SOLVABILITY FOR ONE-DIMENSIONAL PHASE FIELD SYSTEM ASSOCIATED WITH GRAIN BOUNDARY MOTION*

Dedicated to Professor Yoshikazu Giga for the celebration of his winning of the Medal with Purple Ribbon

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Abstract. In this paper, a coupled system of two parabolic equations, subject to the homogeneous Neumann boundary conditions and the initial conditions, is considered, as the one-dimensional version of phase field system of grain boundary motion, proposed by Kobayashi-Warren-Carter [20, 21]. The presented system includes some non-standard situations, which come from a weighted total variation, built in a governing free energy. The main objective of this paper is to give a certain definition method for solution of our system, which can respond to the focused non-standards. Consequently, the existence of solution of our system is demonstrated with help from the general measure theory and the general theory of evolution equations governed by time-dependent subdifferentials.

*AMS Subject Classification: 35K87, 35R06, 35K67; Keywords: parabolic system, parabolic equation with measure, weighted singular diffusion equation

†The first author is supported by Grant-in-Aid for Encouragement of Young Scientists (B) (No. 21740120) JSPS

††The second author is supported by Grant-in-Aid for Scientific Research (C) (No. 23540206) JSPS

‡The third author is supported by Grant-in-Aid for Encouragement of Young Scientists (B) (No. 22740110) JSPS

Introduction

Let $(0,T) \subset \mathbb{R}$ be a bounded time-interval with the terminal time $0 < T < \infty$, and let $\Omega := (-L, L) \subset \mathbb{R}$ be a one-dimensional spatial domain with the boundary points $\pm L \in \mathbb{R}$. Let $Q_T := (0,T) \times \Omega \subset \mathbb{R}^2$ be the product set of the time-space coordinates.

In this paper, a system, denoted by (S), of two parabolic initial-boundary value problems is considered. This system is formally described as follows.

(S):

$$\begin{cases} \eta_t - \eta_{xx} + g(\eta) + \alpha'(\eta)|D\theta| = 0 \ \text{ in } Q_T, \\ \eta_x(t, \pm L) = 0, \ \ t \in (0,T), \\ \eta(0,x) = \eta_0(x), \ \ x \in \Omega; \end{cases} \tag{0.1}$$

$$\begin{cases} \alpha_0(\eta)\theta_t - \left(\alpha(\eta) \dfrac{D\theta}{|D\theta|} \right)_x = 0 \ \text{ in } Q_T, \\ \alpha(\eta(t, \pm L)) \dfrac{D\theta}{|D\theta|}(t, \pm L) = 0, \ \ t \in (0,T), \\ \theta(0,x) = \theta_0(x), \ \ x \in \Omega. \end{cases} \tag{0.2}$$

The presented system (S) is motivated by the phase field model of grain boundary motion, known as Kobayashi-Warren-Carter model (cf. [20, 21]). According to [20, 21], the original Kobayashi-Warren-Carter model is supposed to represent two-dimensional grain boundary motion in a polycrystal, as in Silicon-Carbide. Hence, the system (S) is supposed to be a model case of Kobayashi-Warren-Carter model, and from physical viewpoint, the spatial domain Ω is supposed to be a two-dimensional (or more higher-dimensional) one.

In Kobayashi-Warren-Carter model, the phase of grain is represented by the following vector field of crystal orientation:

$$(t,x) \in Q_T \mapsto \eta(t,x)(\cos\theta(t,x), \sin\theta(t,x));$$

with the use of two unknowns η and θ. In the context, the unknowns $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$ are two order parameters, which describe the orientation order and the mean orientation angle, respectively, at any point $(t,x) \in Q_T$ in polycrystal. Notably, the order parameter η is supposed to satisfy $0 \leq \eta \leq 1$ in Q_T, and then the threshold values 1 and 0 are supposed to indicate the completely oriented phase and the disorder phase of orientation, respectively. In addition, the integrant parts $\alpha = \alpha(\eta)$, $\alpha_0 = \alpha_0(\eta)$ and $g = g(\eta)$ are all given functions under suitable assumptions, and α' is the derivative of α. The functions $\eta_0 = \eta_0(x)$ and $\theta_0 = \theta_0(x)$ are given initial values.

The system (S) is derived from the following governing energy, called "free energy":

$$[\eta, \theta] \mapsto \mathscr{F}(\eta, \theta) := \frac{1}{2} \int_\Omega |\eta_x|^2 \, dx + \int_\Omega \hat{g}(\eta) \, dx + \int_\Omega \alpha(\eta)|D\theta|; \tag{0.3}$$

and the initial-boundary value problems (0.1) and (0.2) are L^2-gradient flows of this free energy, with respect to order parameters η and θ, respectively, where $\hat{g}$ is a nonnegative primitive of g. Then, the PDE expressions in (0.1)-(0.2) are just formal ones, including the homogeneous Neumann type boundary conditions, and their rigorous meanings are going to be given by means of appropriate variational inequalities.

One of characteristics of the free energy $\mathscr{F}$ is in the point that it includes the integral part of the weighted total variation measure $\alpha(\eta)|D\theta|$ of θ. Indeed, due to this term, the spatial regularity of angle θ is within the range of BV-functions, and it implies that the system (S) is equipped to deal with the representation of sharp interfacial phase of grain, as in polycrystal. But, on the other hand, this term also makes the mathematical treatment be quite difficult, and then the major difficulties appear in two non-standard situations, listed below.

(Stn. 1) Mathematical treatment of the term $\alpha'(\eta)|D\theta|$ as in (0.1).

For measure theoretical approach, the weight $\alpha'(\eta)$ (and the unknown η) is expected to be continuous on $\overline{Q_T}$. However it seems to be not favorable within the range of general parabolic regularity, involved in the term $\alpha'(\eta)|D\theta|$ of measure.

(Stn. 2) Mathematical treatment of the singular diffusion $-(\alpha(\eta)\frac{D\theta}{|D\theta|})_x$ as in (0.2).

Under fixed situation of time-variable, a number of existing theories, such as [2, 3, 4, 12, 13, 19, 22, 25, 26], enable to give the representation of the singular diffusion, by means of the subdifferential of the weighted total variation. However, in view of the expected regularities of η and θ, the time-dependence of the energy, caused by the weight $\alpha(\eta(t))$, would not be in the applicable scope of existing theories, such as [17, 24, 27] (see Remark Ap.6 in Appendix, for details).

In order to avoid such non-standards, the authors of [20, 21] also proposed another mathematical model, by using a relaxed free energy. Specifically, under one-dimensional situation of Ω, the relaxed free energy is formulated as:

$$[\eta, \theta] \mapsto \mathscr{F}_\nu(\eta, \theta) := \mathscr{F}(\eta, \theta) + \frac{\nu}{2}\int_\Omega |\theta_x|^2\,dx; \tag{0.4}$$

with the use of small relaxation constant $\nu > 0$.

Now, for any $\nu > 0$, let us denote by $(\mathrm{S})_\nu$ the system of gradient flows, derived from the relaxed free energy $\mathscr{F}_\nu$. Then, the system $(\mathrm{S})_\nu$, for each $\nu > 0$, can be called a relaxation system for the original one (S). Also, from the other point of view, the system (S) can be regarded as a limiting system for the sequence $\{(\mathrm{S})_\nu \,|\, \nu > 0\}$ of relaxation systems, as $\nu \searrow 0$. Furthermore, it is notable that the study results on relaxation models have been reported, recently, from various approaches: the numerical approach [20, 21] and the theoretical approach [14, 15, 16, 18].

In view of such background, let us set the goal of this paper to give a meaningful definition of the solutions of our system (S), which can respond to the non-standards, pointed in (Stn. 1)-(Stn. 2). To this end, the relaxation systems $(\mathrm{S})_\nu$, for $\nu > 0$, will be treated as some kinds of approximation problems for (S). Consequently, the existence of a certain solution of our system (S) will be demonstrated through the limiting observation for $(\mathrm{S})_\nu$, as $\nu \searrow 0$.

The demonstration argument for the existence result will be proceeded according to the following content.

In the first Section 1, the main result of this study will be stated. Although the conclusion is stated in the form of Main Theorem, the essential of this study is not only in the proof of this theorem, but also in the definition method of solution, provided in Definition 1.1. In the next Section 2, the outline of the proof of Main Theorem will be shown. Roughly summarized, the proof will be divided in the following two verification parts.

(Part I) Compactness of the approximation sequence.

More precisely, this part is concerned with finding a limit $[\eta, \theta]$ (cluster point) for the sequence $\{[\eta_\nu, \theta_\nu] \,|\, \nu > 0\}$ of solutions of approximation problems (S)$_\nu$, for $\nu > 0$, in appropriate topologies, as $\nu \searrow 0$.

(Part II) Compatibility of the limit $[\eta, \theta]$ with the system (S).

More precisely, this part is concerned with the verification whether two components η and θ of the limit $[\eta, \theta]$ solve the initial-boundary value problems (0.1) and (0.2), respectively, or not.

In either part, some important matters will be stated in forms of key-lemmas. On that basis, the verifications of (Part I) and (Part II) will be completed in the following Sections 3 and 4, respectively, by giving the proofs of those key-lemmas. Moreover, some technical topics, specific to this study, will be collected in the last Appendix with supplemental remarks.

1 Statement of the main result

Let us begin with the preparation of notations, that are needed for rigorous formulations in mathematics.

Abstract notations. For an abstract Banach Space X, we denote by $|\cdot|_X$ the norm of X. In particular, when X is a Hilbert space, we denote by $(\cdot\,,\,\cdot)_X$ the inner product of X.

Notations for simplicity. Throughout this paper, let us set:

$$\begin{cases} C := C(\Omega), \\ C_\mathrm{c} := C_\mathrm{c}(\Omega) \text{ (the space of functions in } C\text{, with compact supports)}, \\ C_0 := C_0(\Omega) \text{ (the closure of } C_c \text{ in the topology of } C(\overline{\Omega})); \end{cases}$$

and

$$\begin{cases} C^m := C^m(\Omega), \;\; C^m_\mathrm{c} := C^m \cap C_\mathrm{c}, \\ L^p := L^p(\Omega), \;\; W^{m,p} := W^{m,p}(\Omega), \;\; W^{m,p}_0 := W^{m,p}_0(\Omega), \\ H^m := H^m(\Omega) \; (= W^{m,2}(\Omega)), \;\; H^m_0 := H^m_0(\Omega) \; (= W^{m,2}_0(\Omega)), \end{cases}$$
$$\text{for } 1 \leq p \leq \infty \;\text{ and }\; m \in \mathbb{N} \cup \{\infty\}.$$

Also, let us denote by $\langle\,\cdot\,,\,\cdot\,\rangle_*$ the duality pairing between H^1 and its dual $(H^1)^*$. Besides, let $F : H^1 \longrightarrow (H^1)^*$ be the duality mapping, defined as:

$$\langle Fw, v\rangle_* := (w, v)_{H^1} = (w, v)_{L^2} + (w_x, v_x)_{L^2}, \;\text{ for all } v, w \in H^1. \tag{1.1}$$

Notations in basic measure theory. For any dimension $d \in \mathbb{N}$, the d-dimensional Lebesgue measure is denoted by $\mathscr{L}^d$, and the measure theoretical phrases, such as "a.e.", "dt" and "dx", and so on, are all with respect to the one-dimensional Lebesgue measure $\mathscr{L}^1$, if not otherwise specified.

For any open set $U \subset \mathbb{R}^d$, we denote by $\mathcal{M}(U)$ the space of all Radon measures on U, and in particular, we simply denote by $\mathcal{M}$ the space $\mathcal{M}(\Omega)$ of all Radon measures on the one-dimensional domain $\Omega = (-L, L)$. In general, the space $\mathcal{M}(U)$ is known as the dual of the Banach space $C_0(U)$, for any open set $U \subset \mathbb{R}^d$.

Notations in BV-theory. (cf. [1, 6, 10, 11]) Let $d \in \mathbb{N}$ be any number of dimension, and let $U \subset \mathbb{R}^d$ be any open set. Then, a function $v \in L^1(U)$ is called a function of bounded variation, or simply BV-function, on U, if and only if its distributional differential Dv is a Radon measure on U, namely $Dv \in \mathcal{M}(U)^d$.

