) $f(t)=t\cdot f(t^{-1})$ implies $h_a(t)=t\cdot h_a(t^{-1})$.

(3) $a=1$ and $f(t^{-1})=f(t)^{-1}$ implies $h_1(t)=t\cdot h_1(t^{-1})$.

Proof. We can directly prove (1) from theorem 3. Because

$$\begin{aligned}t\cdot h_a(t^{-1})&=\frac{t(t^{-1}-a)(t^{-1}-a^{-1})}{(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1}))}\\&=\frac{(t-a)(t-a^{-1})}{t(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1}))},\end{aligned}$$

we can compute

$$\begin{aligned}&t(f(t^{-1})-f(a))(f^\sharp(t^{-1})-f^\sharp(a^{-1}))-(f(t)-f(a))(f^\sharp(t)-f^\sharp(a^{-1}))\\=&(f(t^{-1})-f(a))(1/f(t^{-1})-t/af(a^{-1}))-(f(t)-f(a))(t/f(t)-1/af(a^{-1}))\\=&0\end{aligned}$$

if it holds $f(t)=t\cdot f(t^{-1})$ or $a=1$, $f(t^{-1})=f(t)^{-1}$. So we have (2) and (3). □

Example 6. *Using this corollary, we can repeatedly construct an operator monotone function $h(t)$ on $[0,\infty)$ satisfying the relation*

$$h(t)=t\cdot h(t^{-1})\qquad t>0. \tag{*}$$

If we choose t^p $(0<p<1)$ as $f(t)$ in Corollary 5(3),

$$h(t)=\frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}.$$

If we choose $\dfrac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}$ *as* $f(t)$ *in Corollary 5(2),*

$$h(t) = \frac{t-a}{\dfrac{(t-1)^2}{(t^p-1)(t^{1-p}-1)} - \dfrac{(a-1)^2}{(a^p-1)(a^{1-p}-1)}} \times \frac{t-a^{-1}}{\dfrac{t(t^p-1)(t^{1-p}-1)}{(t-1)^2)} - \dfrac{a(a^{-p}-1)(a^{p-1}-1)}{(a-1)^2}}$$

for $a > 0$*. If we choose* $t^p + t^{1-p}$ $(0 < p < 1)$ *as* $f(t)$ *in Corollary 5(2),*

$$\begin{aligned} h(t) &= \frac{t-a}{t^p + t^{1-p} - a^p - a^{1-p}} \times \frac{t-a^{-1}}{\dfrac{1}{t^{p-1}+t^{-p}} - \dfrac{1}{a^p + a^{1-p}}} \qquad (a > 0) \\ &= \frac{\sqrt{t}(\cosh(\log t) - \cosh(\log a))}{\cosh(\log\sqrt{t}) - \cosh(\log\sqrt{t} + \log(t^p + t^{1-p}) - \log(a^p + a^{1-p}))}. \end{aligned}$$

These functions, $h \in \mathbb{P}_+[0,\infty)$*, satisfy the relation (*).*

References

[1] R. Bhatia, *Matrix Analysis* , Springer, 1996.

[2] F. Hiai, Matrix Analysis: Matrix monotone functions, matrix means, and majorization, Interdecip. Inform. Sci. 16 (2010) 139–248.

[3] E.A. Morozova, N.N. Chentsov, Markov invariant geometry on state manifolds, Itogi Nauki i Techniki 36 (1990) 69–102, Translated in J. Soviet Math. 56(1991) 2648–2669.

[4] M. Kawasaki, M. Nagisa, Transforms on operator monotone functions, in preparation.

[5] D. Petz, Monotone metric on matrix spaces, Linear Algebra Appl. 244 (1996) 81–96.

[6] D. Petz, H. Hasegawa, On the Riemannian metric of α-entropies of density matrices, Lett. Math. Phys. 38 (1996) 221–225.

[7] V.E.S. Szabo, A class of matrix monotone functions, Linear Algebra Appl. 420 (2007) 79–85.

[8] M. Uchiyama, Majorization and some operator monotone functions, Linear Algebra Appl. 432(2010) 1867–1872.

Graduate School of Science, Chiba University, Chiba 263-8522, Japan
E-mail address: kawasaki_0604abc@yahoo.co.jp
E-mail address: nagisa@math.s.chiba-u.ac.jp