On that basis, we denote by $BV(U)$ the space of all BV-functions on U, and in particular, we simply denote by BV the space $BV(\Omega)$ $(= BV(-L, L))$ of one-dimensional BV-functions on $\Omega = (-L, L)$. For any $v \in BV(U)$, the Radon measure Dv is called the variation measure of v, and its total variation $|Dv|$ is called the total variation measure of v. Additionally, the value $|Dv|(U)$, for any $v \in BV(U)$, is calculated as:

$$|Dv|(U) = \sup \left\{ \int_U v \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_c^1(U) \ \text{ and } \ |\varphi| \leq 1 \text{ on } U \right\}.$$

In general, the space $BV(U)$ is known as a Banach space, endowed with the norm:

$$|v|_{BV(U)} := |v|_{L^1(U)} + |Dv|(U), \quad \text{for any } v \in BV(U);$$

and in principle, it takes over the embedding property of $W^{1,1}(U)$ into the Lebesgue spaces (cf. [1, Corollary 3.49], or [6, Thorems 10.1.3-10.1.4]). Specifically, the space BV of one-dimensional BV-functions is continuously embedded into L^∞ and compactly embedded into L^p, for any $1 \leq p < \infty$.

Moreover, for any $v \in BV$, the both one-sided limits:

$$v(\ell - 0) = \lim_{x \nearrow \ell} v(x) \ \text{ and } \ v(r + 0) = \lim_{x \searrow r} v(x);$$

exist, for all $-L \leq r < \ell \leq L$ (cf, [1, Theorem 3.28]), and hence a natural extension $v^{\mathrm{ex}} \in L^\infty(\mathbb{R})$ of v, with $Dv^{\mathrm{ex}} \in \mathcal{M}(\mathbb{R})$, can be defined as follows:

$$v^{\mathrm{ex}}(x) := \begin{cases} v(x), & \text{if } x \in \Omega = (-L, L), \\ v(L - 0), & \text{if } x \geq L, \\ v(-L + 0), & \text{if } x \leq -L, \end{cases} \qquad \text{for a.e. } x \in \mathbb{R}. \tag{1.2}$$

Incidentally, in the light of coarea formula (cf. [1, Theorem 3.40], [6, Theorem 10.3.3], [10, Section 5.5], or [11, 1.23 Theorem]),

$$\int_\Omega |Dv| = \int_{\mathbb{R}} |Dv^{\mathrm{ex}}|, \quad \text{for any } v \in BV.$$

Other specific notations. For any positive-valued function $\beta \in C(\mathbb{R})$ and any $w \in C(\overline{\Omega})$, let us define a functional $\Phi_\beta(w; \cdot)$ on L^2, by putting:

$$v \in L^2 \mapsto \Phi_\beta(w; v) := \begin{cases} \displaystyle\int_\Omega \beta(w)|Dv|, & \text{if } v \in BV, \\ \infty, & \text{otherwise.} \end{cases}$$

As it is easily seen, the above functional is proper l.s.c. and convex function on L^2. In view of this, we denote by $\partial\Phi_\beta(w; \cdot)$ the subdifferential of each convex function $\Phi_\beta(w; \cdot)$ in the topology of L^2, for any positive-valued $\beta \in C(\mathbb{R})$ and any $w \in C(\overline{\Omega})$ (see Sections Ap.1-Ap.2 of Appendix, for details of the subdifferential).

Next, here are listed the assumptions, imposed to the integrant parts in (0.1)-(0.2).

(A1) The perturbation g, as in (0.1), is settled as a locally Lipschitz continuous function on $\mathbb{R}$, such that $g \leq 0$ on $(-\infty, 0]$ and $g \geq 0$ on $[1, \infty)$, and g has a nonnegative primitive $\hat{g}$.

(A2) The weight α_0, in front of θ_t in (0.2), is settled as a locally Lipschitz continuous function on $\mathbb{R}$.

(A3) The weight α of the singular diffusion $-(\alpha(\eta)\frac{D\theta}{|D\theta|})_x$, as in (0.2), is settled as a C^1-function, such that α is convex on $\mathbb{R}$, and the graph of the derivative α' of α passes the origin, namely $\alpha'(0) = 0$.

(A4) There exists a positive constant δ_α, such that:

$$\alpha_0(\tau) \geq \delta_\alpha \ \text{ and } \ \alpha(\tau) \geq \delta_\alpha, \quad \text{for all } \tau \in \mathbb{R}.$$

(A5) The pair $[\eta_0, \theta_0]$ of initial values, as in (0.1)-(0.2), is assumed to belong to a range $D_0 \subset H^1 \times BV$, defined as:

$$D_0 := \left\{ \ [w, v] \in H^1 \times BV \ \middle| \ 0 \leq w \leq 1 \text{ on } \overline{\Omega} \ \right\} \subset C(\overline{\Omega}) \times L^\infty.$$

Remark 1.1 (Possible choice of integrant parts) Referring to [20, 21], the setting:

$$g(\tau) = \tau - 1 \text{ with } \hat{g}(\tau) := \frac{1}{2}(\tau - 1)^2 \ \text{ and } \ \alpha_0(\tau) = \alpha(\tau) = \tau^2 + \delta_\alpha, \quad \text{for } \tau \in \mathbb{R};$$

will provide a possible choice, that fulfills the above (A1)-(A5).

Remark 1.2 (Exact formulation of free energy) Under the one-dimensional setting $\Omega := (-L, L)$ of spatial domain, the Sobolev space H^1 is compactly embedded into $C(\overline{\Omega})$. Therefore, with the use of the prepared notations and assumptions, the rigorous formulation of the free energy $\mathscr{F}$ in (0.3) can be given by:

$$[w, v] \in L^2 \times L^2 \mapsto \mathscr{F}(w, v) := \begin{cases} \displaystyle\frac{1}{2}\int_\Omega |w_x|^2\,dx + \int_\Omega \hat{g}(w)\,dx + \Phi_\alpha(w; v), \\ \qquad\qquad \text{if } [w, v] \in H^1 \times BV, \\ \infty, \quad \text{otherwise.} \end{cases}$$

On the basis of the above notations and assumptions, the solution of the system (S) is defined as follows.

Definition 1.1 (Definition of solution) A pair $[\eta, \theta] \in L^2(0,T;L^2) \times L^2(0,T;L^2)$ of functions is called a solution of (S), if and only if the components η and θ fulfill the following three conditions.

(S1) $\eta \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \subset C(\overline{Q_T})$, and $0 \leq \eta \leq 1$ on $\overline{Q_T}$;
$\theta \in W^{1,2}(0,T;L^2)$, $|D\theta(\,\cdot\,)|(\Omega) \in L^\infty(0,T)$ and $\alpha'(\eta)|D\theta| \in L^2(0,T;(H^1)^*)$.

(S2) η solves the following Cauchy problem of evolution equation on $(H^1)^*$:

$$\begin{cases} \eta_t(t) + F\eta(t) - \eta(t) + g(\eta(t)) + \alpha'(\eta(t))|D\theta(t)| = 0 \ \text{ in } (H^1)^*, \ \ t \in (0,T), \\ \eta(0) = \eta_0 \ \text{ in } (H^1)^*. \end{cases} \tag{1.3}$$

(S3) θ solves the following Cauchy problem of evolution equation on L^2:

$$\begin{cases} \alpha_0(\eta(t))\theta_t(t) + \partial\Phi_\alpha(\eta(t);\theta(t)) \ni 0 \ \text{ in } L^2, \ \ t \in (0,T), \\ \theta(0) = \theta_0 \ \text{ in } L^2. \end{cases} \tag{1.4}$$

The solutions of Cauchy problems (1.3)-(1.4) are prescribed according to the definition method, mentioned in Section Ap.1 of Appendix.

Remark 1.3 (Treatments of the non-standards) The continuity of η, mentioned in (S1), comes from the one-dimensional setting of Ω, and also, it successfully responds to the non-standard situation, pointed at (Stn. 1) in Introduction. Besides, as a consequence of (S1)-(S2), it is inferred that $\alpha'(\eta(t))|D\theta(t)| \in (H^1)^*$ while it belongs to $\mathcal{M}$ $(= (C_0)^*)$, for a.e. $t \in (0,T)$. Additionally, the subdifferential term $\partial\Phi_\alpha(\eta(t);\theta(t))$ will be the mathematical representation of the singular diffusion $-(\alpha(\eta)\frac{D\theta}{|D\theta|})_x$, as in (0.2). Nowadays, we can check the adequacy of this representation method, by referring to some literatures, such as [2, 3, 4, 12, 13, 19, 22, 25, 26] (see Section Ap.2 in Appendix, for details).

Now, the conclusion of this paper will be summarized in the following Main Theorem.

Main Theorem (Existence of solution) Under the assumptions (A1)-(A5), the system (S) admits at least one solution $[\eta, \theta]$.

Remark 1.4 (Comment on uniqueness) The uniqueness problem for our system (S) is still open, even in one-dimensional setting of Ω. The bottle-neck of this problem appears in the monotonicity pairing for the diffusion:

$$(v_1^* - v_2^*, v_1 - v_2)_{L^2} \left(\approx \int_\Omega \alpha(w) \left(\frac{Dv_1}{|Dv_1|} - \frac{Dv_2}{|Dv_2|} \right) \cdot D(v_1 - v_2) \right),$$
$$\text{for } [v_i, v_i^*] \in \partial\Phi_\alpha(w;\,\cdot\,) \text{ in } L^2 \times L^2,\ i = 1, 2, \text{ with } w \in C(\overline{\Omega}).$$

More precisely, in many of diffusions (e.g. p-Laplacians for $1 < p \leq 2$), the above types of pairings are supported by some norms of the gradient $D(v_1 - v_2)$ with some orders, but such supporting property is not available for our case of singular diffusion (relative to 1-Laplacian).

2 Proof of Main Theorem

In this section, we briefly see the outline of the proof of Main Theorem. As it is mentioned in Introduction, the relaxation system, denoted by $(\mathrm{S})_\nu$, is going to be adopted as the approximation problem for the system (S). Also, the system $(\mathrm{S})_\nu$ is supposed to be derived from the relaxed free energy $\mathscr{F}_\nu$, given in (0.4).

In view of this, let us first clarify the exact formulation of the relaxation system $(\mathrm{S})_\nu$, for any $\nu > 0$. To this end, we add the following notations, for the sake of convenience.

Maximal monotone relative to Laplacian. Let us set:

$$D_N := \left\{ \ z \in H^2 \ \middle| \ z_x(\pm L) = 0 \ \right\};$$

to define an operator $A_N : D_N \subset L^2 \longrightarrow L^2$, by putting:

$$v \in D_N \mapsto A_N v := -v_{xx} + v \in L^2. \tag{2.1}$$

As it is easily seen, A_N has a maximal monotone graph in $L^2 \times L^2$, and it coincides with the restriction $F|_{D_N}$ of the duality map $F : H^1 \longrightarrow (H^1)^*$, onto $D_N \subset H^1$.

Notation for the approximation approach. Let us fix any $\nu > 0$. Here, for any positive-valued function $\beta \in C(\mathbb{R})$ and any $w \in L^2$, we define a proper l.s.c. and convex function $\Phi_{\beta,\nu}(w;\,\cdot\,)$, by putting:

$$v \in L^2 \mapsto \Phi_{\beta,\nu}(w; v) := \begin{cases} \displaystyle\int_\Omega \beta(w)|v_x|\,dx + \frac{\nu}{2}\int_\Omega |v_x|^2\,dx, & \text{if } v \in H^1, \\ \infty, \quad \text{otherwise}; \end{cases} \tag{2.2}$$

and we denote by $\partial\Phi_{\beta,\nu}(w;\,\cdot\,)$ its subdifferential in the topology of L^2.

By using the above notations, the relaxation system $(\mathrm{S})_\nu$, for any $\nu > 0$, is formulated as a system of the following Cauchy problems of two evolution equations.

$(\mathrm{S})_\nu$:

$$\begin{cases} (\eta_\nu)_t(t) + A_N\eta_\nu(t) - \eta_\nu(t) + g(\eta_\nu(t)) + \alpha'(\eta_\nu(t))|(\theta_\nu)_x(t)| = 0 \ \text{ in } L^2, \ t \in (0,T), \\ \eta_\nu(0) = \eta_{0,\nu} \ \text{ in } L^2; \end{cases} \tag{2.3}$$

$$\begin{cases} \alpha_0(\eta_\nu(t))(\theta_\nu)_t(t) + \partial\Phi_{\alpha,\nu}(\eta_\nu(t); \theta_\nu(t)) \ni 0 \ \text{ in } L^2, \ t \in (0,T), \\ \theta_\nu(0) = \theta_{0,\nu} \ \text{ in } L^2. \end{cases} \tag{2.4}$$

In each system $(\mathrm{S})_\nu$, the solution is defined as a pair $[\eta_\nu, \theta_\nu] \in L^2(0,T;L^2) \times L^2(0,T;L^2)$ of functions, such that:

$$\begin{cases} \eta_\nu \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2) \subset C(\overline{Q_T}), \ 0 \le \eta_\nu \le 1 \text{ on } \overline{Q_T}, \\ \theta_\nu \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \subset C(\overline{Q_T}); \end{cases} \tag{2.5}$$

and η_ν and θ_ν fulfill the Cauchy problems (2.3) and (2.4), respectively. Also, the rigorous expression of the relaxed free energy $\mathscr{F}_\nu$ in (0.4), can be given as follows:

$$[w,v] \in L^2 \times L^2 \mapsto \mathscr{F}_\nu(w,v) := \begin{cases} \displaystyle \frac{1}{2}\int_\Omega |w_x|^2\,dx + \int_\Omega \hat{g}(w)\,dx + \Phi_{\alpha,\nu}(w;v), \\ \qquad\qquad \text{if } [w,v] \in H^1 \times H^1, \\ \infty, \quad \text{otherwise.} \end{cases} \tag{2.6}$$

Now, the proof of Main Theorem will be divided into two parts (Part I) and (Part II), mentioned in Introduction.

(Part I) Compactness of the approximation sequence

As previous studies for the relaxation system (S)$_\nu$, we can now refer to several papers, such as [14, 15, 16, 18]. In either study, the domain of the free energy $\mathscr{F}_\nu$ is settled by $L^2 \times H^1_0$, so that the homogeneous Dirichlet boundary condition is imposed to the Cauchy problem, corresponding to the second one (2.4). However, from (2.2) and (2.6), it is inferred that the homogeneous Neumann type boundary condition is inherent to the Cauchy problem (2.4).

In view of this, we have to start with checking some basic properties for the relaxation system (S)$_\nu$, as in the following key-lemmas.

Lemma 2.1 *(Solvability of the relaxation system) Let us fix any $\nu > 0$. Let us assume the conditions (A1)-(A4), and instead of (A5), let us assume:*

$$[\eta_{0,\nu}, \theta_{0,\nu}] \in D_0 \cap (H^1 \times H^1) \subset C(\overline{\Omega}) \times C(\overline{\Omega}).$$

Then, the relaxation system (S)$_\nu$ admits a unique solution $[\eta_\nu, \theta_\nu]$.

Lemma 2.2 *(Dissipation of the relaxed energy) Under the same assumptions and notations with Lemma 2.1, the both functions:*

$$t \in [0,T] \mapsto \Phi_{\alpha,\nu}(\eta_\nu(t); \theta_\nu(t)) \in \mathbb{R}, \quad \textit{and} \quad t \in [0,T] \mapsto \mathscr{F}_\nu(\eta_\nu(t), \theta_\nu(t)) \in \mathbb{R};$$

are absolutely continuous, and furthermore:

$$\begin{aligned} \frac{1}{2}\int_s^t \left(|(\eta_\nu)_t(\tau)|^2_{L^2} + |\sqrt{\alpha_0(\eta_\nu(\tau))}(\theta_\nu)_t(\tau)|^2_{L^2} \right) d\tau + \mathscr{F}_\nu(\eta_\nu(t), \theta_\nu(t)) \\ = \mathscr{F}_\nu(\eta_\nu(s), \theta_\nu(s)), \quad \textit{for all } s, t \in [0,T]. \end{aligned} \tag{2.7}$$

In addition to the above, we need to prepare the following lemma, concerned with continuous dependence of convex energies, in the sense of Mosco [23].

Lemma 2.3 *(Mosco convergence) Let $\beta \in C(\mathbb{R})$, such that $\beta \geq \delta_\beta$ on $\mathbb{R}$ for a certain constant $\delta_\beta > 0$, let $w_0 \in C(\overline{\Omega})$, and let $\{w_\nu \,|\, \nu > 0\} \subset C(\overline{\Omega})$ be a sequence, such that:*

$$w_\nu \to w_0 \quad \textit{in } C(\overline{\Omega}) \textit{ as } \nu \searrow 0. \tag{2.8}$$

Then, the sequence $\{\Phi_{\beta,\nu}(w_\nu; \cdot\,) \,|\, \nu > 0\}$ of convex functions converges to the convex function $\Phi_\beta(w_0; \cdot\,)$ on L^2, in the sense of Mosco, as $\nu \searrow 0$. More precisely:

(m1) $\liminf_{\nu \searrow 0} \Phi_{\beta,\nu}(w_\nu; \check{v}_\nu) \geq \Phi_\beta(w_0; \check{v}_0)$, *if* $\{\check{v}_\nu \,|\, \nu > 0\} \subset L^2$, $\check{v}_0 \in L^2$ *and* $\check{v}_\nu \to \check{v}_0$ *weakly in* L^2 *as* $\nu \searrow 0$;

(m2) *for any* $\hat{v}_0 \in D(\Phi_\beta(w_0; \cdot))$ $(= BV)$, *there exists a sequence* $\{\hat{v}_\nu \,|\, \nu > 0\} \subset H^1$, *such that* $\hat{v}_\nu \to \hat{v}_0$ *in* L^2 *and* $\Phi_{\beta,\nu}(w_\nu; \hat{v}_\nu) \to \Phi_\beta(w_0; \hat{v}_0)$, *as* $\nu \searrow 0$.

After the proofs of the above Lemmas 2.1-2.3, the verification of (Part I) will be demonstrated as follows.

The first, we fix any $[\eta_0, \theta_0] \in D_0$. Then, applying (m2) of Lemma 2.3 to the case when:

$$\beta = \alpha \text{ in } C(\mathbb{R}), \ \ w_\nu = w_0 = \eta_0 \text{ in } C(\overline{\Omega}) \text{ for } \nu > 0, \ \text{ and } \ \hat{v}_0 = \theta_0 \text{ in } L^2;$$

we can prepare an approximation sequence $\{\bar{\theta}_{0,\nu} \,|\, \nu > 0\} \subset H^1$ of $\theta_0 \in BV$, such that:

$$\bar{\theta}_{0,\nu} \to \theta_0 \ \text{ in } L^2 \ \text{ and } \ \Phi_{\alpha,\nu}(\eta_0; \bar{\theta}_{0,\nu}) \to \Phi_\alpha(\eta_0; \theta_0), \ \text{ as } \nu \searrow 0. \tag{2.9}$$

Hence:

$$\mathscr{F}_\nu(\eta_0, \bar{\theta}_{0,\nu}) \to \mathscr{F}(\eta_0, \theta_0) \ \text{ as } \nu \searrow 0;$$

and we may assume that:

$$R_0 := \sup_{0<\nu\leq\nu_0} \mathscr{F}_\nu(\eta_0, \bar{\theta}_{0,\nu}) < \infty, \ \text{ for a certain small } 0 < \nu_0 < 1. \tag{2.10}$$

Secondly, let $\{[\bar{\eta}_\nu, \bar{\theta}_\nu] \,|\, \nu > 0\}$ be a sequence, consisting of the solutions $[\bar{\eta}_\nu, \bar{\theta}_\nu]$ of $(\mathrm{S})_\nu$ when $[\eta_{0,\nu}, \theta_{0,\nu}] = [\eta_0, \bar{\theta}_{0,\nu}]$ for $\nu > 0$. Then, by virtue of (A1)-(A4), (2.7) and (2.10), it is deduced that:

$$\begin{aligned} &\frac{1}{2}\left(|(\bar{\eta}_\nu)_t|^2_{L^2(0,T;L^2)} + \sup_{0\leq t\leq T} |(\bar{\eta}_\nu)_x(t)|^2_{L^2} \right) \\ &+\delta_\alpha \left(\frac{1}{2}|(\bar{\theta}_\nu)_t|^2_{L^2(0,T;L^2)} + \sup_{0\leq t\leq T} |(\bar{\theta}_\nu)_x(t)|_{L^1} \right) \leq 4R_0, \ \text{ for any } 0 < \nu \leq \nu_0. \end{aligned} \tag{2.11}$$

Under the one-dimensional setting of Ω, we see from (2.5) and (2.11) that:

$$\begin{cases} \bullet \ \{\bar{\eta}_\nu \,|\, 0 < \nu \leq \nu_0\} \text{ is a bounded sequence in } W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \subset C(\overline{Q_T}), \text{ satisfying } 0 \leq \bar{\eta}_\nu \leq 1 \text{ on } \overline{Q_T} \text{ for all } 0 < \nu \leq \nu_0, \text{ and hence it is compact in } C(\overline{Q_T}); \\ \bullet \ \{\bar{\theta}_\nu \,|\, 0 < \nu \leq \nu_0\} \text{ is a bounded sequence in } W^{1,2}(0,T;L^2) \cap L^\infty(0,T;W^{1,1}) \subset L^\infty(Q_T), \text{ and hence it is compact in } C([0,T];L^2). \end{cases} \tag{2.12}$$

Consequently, for a certain decreasing sequence $\{\nu_n \,|\, n = 1, 2, 3, \cdots\} \subset (0, \nu_0)$, having the zero-convergence property:

$$\nu_n \searrow 0 \ \text{ as } n \to \infty; \tag{2.13}$$

we find a pair $[\eta, \theta] \in L^2(0,T;L^2) \times L^2(0,T;L^2)$ of functions, such that:

$$\begin{cases} \eta \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \subset C(\overline{Q_T}) \text{ with } 0 \leq \eta \leq 1 \text{ on } \overline{Q_T}, \\ \theta \in W^{1,2}(0,T;L^2) \cap L^\infty(Q_T) \text{ with } |D\theta(\cdot)|(\Omega) \in L^\infty(0,T); \end{cases} \tag{2.14}$$

and the approximation sequence $\{[\eta_n, \theta_n]\} := \{[\bar{\eta}_{\nu_n}, \bar{\theta}_{\nu_n}] \,|\, n = 1, 2, 3, \cdots\}$, fulfills:

$$\eta_n \to \eta \quad \text{in } C(\overline{Q_T}), \text{ weakly in } W^{1,2}(0,T;L^2), \text{ and weakly-}* \text{ in } L^\infty(0,T;H^1), \tag{2.15}$$

$$\theta_n \to \theta \quad \text{in } C([0,T];L^2), \text{ weakly in } W^{1,2}(0,T;L^2), \text{ and weakly-}* \text{ in } L^\infty(Q_T), \tag{2.16}$$

$$F\eta_n \ (= A_N \eta_n) \to F\eta \quad \text{weakly in } L^2(0,T;(H^1)^*), \tag{2.17}$$

$$\begin{cases} \eta_n(t) \to \eta(t) & \text{weakly in } H^1, \\ \theta_n(t) \to \theta(t) & \text{weakly-}* \text{ in } BV, \end{cases} \quad \text{for any } t \in [0,T]; \tag{2.18}$$

as $n \to \infty$.

Thus, we conclude the compactness of the approximation sequence $\{[\eta_\nu, \theta_\nu]\}$ in the topologies, shown in (2.15)-(2.18). ∎

(Part II) Compatibility of the limit $[\boldsymbol{\eta}, \boldsymbol{\theta}]$ with the system (S)

Throughout this (Part II), we fix the (decreasing) sequence $\{\nu_n\} \subset (0,1)$, the sequence $\{[\eta_n, \theta_n]\} \subset L^2(0,T;L^2) \times L^2(0,T;L^2)$ and the pair $[\eta, \theta] \in L^2(0,T;L^2) \times L^2(0,T;L^2)$ of functions, found in (2.13)-(2.18). On that basis, the discussion will be proceeded in the order of verifications, from the compatibility with the second Cauchy problem (1.4) to that with the first one (1.3).

Verification of the compatibility with (1.4). For the sake of convenience, we start with adding some more notations.

Notation for the limiting observation. Let $\nu_0 \in (0,1)$, $\{\nu_n\} \subset (0, \nu_0)$, $\{\eta_n\} \subset W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1)$ and $\eta \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1)$ be the same as in (2.10)-(2.15). Also, let $I \subset (0,T)$ be any open interval. Here, we define a functional $\hat{\Phi}(\,\cdot\,)_I$ on $L^2(I;L^2)$, by putting:

$$\zeta \in L^2(I;L^2) \mapsto \hat{\Phi}(\zeta)_I := \begin{cases} \displaystyle\int_I \Phi_\alpha(\eta(t);\zeta(t))\,dt, & \text{if } \Phi_\alpha(\eta(\,\cdot\,);\zeta(\,\cdot\,)) \in L^1(I), \\ \infty, & \text{otherwise.} \end{cases} \tag{2.19}$$

As well as, for any $n \in \mathbb{N}$, we define a functional $\hat{\Phi}_n(\,\cdot\,)_I$ on $L^2(I;L^2)$, by putting:

$$\zeta \in L^2(I;L^2) \mapsto \hat{\Phi}_n(\zeta)_I := \begin{cases} \displaystyle\int_I \Phi_{\alpha,\nu_n}(\eta_n(t);\zeta(t))\,dt, \\ \qquad \text{if } \Phi_{\alpha,\nu_n}(\eta_n(\,\cdot\,);\zeta(\,\cdot\,)) \in L^1(I), \\ \infty, \quad \text{otherwise.} \end{cases} \tag{2.20}$$

Remark 2.1 (Key-properties for $\hat{\Phi}(\,\cdot\,)_I$ and $\hat{\Phi}_n(\,\cdot\,)_I$) Let $I \subset (0,T)$ be any open interval. Then, clearly, $\hat{\Phi}(\,\cdot\,)_I$ and $\hat{\Phi}_n(\,\cdot\,)_I$, $n \in \mathbb{N}$, are proper functionals on $L^2(I;L^2)$, because:

$$\Phi_\alpha(\eta(t);0) = \Phi_{\alpha,\nu_n}(\eta_n(t);0) = 0, \quad \text{for all } t \in I \text{ and } n = 1, 2, 3, \cdots.$$

Subsequently, according to Remark Ap.6 in Appendix, the lower semi-continuity and convexity of functionals $\hat{\Phi}_n(\,\cdot\,)_I$, $n \in \mathbb{N}$, are guaranteed by [17, Lemma 1.2.2]. But, as it is also mentioned in Remark Ap.6, the functional $\hat{\Phi}(\,\cdot\,)_I$ is not in applicable scope of the general theory [17], and hence the lower semi-continuity and convexity of $\hat{\Phi}(\,\cdot\,)_I$ are not derived, so immediately.

On the basis of the above notation and remark, we next prepare the following key-lemmas.

Lemma 2.4 *(Preliminaries) Let us fix any open interval $I \subset (0,T)$, and let us take any function $\xi \in L^2(I;L^2)$, such that $\xi(t) \in BV$ for a.e. $t \in I$. Then, the following three items hold.*

(I) If $\psi \in C(\overline{I \times \Omega})$, then the function $t \in I \mapsto \int_\Omega \psi(t)|D\xi(t)| \in \mathbb{R}$ *is measurable on I.*

(II) If $|D\xi(\,\cdot\,)|(\Omega) \in L^1(I)$, and if $\gamma \in C(\overline{I \times \Omega})$, then a continuous linear functional $[\gamma|D\xi|]$ on $C(\overline{I \times \Omega})$, defined as:

$$[\gamma|D\xi|] : \psi \in C(\overline{I \times \Omega}) \mapsto \int_I \int_\Omega \psi(t)\gamma(t)|D\xi(t)|\,dt \in \mathbb{R}; \tag{2.21}$$

forms a Radon measure on $I \times \Omega$, such that:

$$[\gamma|D\xi|](U) = \int_I \gamma(t)|D\xi(t)|(\{x \in \Omega \,|\, (t,x) \in U\})\,dt, \quad \text{for any open set } U \subset I \times \Omega. \tag{2.22}$$

Lemma 2.5 *(Auxiliary observations) Let $I \subset (0,T)$ be any open interval. Let $\psi_\infty \in C(\overline{I \times \Omega})$, and let $\{\psi_n \,|\, n = 1,2,3,\cdots\} \subset C(\overline{I \times \Omega})$ be any sequence of functions, such that:*

$$\psi_n \to \psi_\infty \text{ in } C(\overline{I \times \Omega}) \quad \text{as } n \to \infty. \tag{2.23}$$

Let $\xi_\infty \in L^2(I;L^2)$, and let $\{\xi_n \,|\, n = 1,2,3,\cdots\} \subset L^2(I;L^2)$ be a sequence of functions, such that:

$$\begin{cases} \xi_\infty(t) \in BV, \quad \xi_n(t) \in BV, \ n = 1,2,3,\cdots, \\ D\xi_n(t) \to D\xi_\infty(t) \quad \text{weakly-}* \text{ in } \mathcal{M} \text{ as } n \to \infty, \end{cases} \text{for a.e. } t \in I. \tag{2.24}$$

Then, the following two items hold.

(III) $\displaystyle\liminf_{n\to\infty} \int_I \int_\Omega |\psi_n(t)|\,|D\xi_n(t)|\,dt \geq \int_I \int_\Omega |\psi_\infty(t)|\,|D\xi_\infty(t)|\,dt;$

(IV) If:

$$\begin{cases} \bullet\ |D\xi_\infty(\,\cdot\,)|(\Omega) \in L^1(I), \ |D\xi_n(\,\cdot\,)|(\Omega) \in L^1(I), \ n = 1,2,3,\cdots, \\ \bullet\ \gamma_\infty \in C(\overline{I \times \Omega}), \ \{\gamma_n \,|\, n = 1,2,3,\cdots\} \subset C(\overline{I \times \Omega}), \\ \bullet\ \gamma_\infty \geq 0 \text{ on } \overline{I \times \Omega}, \text{ and } \gamma_n \geq 0 \text{ on } \overline{I \times \Omega}, \ n = 1,2,3,\cdots, \\ \bullet\ \gamma_n \to \gamma_\infty \text{ in } C(\overline{I \times \Omega}) \text{ as } n \to \infty, \\ \bullet\ \displaystyle\limsup_{n\to\infty} \Bigl|\gamma_n(\,\cdot\,)|D\xi_n(\,\cdot\,)|(\Omega)\Bigr|_{L^1(I)} \leq \Bigl|\gamma_\infty(\,\cdot\,)|D\xi_\infty(\,\cdot\,)|(\Omega)\Bigr|_{L^1(I)}; \end{cases} \tag{2.25}$$

then:

$$\lim_{n \to \infty} \int_I \int_\Omega \psi_n(t)\gamma_n(t)|D\xi_n(t)|\,dt = \int_I \int_\Omega \psi_\infty(t)\gamma_\infty(t)\,|D\xi_\infty(t)|\,dt.$$

Lemma 2.6 *(Approximation for time-dependent BV-function) Let $I \subset (0, T)$ be any open interval, and let $\breve{\zeta} \in L^2(I; L^2)$ be any function, such that $|D\breve{\zeta}(\,\cdot\,)|(\Omega) \in L^1(I)$. Then, there exists an approximation sequence $\{\breve{\psi}_i \,|\, i = 1, 2, 3, \cdots\} \subset C^\infty(\mathbb{R}^2)$ of the function $\breve{\zeta}$, in the sense that:*

$$\left\{ \begin{array}{l} \breve{\psi}_i \to \breve{\zeta} \text{ in } L^2(I; L^2), \\ \displaystyle \int_I \int_\Omega |(\breve{\psi}_i)_x(t)|\,dxdt \to \int_I \int_\Omega |D\breve{\zeta}(t)|\,dt, \end{array} \right. \qquad \text{as } i \to \infty; \tag{2.26}$$

and

$$\breve{\psi}_i(t) \to \breve{\zeta}(t) \text{ in } L^2 \text{ and strictly in } BV \quad \text{as } i \to \infty, \quad \text{for a.e. } t \in I. \tag{2.27}$$

Lemma 2.7 *(Γ-convergence) Let $I \subset (0, T)$ be any open interval, and let $\hat{\Phi}(\,\cdot\,)_I$ and $\hat{\Phi}_n(\,\cdot\,)_I$, $n = 1, 2, 3, \cdots$, be the functionals, defined in (2.19) and (2.20), respectively. Then, the functional $\hat{\Phi}(\,\cdot\,)_I$ is a proper l.s.c. and convex function on $L^2(I; L^2)$, such that:*

$$D(\hat{\Phi}(\,\cdot\,)_I) = \left\{ \; \tilde{\zeta} \in L^2(I; L^2) \;\middle|\; |D\tilde{\zeta}(\,\cdot\,)|(\Omega) \in L^1(I) \; \right\}. \tag{2.28}$$

Moreover, the sequence $\{\hat{\Phi}_n(\,\cdot\,)_I \,|\, n = 1, 2, 3, \cdots\}$ of convex functions on $L^2(I; L^2)$ converges to $\hat{\Phi}(\,\cdot\,)_I$ on $L^2(I; L^2)$, in the sense of Γ-convergence [9], as $n \to \infty$. More precisely:

(γ1) $\liminf_{n \to \infty} \hat{\Phi}_n(\breve{\zeta}_n)_I \geq \hat{\Phi}(\breve{\zeta}_\infty)_I$, if $\breve{\zeta}_\infty \in L^2(I; L^2)$, $\{\breve{\zeta}_n \,|\, n = 1, 2, 3, \cdots\} \subset L^2(I; L^2)$ and $\breve{\zeta}_n \to \breve{\zeta}_\infty$ in $L^2(I; L^2)$ as $n \to \infty$;

(γ2) for any $\hat{\zeta}_\infty \in D(\hat{\Phi}(\,\cdot\,)_I)$, there exists a sequence $\{\hat{\zeta}_n \,|\, n = 1, 2, 3, \cdots\} \subset L^2(I; H^1)$, such that $\hat{\zeta}_n \to \hat{\zeta}_\infty$ in $L^2(I; L^2)$ and $\hat{\Phi}_n(\hat{\zeta}_n)_I \to \hat{\Phi}(\hat{\zeta}_\infty)_I$, as $n \to \infty$.

Now, let us see the verification argument, after we obtain the above key-lemmas. We first take any open interval $I \subset (0, T)$ and any $z \in BV$ ($z \in D(\hat{\Phi}(\,\cdot\,)_I)$), to find a sequence $\{\zeta_n \,|\, n = 1, 2, 3, \cdots\} \subset L^2(I; H^1)$ of functions, such that:

$$\zeta_n \to z \text{ in } L^2(I; L^2) \quad \text{and} \quad \hat{\Phi}_n(\zeta_n)_I \to \hat{\Phi}(z)_I, \quad \text{as } n \to \infty.$$

Such sequence $\{\zeta_n\}$ can be obtained by applying (γ2) of Lemma 2.7, to the case when $\hat{\zeta}_\infty(t) = z$ in L^2 for $t \in [0, T]$.

On the other hand, since the pair $[\eta_n, \theta_n]$, for each $n \in \mathbb{N}$, fulfills the Cauchy problem (2.4) in the case of $\nu = \nu_n$,

$$\begin{gathered} (\alpha_0(\eta_n(t))(\theta_n)_t(t), \theta_n(t) - \zeta_n(t))_{L^2} + \Phi_{\alpha,\nu_n}(\eta_n(t); \theta_n(t)) \leq \Phi_{\alpha,\nu_n}(\eta_n(t); \zeta_n(t)), \\ \text{for a.e. } t \in I \text{ and } n = 1, 2, 3, \cdots. \end{gathered} \tag{2.29}$$

Here, let us integrate the both sides of (2.29) over I. Then, taking into account of (2.13)-(2.18) and (γ1) of Lemma 2.7, we deduce that:

$$\begin{aligned} & \int_I (\alpha_0(\eta(t))\theta_t(t), \theta(t) - z)_{L^2}\,dt + \int_I \Phi_\alpha(\eta(t); \theta(t))\,dt \\ \leq \;& \lim_{n \to \infty} \int_I (\alpha(\eta_n(t))(\theta_n)_t(t), \theta_n(t) - \zeta_n(t))_{L^2}\,dt + \liminf_{n \to \infty} \int_I \Phi_{\alpha,\nu_n}(\eta_n(t); \theta_n(t))\,dt \\ \leq \;& \lim_{n \to \infty} \hat{\Phi}_n(\zeta_n)_I = \hat{\Phi}(z)_I = \int_I \Phi_\alpha(\eta(t); z)\,dt. \end{aligned} \tag{2.30}$$

Since the selection of the open interval $I \subset (0,T)$ is arbitrary, it follows from (2.30) that:

$$\begin{aligned} &(\alpha_0(\eta(t))\theta_t(t), \theta(t) - z)_{L^2} + \Phi_\alpha(\eta(t);\theta(t)) \leq \Phi_\alpha(\eta(t); z), \\ &\qquad \text{for any } z \in BV = D(\Phi_\alpha(\eta(t); \cdot)); \end{aligned} \tag{2.31}$$

and for a.e. $t \in (0,T)$, that is the Lebesgue point of densities in (2.30). Moreover, by (2.9) and (2.16),

$$\theta_n(0) = \bar{\theta}_{0,\nu_n} \to \theta(0) = \theta_0 \text{ in } L^2, \text{ as } n \to \infty. \tag{2.32}$$

(2.31) and (2.32) imply that the pair $[\eta, \theta]$ solves the second Cauchy problem (1.4). ■

Verification of the compatibility with (1.3). With regard to the compatibility with (1.3), the principal part of the verification argument will be reduced to the following lemma.

Lemma 2.8 *(Convergence of time-integrals of weighted total variations) Let $I \subset (0,T)$ be any open interval, let $\psi_\infty \in C(\overline{I \times \Omega})$ and $\{\psi_n \,|\, n = 1,2,3,\cdots\} \subset C(\overline{I \times \Omega})$ be the same as in Lemma 2.5, and let $[\eta, \theta] \in L^2(I; L^2) \times L^2(I; L^2)$ and $\{[\eta_n, \theta_n] \,|\, n = 1,2,3,\cdots\} \subset L^2(I; L^2) \times L^2(I; L^2)$ be the same as in (2.14)-(2.18). Then:*

$$\int_I \int_\Omega \psi_n(t)\alpha(\eta_n(t))|(\theta_n)_x(t)|\, dxdt \to \int_I \int_\Omega \psi_\infty(t)\alpha(\eta(t))|D\theta(t)|\, dt \quad \text{as } n \to \infty. \tag{2.33}$$

On the basis of this lemma, the compatibility with the first Cauchy problem (1.3) will be verified as follows.

From (A1), (2.15) and (2.17), we immediately see that:

$$\begin{aligned} \mu_n^* := \alpha'(\eta_n)|(\theta_n)_x| \quad &= \quad -(\eta_n)_t - A_N\eta_n + \eta_n - g(\eta_n) \\ &\to \quad \mu^* := -\eta_t - F\eta + \eta - g(\eta) \end{aligned} \tag{2.34}$$

$$\text{weakly in } L^2(I; (H^1)^*) \quad \text{as } n \to \infty, \quad \text{for any open interval } I \subset (0,T).$$

In the meantime, applying Lemma 2.8, to the case when:

$$\psi_\infty = \psi \, \frac{\alpha'(\eta)}{\alpha(\eta)} \text{ in } C(\overline{I \times \Omega}), \text{ and } \psi_n = \psi \, \frac{\alpha'(\eta_n)}{\alpha(\eta_n)} \text{ in } C(\overline{I \times \Omega}),\ n = 1,2,3,\cdots,$$
$$\text{with any open interval } I \subset \Omega \text{ and any } \psi \in C(\overline{I \times \Omega});$$

we also have:

$$\begin{aligned} &\int_I \int_\Omega \psi(t)\, \mu_n^*(t)\, dxdt \to \int_I \int_\Omega \psi(t)\, \alpha'(\eta(t))|D\theta(t)|\, dt \quad \text{as } n \to \infty, \\ &\qquad \text{for any open interval } I \subset (0,T) \text{ and any } \psi \in C(\overline{I \times \Omega}). \end{aligned} \tag{2.35}$$

On account of (2.34)-(2.35),

$$\alpha'(\eta(t))|D\theta(t)| = \mu^*(t) = -\eta_t(t) - F\eta(t) + \eta(t) - g(\eta(t))$$
$$\text{in } \mathcal{D}'(\Omega) \text{ (in the distribution sense)}, \quad \text{for a.e. } t \in (0,T).$$

Thus, noting that:

$$\eta_n(0) = \eta(0) = \eta_0 \text{ in } H^1, \text{ for } n = 1,2,3,\cdots.$$

we can conclude that the limit $[\eta, \theta]$ solves the first Cauchy problem (1.3). ■

3 Verification of (Part I) in proof of Main Theorem

In this section, we complete the verification of (Part I), by giving the proofs of Lemmas 2.1-2.3.

Proof of Lemma 2.1. As it is already mentioned, the difference between our study and the previous ones [14, 15, 16, 18] is found only in the boundary condition, inherent in the second Cauchy problem (2.4). So, in principle, we can prove this lemma just as in [14, Proofs of Theorems 2.1-2.2], and all we have to care is in the situation such that we essentially rely on Poincaré's inequality.

In order to make clear such situations, let us first overview the outline of the proof. Referring to [14, Sections 3-5], the proof of Lemma 2.1 will be largely divided in three steps, summarized below.

(Step 1) Study of an auxiliary problem, denoted by $(\mathrm{P1};\bar{\theta})_\nu$:

$$(\mathrm{P1};\bar{\theta})_\nu \quad \begin{cases} (\eta_\nu)_t(t) + A_N \eta_\nu(t) - \eta_\nu(t) + g(\eta_\nu(t)) + \alpha'(\eta_\nu(t))|\bar{\theta}_x(t)| = 0 \\ \qquad\qquad\qquad \text{in } L^2, \ \ t \in (0,T), \\ \eta_\nu(0) = \eta_{0,\nu} \text{ in } L^2; \end{cases}$$

under given setting of the function $\bar{\theta} \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \subset C(\overline{Q_T})$. The discussion in this step is proceeded through the following two small steps:

(step 1-1) the existence and uniqueness of solution $\eta_\nu \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1)$;

(step 1-2) the range constraint property, described as $0 \leq \eta_\nu \leq 1$ on $\overline{Q_T}$.

(Step 2) Study of an auxiliary problem, denoted by $(\mathrm{P2};\bar{\alpha}_0,\bar{\eta})_\nu$:

$$(\mathrm{P2};\bar{\alpha}_0,\bar{\eta})_\nu \quad \begin{cases} \bar{\alpha}_0(t)(\theta_\nu)_t(t) + \partial\Phi_{\alpha,\nu}(\bar{\eta}(t);\theta_\nu(t)) \ni 0 \ \text{ in } L^2, \ \ t \in (0,T), \\ \theta_\nu(0) = \theta_{0,\nu} \text{ in } L^2; \end{cases}$$

under given setting of the function $\bar{\alpha}_0 \in L^\infty(Q_T)$, satisfying $\bar{\alpha}_0 \geq \delta_\alpha$ $\mathscr{L}^2$-a.e. in Q_T, and the function $\bar{\eta} \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \subset C(\overline{Q_T})$. The solution θ_ν of the auxiliary problem $(\mathrm{P2};\bar{\alpha}_0,\bar{\eta})_\nu$ is found in the range of $W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1)$. On that basis, this step is further subdivided in the following small steps:

(step 2-1) the existence and uniqueness of solution in case of $\bar{\alpha}_0 \in C^2(\overline{Q_T})$;

(step 2-2) the existence and uniqueness of solution in case of $(\bar{\alpha}_0)_t \in L^\infty(Q_T)$;

(step 2-3) the existence of solution in general case of $\bar{\alpha}_0 \in L^\infty(Q_T)$.

(Step 3) Study of coupled system $\{(\mathrm{P1};\theta_\nu)_\nu, (\mathrm{P2};\bar{\alpha}_0,\eta_\nu)_\nu\}$, under the setting such that the function $\bar{\alpha}_0$ also depends on the unknown η_ν. More precisely:

(step 3-1) the existence of solution in the case when $\bar{\alpha}_0$ is given as

$$\bar{\alpha}_0(t) := \alpha_0 \left(\int_{\mathbb{R}} \varrho_\varepsilon(t-\tau)\eta_\nu(\min\{T, \max\{\tau, 0\}\})\, d\tau \right) \ \text{ in } L^2, \ \ t \in [0,T],$$

by using the unknown η_ν, and the usual (one-dimensional) mollifier ϱ_ε with a small constant $\varepsilon > 0$;

(step 3-2) the existence of solution in the required case when $\bar{\alpha}_0 = \alpha_0(\eta_\nu) \in L^\infty(Q_T)$;

(step 3-3) the uniqueness of solution of $(\mathrm{S})_\nu$.

In these steps, Poincaré's inequality will be involved only in the proof of (step 2-1), and the proofs of all other steps will be slight modifications of those, found in the previous study [14, Sections 3-5].

Hence, we give here only the proof of (step 2-1). The existence and uniqueness problem in this step will be a direct consequence of one of general theories, e.g. [24, 27], which is kindred to the study of Kenmochi [17] (see Section Ap.3 of Appendix, for details).

Let us begin with the preparation of some auxiliary notations, for the application of the general theory. Let us fix any large constant $A_0 > 0$, such that:

$$\delta_\alpha^3 A_0 \geq |(\bar{\alpha}_0)_x|^2_{C(\overline{Q_T})};$$

and for any $t \in [0, T]$, let us define a proper l.s.c. and convex function Ψ^t on L^2, by putting:

$$v \in L^2 \mapsto \Psi^t(v) := \Phi_{\alpha,\nu}\left(\bar{\eta}(t); \frac{v}{\sqrt{\bar{\alpha}_0(t)}}\right) + \frac{A_0}{2}|v|^2_{L^2}. \tag{3.1}$$

Then, the auxiliary problem $(\mathrm{P2}; \bar{\alpha}_0, \bar{\eta})_\nu$ will be reformulated, as follows:

$$\begin{cases} (u_\nu)_t(t) + \partial\Psi^t(u_\nu(t)) - \left(\dfrac{(\bar{\alpha}_0)_t(t, \cdot)}{2\bar{\alpha}_0(t, \cdot)} + A_0\right) u_\nu(t) \ni 0 \ \text{ in } L^2, \ \ t \in (0, T), \\ u_\nu(0) = \sqrt{\bar{\alpha}_0(0, \cdot)}\, \theta_{0,\nu} \ \text{ in } L^2; \end{cases} \tag{3.2}$$

by using an equivalent transform:

$$\theta_\nu \in W^{1,2}(0, T; L^2) \cap L^\infty(0, T; H^1) \mapsto u_\nu := \sqrt{\bar{\alpha}_0}\, \theta_\nu \in W^{1,2}(0, T; L^2) \cap L^\infty(0, T; H^1);$$

where $\partial\Psi^t$ is the subdifferential of Ψ^t in the topology of L^2, for any $t \in [0, T]$. Additionally, by basic (but technical) calculations, we will find a large constant $A_1 > 0$, to realize that:

$$|\Psi^t(z) - \Psi^s(z)| \leq A_1\left(1 + \frac{1}{\nu}\right)\left|\int_s^t (1 + |\bar{\eta}_t(\tau)|_{L^2})\, d\tau\right|(1 + \Psi^s(z)), \tag{3.3}$$
$$\text{for all } s, t \in [0, T] \text{ and all } z \in H^1 \ (= D(\Psi^s)).$$

Thus, taking into account of Lemma Ap.2 and Remarks Ap.4-Ap.5 in Appendix, we can apply some certain general theories, such as [24, 27], to conclude the existence and uniqueness of solution of $(\mathrm{P2}; \bar{\alpha}_0, \bar{\eta})_\nu$ in (step 2-1). ■

Proof of Lemma 2.2. We can prove this lemma, by using almost the same demonstration method, as in [14, Section 5]. Incidentally, the absolute continuity of the energies will be direct consequences of (B) of Lemma Ap.3 and Remark Ap.6 in Appendix. ■

Proof of Lemma 2.3. First, we verify the condition (m1). If $\liminf_{\nu\searrow 0} \Phi_{\beta,\nu}(w_\nu; \check{v}_\nu) = \infty$, then it is trivial, otherwise, there is a decreasing sequence $\{\check{\nu}_i \,|\, i = 1, 2, 3, \cdots\} \subset (0, 1)$, such that:

$$\check{\nu}_i \searrow 0 \text{ as } i \to \infty, \ \text{ and } \ \liminf_{\nu\searrow 0} \Phi_{\beta,\nu}(w_\nu; \check{v}_\nu) = \lim_{i\to\infty} \Phi_{\beta,\check{\nu}_i}(w_{\check{\nu}_i}; \check{v}_{\check{\nu}_i}).$$

Besides, let us note that:

$$R_1 := \sup_{i\in\mathbb{N}} |(\check{v}_{\check{\nu}_i})_x|_{L^1} \leq \frac{1}{\delta_\beta} \sup_{i\in\mathbb{N}} \Phi_{\beta,\check{\nu}_i}(w_{\check{\nu}_i}; \check{v}_{\check{\nu}_i}) < \infty.$$

Thus, by virtue of (2.8) and the lower semi-continuity of $\Phi_\beta(w_0;\,\cdot\,)$, it is deduced that:

$$
\begin{aligned}
\liminf_{\nu\searrow 0}\Phi_{\beta,\nu}(w_\nu;\check{v}_\nu) &= \lim_{i\to\infty}\int_\Omega\left(\beta(w_{\check{\nu}_i})|(\check{v}_{\check{\nu}_i})_x| + \frac{\check{\nu}_i}{2}|(\check{v}_{\check{\nu}_i})_x|^2\right)dx \\
&\geq \liminf_{i\to\infty}\Phi_\beta(w_0;\check{v}_{\check{\nu}_i}) - R_1\lim_{i\to\infty}|\beta(w_{\check{\nu}_i})-\beta(w_0)|_{C(\overline{\Omega})} \\
&\geq \Phi_\beta(w_0;\check{v}_0).
\end{aligned}
$$

Secondly, we verify the condition (m2). For any $\hat{v}_0 \in D(\Phi_\beta(w_0;\,\cdot\,)) = BV$, let us take a sequence $\{\hat{\varphi}_i \,|\, i = 1,2,3,\cdots\} \subset H^1$, such that:

$$
\hat{\varphi}_i \to \hat{v}_0 \text{ in } L^2 \quad\text{and}\quad \int_\Omega |(\hat{\varphi}_i)_x|\,dx \to \int_\Omega |D\hat{v}_0|, \quad \text{as } i\to\infty;
$$

by using the usual regularization method of BV-function (cf. [1, Theorem 3.9], [6, Theorem 10.1.2], [10, Section 5.2.2], or [11, 1.17 Theorem]).

Next, for any $i \in \mathbb{N}$, let us take a decreasing sequence $\{\hat{\nu}_i \,|\, i = 1,2,3,\cdots\} \subset (0,1)$, such that:

$$
\hat{\nu}_i \searrow 0 \text{ as } i\to\infty, \quad\text{and}\quad \frac{\nu}{2}\int_\Omega |(\hat{\varphi}_i)_x|^2\,dx \leq \frac{1}{i} \quad\text{for any } i\in\mathbb{N} \text{ and any } 0<\nu\leq\hat{\nu}_i.
$$

Now, the required sequence $\{\hat{v}_\nu \,|\, \nu > 0\}$ will be obtained in the following way:

$$
\hat{v}_\nu := \begin{cases} \hat{\varphi}_i, \text{ if } \hat{\nu}_{i+1} < \nu \leq \hat{\nu}_i, \text{ for some } i\in\mathbb{N}, \\ \hat{\varphi}_1, \text{ if } \nu > \hat{\nu}_1. \end{cases}
$$

In fact, we immediately see from the definition of $\{\hat{v}_\nu \,|\, \nu > 0\}$ that:

$$
\begin{cases} R_2 := \sup_{\nu>0}\int_\Omega |(\hat{v}_\nu)_x|\,dx < \infty, \\ \hat{v}_\nu \to \hat{v}_0 \text{ in } L^2, \ \int_\Omega |(\hat{v}_\nu)_x|\,dx \to \int_\Omega |D\hat{v}_0| \text{ and } \frac{\nu}{2}\int_\Omega |(\hat{v}_\nu)_x|^2\,dx \to 0, \ \text{ as } \nu\searrow 0. \end{cases} \tag{3.4}
$$

Also, by the lower semi-continuity of the total variation,

$$
\liminf_{\nu\searrow 0}\int_W |(\hat{v}_\nu)_x|\,dx \geq \int_W |D\hat{v}_0|, \quad \text{for any open set } W\subset\Omega. \tag{3.5}
$$

On account of (2.8) and (3.4)-(3.5), we can apply [1, Proposition 1.80] to calculate that:

$$
\begin{aligned}
& |\Phi_{\beta,\nu}(w_\nu;\hat{v}_\nu) - \Phi_\beta(w_0;\hat{v}_0)| \\
\leq\ & R_2|\beta(w_\nu)-\beta(w_0)|_{C(\overline{\Omega})} + \left|\int_\Omega \beta(w_0)|(\hat{v}_\nu)_x|\,dx - \int_\Omega \beta(w_0)|D\hat{v}_0|\right| + \frac{\nu}{2}\int_\Omega |(\hat{v}_\nu)_x|^2\,dx \\
\to\ & 0, \quad \text{as } \nu\searrow 0.
\end{aligned}
$$

■

4 Verification of (Part II) in proof of Main Theorem

In this section, the proofs of Lemmas 2.4-2.8 are going to be given, to complete the verification of (Part II).

Proof of Lemma 2.4. First, we give the proof of the item (I). Let us fix any $\xi \in L^2(I; L^2)$. Besides, let us set $t_* := \inf I$ and $t^* := \sup I$, and let us prepare a class $\{\Delta_m \,|\, m = 1, 2, 3, \cdots\} \subset 2^I$ of equal-partitionings $\Delta_m := \{t_i^{(m)} = t_* + ih_m \,|\, i = 0, 1, \cdots, 2^m\} \subset I$ with the constant partition-widths $h_m = (t^* - t_*)/2^m$, $m = 1, 2, 3, \cdots$. Then, it is inferred from [7, Proof of Proposition 2.16] (or [8, Proposition 7 in Section 5 of Chpater IV]) that the functions:

$$\begin{aligned} t \in I \mapsto \lambda_i^{(m)}(t) &:= \int_\Omega (\psi^+(t_i^{(m)}) + 1)\,|D\xi(t)| - \int_\Omega (\psi^-(t_i^{(m)}) + 1)\,|D\xi(t)| \\ &= \int_\Omega \psi(t_i^{(m)})|D\xi(t)| \in \mathbb{R}, \quad \text{for } m = 1, 2, 3, \cdots, \text{ and } i = 1, \cdots, 2^m; \end{aligned}$$

are measurable, where the superscripts "+" and "−" denote the positive part and the negative part of functions, respectively. Here, for any $m \in \mathbb{N}$, let us define a step function $\bar{\psi}_m : \overline{I} \longrightarrow C(\overline{\Omega})$, by putting:

$$t \in \overline{I} \mapsto \bar{\psi}_m(t) := \sum_{i=1}^{2^m} \chi_{(t_{i-1}^{(m)}, t_i^{(m)}]}(t)\psi(t_i^{(m)}) \in C(\overline{\Omega});$$

where for any Borel set $B \subset \mathbb{R}$, $\chi_B : \mathbb{R} \longrightarrow \{0, 1\}$ is the characteristic function of B. Then, the functions:

$$t \in I \mapsto \bar{\lambda}_m(t) := \sum_{i=1}^{2^m} \chi_{(t_{i-1}^{(m)}, t_i^{(m)}]}(t)\lambda_i^{(m)}(t) = \int_\Omega \bar{\psi}_m(t)\,|D\xi(t)| \in \mathbb{R}, \quad m = 1, 2, 3, \cdots;$$

are also measurable on I.

In the meantime, by the uniform continuity of $\psi \in C(\overline{I \times \Omega})$, it is easily checked that:

$$\bar{\psi}_m(t) \to \psi(t) \text{ in } C(\overline{\Omega}) \text{ as } m \to \infty, \text{ for any } t \in \overline{I}.$$

Hence, applying Lebesgue's dominated convergence theorem, we calculate that:

$$\bar{\lambda}_m(t) = \int_\Omega \bar{\psi}_m(t)\,|D\xi(t)| \to \int_\Omega \psi(t)\,|D\xi(t)| \in \mathbb{R} \text{ as } m \to \infty, \text{ for a.e. } t \in I.$$

It implies the validity of the item (I).

Secondly, let us look toward the remaining item (II). Since:

$$[\gamma|D\xi|] \in C(I \times \Omega)^* \subset C_0(I \times \Omega)^* = \mathcal{M}(I \times \Omega);$$

we can immediately regard $[\gamma|D\xi|]$ as a Radon measure on $I \times \Omega$.

On that basis, let us take any open set $U \subset I \times \Omega$, and let us take an approximation sequence $\{\bar{\chi}_i \,|\, i = 1, 2, 3, \cdots\} \subset C_c^\infty(I \times \Omega)$ of the characteristic function $\chi_U : \mathbb{R}^2 \longrightarrow \{0, 1\}$ of the open set $U \subset I \times \Omega$, such that:

$$\begin{cases} \bar{\chi}_1 \leq \cdots \leq \bar{\chi}_i \leq \cdots \leq \chi_U \text{ on } \mathbb{R}^2, \\ \bar{\chi}_i(t,x) \nearrow \chi_U(t,x) \text{ as } i \to \infty, \text{ for any } (t,x) \in I \times \Omega. \end{cases}$$

Then, we can apply Lebesgues's dominated convergence theorem, to deduce that the function:

$$\begin{aligned} t \in I \;\; &\mapsto \;\; \gamma(t)|D\xi(t)|(\{x \in \Omega \,|\, (t,x) \in U\}) \\ &= \int_\Omega \chi_U(t)\gamma(t)|D\xi(t)| = \lim_{i \to \infty} \int_\Omega \bar{\chi}_i(t)\gamma(t)|D\xi(t)| \in \mathbb{R}; \end{aligned}$$

is measurable on I, and:

$$\begin{aligned} [\gamma|D\xi|](U) &= \int_{I \times \Omega} \chi_U \, d[\gamma|D\xi|] = \lim_{i \to \infty} \int_{I \times \Omega} \bar{\chi}_i \, d[\gamma|D\xi|] \\ &= \lim_{i \to \infty} \int_I \int_\Omega \bar{\chi}_i(t)\gamma(t)|D\xi(t)| \, dt = \int_I \gamma(t)|D\xi(t)|(\{x \in \Omega \,|\, (t,x) \in U\}) \, dt; \end{aligned}$$

as it is asserted in (II). ∎

Proof of Lemma 2.5. This lemma is obtained by means of the general measure theory, as in [1, Chapter 1].

Let us start with verifying the item (III). By virtue of (2.23), (2.24) and the uniform boundedness theorem, we have:

$$\begin{aligned} &\left| \int_\Omega \varphi \, \psi_n(t) \, D\xi_n(t) - \int_\Omega \varphi \, \psi_\infty(t) \, D\xi_\infty(t) \right| \\ \leq \;\; &|\varphi|_{C(\overline{\Omega})} |\psi_n - \psi_\infty|_{C(\overline{Q_T})} \sup_{n \in \mathbb{N}} |D\xi_n(t)|_{\mathcal{M}} + \left| \int_\Omega \varphi \, \psi_\infty(t) \, D(\xi_n - \xi_\infty)(t) \right| \\ \to \;\; &0, \;\; \text{as } n \to \infty, \;\; \text{for any } \varphi \in C_0 \text{ and a.e. (fixed) } t \in I. \end{aligned} \tag{4.1}$$

Also, by [1, Proposition 1.23]:

$$\begin{cases} |\psi_\infty(t)D\xi_\infty(t)| = \left|\psi_\infty(t)\frac{D\xi_\infty(t)}{|D\xi_\infty(t)|}\right| |D\xi_\infty(t)| = |\psi_\infty(t)||D\xi_\infty(t)|, \\ |\psi_n(t)D\xi_n(t)| = \left|\psi_n(t)\frac{D\xi_n(t)}{|D\xi_n(t)|}\right| |D\xi_n(t)| = |\psi_n(t)||D\xi_n(t)|, \;\; n = 1,2,3,\cdots, \end{cases} \tag{4.2}$$
$$\text{in } \mathcal{M}, \;\; \text{for a.e. } t \in I;$$

where for a.e. $t \in I$, $\frac{D\xi_\infty(t)}{|D\xi_\infty(t)|}$ and $\frac{D\xi_n(t)}{|D\xi_n(t)|}$, $n \in \mathbb{N}$, are the Radon-Nikodým densities of $D\xi_\infty(t)$ and $D\xi_n(t)$, $n \in \mathbb{N}$, respectively, for their total variations.

By virtue of (4.1), (4.2) and [1, Theorem 1.59],

$$\begin{aligned} &\liminf_{n \to \infty} \int_\Omega |\psi_n(t)| \, |D\xi_n(t)| = \liminf_{n \to \infty} |\psi_n(t)D\xi_n(t)|(\Omega) \\ &\geq |\psi_\infty(t)D\xi_\infty(t)|(\Omega) = \int_\Omega |\psi_\infty(t)| \, |D\xi_\infty(t)|, \quad \text{for a.e. } t \in I. \end{aligned}$$

Hence, the item (III) is obtained by taking into account of (I) of Lemma 2.4 and Fatou's lemma.

Next, let us assume (2.25) to verify the item (IV). In (4.1), we can replace the functions ψ_∞ and ψ_n, $n \in \mathbb{N}$, by γ_∞ and γ_n, $n \in \mathbb{N}$, respectively. So, we easily see from [1, Propositoin 1.62] that:

$$\liminf_{n\to\infty} \gamma_n(t)|D\xi_n(t)|(W) \geq \gamma_\infty(t)|D\xi_\infty(t)|(W), \\ \text{for any open } W \subset \Omega \text{ and a.e. } t \in I. \tag{4.3}$$

Besides, let $[\gamma_\infty|D\xi_\infty|] \in \mathcal{M}(I \times \Omega)$ and $[\gamma_n|D\xi_n|] \in \mathcal{M}(I \times \Omega)$, $n \in \mathbb{N}$, be the Radon measures, that are defined according to (2.21) in Lemma 2.4. Then, it follows from (4.3), (II) of Lemma 2.4 and Fatou's lemma that:

$$\begin{aligned} \liminf_{n\to\infty}\ [\gamma_n|D\xi_n|](U) &= \liminf_{n\to\infty} \int_I \gamma_n(t)|D\xi_n(t)|(\{x \in \Omega \,|\, (t,x) \in U\})\, dt \\ \geq & \int_I \liminf_{n\to\infty} \gamma_n(t)|D\xi_n(t)|(\{x \in \Omega \,|\, (t,x) \in U\})\, dt \\ \geq & \int_I \gamma_\infty(t)|D\xi_\infty(t)|(\{x \in \Omega \,|\, (t,x) \in U\})\, dt = [\gamma_\infty|D\xi_\infty|](U), \\ & \text{for any open } U \subset I \times \Omega. \end{aligned} \tag{4.4}$$

In particular, combining (2.25) and (4.4), it follows that:

$$\lim_{n\to\infty} [\gamma_n|D\xi_n|](I \times \Omega) = [\gamma_\infty|D\xi_\infty|](I \times \Omega). \tag{4.5}$$

Now, on account of (2.21), (4.4), (4.5) and [1, Proposition 1.80], we deduce that:

$$\lim_{n\to\infty} \int_{I\times\Omega} \psi\ d[\gamma_n|D\xi_n|] = \int_{I\times\Omega} \psi\ d[\gamma_\infty|D\xi_\infty|], \quad \text{for any } \psi \in C(\overline{I \times \Omega});$$

and hence:

$$\begin{aligned} & \left| \int_I \int_\Omega \psi_n(t)\gamma_n(t)|D\xi_n(t)|\, dt - \int_I \int_\Omega \psi_\infty(t)\gamma_\infty(t)|D\xi_\infty(t)|\, dt \right| \\ \leq\ & |\psi_n - \psi_\infty|_{C(\overline{I\times\Omega})} \sup_{n\in\mathbb{N}} [\gamma_n|D\xi_n|](I \times \Omega) + \left| \int_{I\times\Omega} \psi_\infty\ d([\gamma_n|D\xi_n|] - [\gamma_\infty|D\xi_\infty|]) \right| \\ \to\ & 0, \quad \text{as } n \to \infty. \end{aligned}$$

Thus, we conclude the item (IV). ∎

Proof of Lemma 2.6. Let us fix any $\breve{\zeta} \in L^2(I; L^2)$, satisfying $|D\breve{\zeta}(\,\cdot\,)|(\Omega) \in L^1(I)$. Then, by Lusin's Theorem [10, Theorem 2 in Section 1.2], we find a class $\{J_\kappa \,|\, \kappa > 0\} \subset 2^I$ of compact sets $J_\kappa \subset I$, such that:

$$\mathscr{L}^1(I \setminus J_\kappa) \leq \kappa, \ \breve{\zeta} \in C(J_\kappa; L^2), \ |D\breve{\zeta}(\,\cdot\,)|(\Omega) \in C(J_\kappa) \\ \text{and } \Lambda_\kappa := \sup_{t\in J_\kappa} |\breve{\zeta}(t)|_{L^\infty} < \infty, \ \text{for any } \kappa > 0.$$

On that basis, let us define a sequence $\{\breve{\xi}_\kappa \,|\, \kappa > 0\} \subset L^\infty(\mathbb{R}^2)$ of bounded functions, by putting:

$$\breve{\xi}_\kappa(t) := \begin{cases} \breve{\zeta}(t)^{\mathrm{ex}} \text{ in } L^\infty(\mathbb{R}), & \text{if } t \in J_\kappa, \\ 0 \text{ in } L^\infty(\mathbb{R}), & \text{otherwise,} \end{cases} \quad \text{for a.e. } t \in \mathbb{R};$$

where $\breve{\zeta}(t)^{\rm ex} \in L^\infty(\mathbb{R})$ is the natural extension of $\breve{\zeta}(t) \in BV$, prescribed in (1.2), for a.e. $t \in I$. Additionally, for any $\kappa > 0$, let us define a sequence $\{\psi_\varepsilon^{(\kappa)} \,|\, \varepsilon > 0\} \subset C^\infty(\mathbb{R}^2)$, by putting:

$$\psi_\varepsilon^{(\kappa)}(t,x) := \int_{\mathbb{R}} \int_{\mathbb{R}} \varrho_\varepsilon(t-\tau)\varrho_\varepsilon(x-y)\, \breve{\xi}_\kappa(\tau,y)\, dyd\tau, \quad \text{for all } (t,x) \in \mathbb{R}^2;$$

where for any $\varepsilon > 0$, ϱ_ε is the usual (one-dimensional) mollifier.

Hereafter, we prove this lemma by applying some diagonal argument to the class $\{\psi_\varepsilon^{(\kappa)} \,|\, \kappa > 0, \varepsilon > 0\} \subset C^\infty(\mathbb{R}^2)$.

First, for the sequence $\{\breve{\xi}_\kappa \,|\, \kappa > 0\}$, it is immediately seen that:

$$\begin{cases} \breve{\xi}_\kappa \to \breve{\zeta} \text{ in } L^2(I;L^2), \\ \breve{\xi}_\kappa(t) \to \breve{\zeta}(t) \text{ in } L^2, \text{ for a.e. } t \in I, \end{cases} \quad \text{as } \kappa \searrow 0. \tag{4.6}$$

Besides, since:

$$\int_\Omega |D\breve{\zeta}(t)| \le \liminf_{\kappa \searrow 0} \int_{\mathbb{R}} |D\breve{\xi}_\kappa(t)| \ \text{ and } \ \sup_{\kappa>0} \int_{\mathbb{R}} |D\breve{\xi}_\kappa(t)| \le \int_{\mathbb{R}} |D\breve{\zeta}(t)^{\rm ex}| = \int_\Omega |D\breve{\zeta}(t)|,$$
$$\text{for a.e. } t \in I;$$

we can apply Lebesgue's dominated convergence theorem, to deduce that:

$$\int_I \left| \int_{\mathbb{R}} |D\breve{\xi}_\kappa(t)| - \int_\Omega |D\breve{\zeta}(t)| \right| dt \to 0 \ \text{ as } \kappa \searrow 0. \tag{4.7}$$

Secondly, for any $\kappa > 0$, the sequence $\{\psi_\varepsilon^{(\kappa)} \,|\, \varepsilon > 0\}$ can be supposed to fulfill that:

$$|\psi_\varepsilon^{(\kappa)}(t,x)| \le \Lambda_\kappa \ \text{ and } \ \psi_\varepsilon^{(\kappa)}(t,x) \to \breve{\xi}_\kappa(t,x) \ \text{ in pointwise sense, } \ \text{as } \varepsilon \searrow 0,$$
$$\text{for } \mathscr{L}^2\text{-a.e. } (t,x) \in \mathbb{R}^2.$$

So, due to Lebesgue's dominated convergence theorem and the lower semi-continuity of the total variation,

$$\psi_\varepsilon^{(\kappa)} \to \breve{\xi}_\kappa \text{ in } L^2(I;L^2) \text{ as } \varepsilon \searrow 0, \text{ for any } \kappa > 0, \tag{4.8}$$

and

$$\begin{cases} \psi_\varepsilon^{(\kappa)}(t) \to \breve{\xi}_\kappa(t) \text{ in } L^2 \text{ as } \varepsilon \searrow 0, \\ \displaystyle \liminf_{\varepsilon \searrow 0} \int_\Omega |(\psi_\varepsilon^{(\kappa)})_x(t)|\, dx \ge \int_\Omega |D\breve{\xi}_\kappa(t)|, \end{cases} \quad \text{for a.e. } t \in I \text{ and any } \kappa > 0. \tag{4.9}$$

Here, for any $\kappa > 0$, any $\varepsilon > 0$, any $t \in I$ and any $\varphi \in C_{\rm c}^1$ with $|\varphi|_{C(\overline{\Omega})} \le 1$, Fubini's theorem enables us to calculate that:

$$\begin{aligned} \int_\Omega \psi_\varepsilon^{(\kappa)}(t,x)\varphi_x(x)\, dx &= \int_{\mathbb{R}} \varphi_x(x) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \varrho_\varepsilon(t-\tau)\varrho_\varepsilon(x-y)\breve{\xi}_\kappa(\tau,y)\, dyd\tau \right) dx \\ &= \int_{\mathbb{R}} \varrho_\varepsilon(t-\tau) \int_{\mathbb{R}} \breve{\xi}_\kappa(\tau,y) \left(\int_{\mathbb{R}} \varrho_\varepsilon(y-x)\varphi_x(x)\, dx \right) dy\, d\tau \\ &= \int_{\mathbb{R}} \varrho_\varepsilon(t-\tau) \int_{\mathbb{R}} \breve{\xi}_\kappa(\tau,y)(\varrho_\varepsilon * \varphi)_x(y)\, dy\, d\tau. \end{aligned} \tag{4.10}$$

Therefore, noting that:

$$|\varrho_\varepsilon * \varphi|_{C(\overline{\Omega})} \leq 1, \quad \text{for any } \varphi \in C_{\rm c}^1 \text{ with } |\varphi|_{C(\overline{\Omega})} \leq 1; \tag{4.11}$$

we have:

$$\begin{aligned}\int_\Omega |(\psi_\varepsilon^{(\kappa)})_x(t)|\,dx &= \sup\left\{ \int_\Omega \psi_\varepsilon^{(\kappa)}(t,x)\tilde{\varphi}_x(x)\,dx \;\middle|\; \tilde{\varphi} \in C_{\rm c}^1 \text{ with } |\tilde{\varphi}|_{C(\overline{\Omega})} \leq 1 \right\} \\ &\leq \left(\sup_{\varsigma \in \mathbb{R}} \int_{\mathbb{R}} |D\breve{\xi}_\kappa(\varsigma)| \right) \left(\int_{\mathbb{R}} \varrho_\varepsilon(\tau - t)\,d\tau \right) \leq \Bigl| |D\breve{\zeta}(\,\cdot\,)|(\Omega) \Bigr|_{C(J_\kappa)}, \\ &\text{for any } \varepsilon > 0, \text{ any } t \in I \text{ and any } \kappa > 0.\end{aligned} \tag{4.12}$$

On the other hand, it follows from (4.10)-(4.11) and Lemma 2.4 that:

$$\begin{aligned}\limsup_{\varepsilon \searrow 0} \int_\Omega |(\psi_\varepsilon^{(\kappa)})_x(t)|\,dx \leq \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \varrho_\varepsilon(t-\tau) \int_{\mathbb{R}} |D\breve{\xi}_\kappa(\tau)|\,d\tau = \int_{\mathbb{R}} |D\breve{\xi}_\kappa(t)|, \\ \text{for a.e. } t \in I \text{ (Lebesgue point of } |D\xi(\,\cdot\,)|(\mathbb{R}) \in L^1(I)) \text{ and any } \kappa > 0.\end{aligned} \tag{4.13}$$

On account of (4.9) and (4.12)-(4.13), we can apply Lebesgue's dominated convergence theorem, to obtain that:

$$\int_I \left| \int_\Omega |(\psi_\varepsilon^{(\kappa)})_x(t)|\,dx - \int_{\mathbb{R}} |D\breve{\xi}_\kappa(t)| \right| dt \to 0 \text{ as } \varepsilon \searrow 0, \text{ for any (fixed) } \kappa > 0. \tag{4.14}$$

In the light of (4.6)-(4.7), we find a decreasing sequence $\{\kappa_m \,|\, m = 1,2,3,\cdots\} \subset (0,1)$, such that:

$$\begin{cases} 0 < \kappa_m < \dfrac{1}{2^m}, \quad |\breve{\xi}_{\kappa_m} - \breve{\zeta}|_{L^2(I;L^2)} \leq \dfrac{1}{2^m}, \\ \Bigl| |D\breve{\xi}_{\kappa_m}(\,\cdot\,)|(\Omega) - |D\breve{\zeta}(\,\cdot\,)|(\Omega) \Bigr|_{L^1(I)} \leq \dfrac{1}{2^m}, \end{cases} \quad \text{for } m = 1,2,3,\cdots.$$

Subsequently, by using (4.8) and (4.14), we further find a decreasing sequence $\{\varepsilon_m \,|\, m = 1,2,3,\cdots\} \subset (0,1)$, such that:

$$\begin{cases} 0 < \varepsilon_m < \dfrac{1}{2^m}, \quad |\psi_{\varepsilon_m}^{(\kappa_m)} - \breve{\xi}_{\kappa_m}|_{L^2(I;L^2)} \leq \dfrac{1}{2^m}, \\ \Bigl| |D\psi_{\varepsilon_m}^{(\kappa_m)}(\,\cdot\,)|(\Omega) - |D\breve{\xi}_{\kappa_m}(\,\cdot\,)|(\Omega) \Bigr|_{L^1(I)} \leq \dfrac{1}{2^m}, \end{cases} \quad \text{for } m = 1,2,3,\cdots.$$

Now, the sequence $\{\breve{\psi}_i \,|\, i = 1,2,3,\cdots\} \subset C^\infty(\mathbb{R}^2)$, required here, will be obtained as a subsequence of $\{\psi_{\varepsilon_m}^{(\kappa_m)} \,|\, m = 1,2,3,\cdots\} \subset C^\infty(\mathbb{R}^2)$, such that:

$$\begin{cases} \breve{\psi}_i(t) \to \breve{\zeta}(t) \text{ in } L^2 \\ |D\breve{\psi}_i(t)|(\Omega) \to |D\breve{\zeta}(t)|(\Omega), \end{cases} \quad \text{as } i \to \infty, \text{ for a.e. } t \in I.$$

■

Proof of Lemma 2.7. Let $\nu_0 \in (0,1)$, $\{\nu_n\} \subset (0,\nu_0)$, $\{\eta_n\} \subset W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1)$ and $\eta \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1)$ be the same as in (2.10)-(2.15). Also, let $I \subset (0,T)$ be any open interval, and let $\hat{\Phi}(\,\cdot\,)_I$ and $\hat{\Phi}_n(\,\cdot\,)_I$, $n = 1, 2, 3, \cdots$, be the functionals, defined in (2.19) and (2.20), respectively.

As it is already mentioned in Remark 2.1, $\hat{\Phi}(\,\cdot\,)_I$ is a proper functional on $L^2(I; L^2)$. Also, for any $\zeta \in D(\hat{\Phi}(\,\cdot\,)_I)$, we easily check from (A4) that $\zeta(t) \in BV$ for a.e. $t \in I$, and we infer from (I) of Lemma 2.4 that the function:

$$t \in I \mapsto \Phi_\alpha(\eta(t); \zeta(t)) \in \mathbb{R};$$

is measurable on I. Hence, "the equality (2.28)", and "the lower semi-continuity and convexity of $\hat{\Phi}(\,\cdot\,)_I$" turn out to be direct consequences of "(A3)-(A4) and (2.14)", and "Fatou's lemma and the triangle inequality", respectively.

As well as, the condition (γ1) is immediately verified by taking a subsequence $\{\check{\zeta}_{n_i} \,|\, i = 1, 2, 3, \cdots\} \subset \{\check{\zeta}_n \,|\, n = 1, 2, 3, \cdots\}$, such that:

$$\check{\zeta}_{n_i}(t) \to \check{\zeta}_\infty(t) \text{ in } L^2 \text{ for a.e. } t \in I, \text{ and } \lim_{i \to \infty} \hat{\Phi}_{n_i}(\check{\zeta}_{n_i}) = \liminf_{n \to \infty} \hat{\Phi}_n(\check{\zeta}_n);$$

and applying (m1) of Lemma 2.3 and Fatou's lemma.

For the verification of the condition (γ2), let us first apply Lemma 2.6, to prepare an approximation sequence $\{\hat{\psi}_i\} \subset C^\infty(\mathbb{R}^2)$ of the function $\hat{\zeta}_\infty \in D(\hat{\Phi}(\,\cdot\,)_I)$, in the same sense as in (2.26)-(2.27), and secondly, let us take an increasing sequence $\{\hat{n}_i \,|\, i = 1, 2, 3, \cdots\} \subset \mathbb{N}$, such that:

$$\hat{n}_{i+i} > \hat{n}_i \geq i \text{ and } \frac{\nu}{2} \int_I \int_\Omega |(\hat{\psi}_i)_x|^2 \, dx dt \leq \frac{1}{i} \text{ for any } 0 < \nu < \nu_{\hat{n}_i}, \; i = 1, 2, 3, \cdots.$$

On that basis, the required sequence $\{\hat{\zeta}_n \,|\, n = 1, 2, 3, \cdots\}$ will be obtained as follows:

$$\hat{\zeta}_n := \begin{cases} \hat{\psi}_i \text{ in } L^2(I; L^2), & \text{if } \hat{n}_i < n \leq \hat{n}_{i+1} \text{ for some } i \in \mathbb{N}, \\ \hat{\psi}_1 \text{ in } L^2(I; L^2), & \text{if } 1 \leq n \leq \hat{n}_1. \end{cases}$$

In fact, from the constitution method of $\{\hat{\zeta}_n\}$, we immediately see that:

$$\begin{cases} \hat{\zeta}_n \to \hat{\zeta}_\infty \text{ in } L^2(I; L^2), \\ \displaystyle\int_I \int_\Omega |(\hat{\zeta}_n)_x(t)| \, dx dt \to \int_I \int_\Omega |D\hat{\zeta}_\infty(t)| \, dt, \\ \displaystyle\frac{\nu_n}{2} \int_I \int_\Omega |(\hat{\zeta}_n)_x(t)|^2 \, dx dt \to 0, \end{cases} \text{ as } n \to \infty, \tag{4.15}$$

and

$$D\hat{\zeta}_n(t) \to D\hat{\zeta}_\infty(t) \text{ weakly-}* \text{ in } \mathcal{M}, \text{ as } n \to \infty, \text{ for a.e. } t \in I.$$

So, applying (IV) of Lemma 2.5 to the case when:

$$\begin{cases} \psi_\infty = \alpha(\eta) \text{ in } C(\overline{I \times \Omega}), \; \psi_n = \alpha(\eta_n) \text{ in } C(\overline{I \times \Omega}), \; n = 1, 2, 3, \cdots, \\ \gamma_\infty = \gamma_n \equiv 1 \text{ on } \overline{I \times \Omega}, \; n = 1, 2, 3, \cdots, \\ \xi_\infty = \hat{\zeta}_\infty \text{ in } L^2(I; L^2), \; \xi_n = \hat{\zeta}_n \text{ in } L^2(I; L^2), \; n = 1, 2, 3, \cdots; \end{cases}$$

it is also seen that:

$$\int_I \int_\Omega \alpha(\eta_n(t))|(\hat{\zeta}_n)_x(t)|\,dxdt \to \int_I \int_\Omega \alpha(\eta(t))|D\hat{\zeta}_\infty(t)|\,dt \quad \text{as } n \to \infty. \tag{4.16}$$

Now, the convergences, asserted in ($\gamma 2$), will follow from (4.15)-(4.16). ■

Proof of Lemma 2.8. Let $\{\hat{\theta}_n \,|\, n = 1, 2, 3, \cdots\} \subset L^2(I; H^1)$ be a sequence, such that:

$$\hat{\theta}_n \to \theta \text{ in } L^2(I; L^2) \ \text{ and } \ \hat{\Phi}_n(\hat{\theta}_n)_I \to \hat{\Phi}(\theta)_I, \ \text{ as } n \to \infty.$$

Such $\{\hat{\theta}_n\}$ can be taken by applying ($\gamma 2$) of Lemma 2.7 to the case of $\hat{\zeta}_\infty = \theta \in D(\hat{\Phi}(\,\cdot\,)_I)$.

On the other hand, since the pair $[\eta_n, \theta_n]$, for each $n \in \mathbb{N}$, fulfills the Cauchy problem (2.4) in case of $\nu = \nu_n$, it is deduced that:

$$\int_I (\alpha_0(\eta_n(t))(\theta_n)_t(t), \theta_n(t) - \hat{\theta}_n(t))_{L^2}\,dt + \hat{\Phi}_n(\theta_n) \le \hat{\Phi}_n(\hat{\theta}_n), \quad n = 1, 2, 3, \cdots.$$

So, letting $n \to \infty$, with the use of (2.16) and ($\gamma 1$) of Lemma 2.7, yield that:

$$\hat{\Phi}(\theta)_I \le \liminf_{n\to\infty} \hat{\Phi}_n(\theta_n)_I \le \limsup_{n\to\infty} \hat{\Phi}_n(\theta_n)_I \le \lim_{n\to\infty} \hat{\Phi}_n(\hat{\theta}_n)_I = \hat{\Phi}(\theta)_I. \tag{4.17}$$

Additionally, by (2.12), (2.14), (2.15), (2.18) and (4.17), we can apply (III) of Lemma 2.5 to the case when:

$$\begin{cases} \psi_\infty = \alpha(\eta) \text{ in } C(\overline{I \times \Omega}), \ \psi_n = \alpha(\eta_n) \text{ in } C(\overline{I \times \Omega}), \ n = 1, 2, 3, \cdots, \\ \xi_\infty = \theta \text{ in } L^2(I; L^2), \ \xi_n = \theta_n \text{ in } L^2(I; L^2), \ n = 1, 2, 3, \cdots; \end{cases}$$

to calculated that:

$$\begin{aligned} &\limsup_{n\to\infty} \left(\frac{\nu_n}{2} \int_I \int_\Omega |(\theta_n)_x(t)|^2\,dxdt \right) \\ \le\ & \lim_{n\to\infty} \hat{\Phi}_n(\theta_n)_I - \liminf_{n\to\infty} \int_I \int_\Omega \alpha(\eta_n(t))|(\theta_n)_x(t)|\,dxdt \\ \le\ & \hat{\Phi}(\theta)_I - \hat{\Phi}(\theta)_I = 0. \end{aligned} \tag{4.18}$$

Combining (4.17) and (4.18),

$$\int_I \int_\Omega \alpha(\eta_n(t))|(\theta_n)_x(t)|\,dxdt \to \int_I \int_\Omega \alpha(\eta(t))|D\theta(t)|\,dt, \quad \text{as } n \to \infty. \tag{4.19}$$

On account of (2.12), (2.14), (2.15), (2.18) and (4.19), the assertion (2.33) of this lemma will be obtained by applying (IV) of Lemma 2.5 to the case when:

$$\begin{cases} \gamma_\infty = \alpha(\eta) \text{ in } C(\overline{I \times \Omega}), \ \gamma_n = \alpha(\eta_n) \text{ in } C(\overline{I \times \Omega}), \ n = 1, 2, 3, \cdots, \\ \xi_\infty = \theta \text{ in } L^2(I; L^2), \ \xi_n = \theta_n \text{ in } L^2(I; L^2), \ n = 1, 2, 3, \cdots. \end{cases}$$

■

Appendix

This Appendix is devoted to the summary of some specific topics, that are often used throughout this paper. All topics are related to the notion of subdifferential of convex function, on a Hilbert space.

Ap.1 Notion of subdifferential

Let H be an abstract Hilbert space, and let $\Phi : D(\Phi) \subset H \longrightarrow (-\infty, \infty]$ be a proper l.s.c. and convex function on H, with the effective domain $D(\Phi)$. Here, let us overview some elemental mattes of the subdifferential of Φ, denoted by $\partial\Phi$.

Definition Ap.1 (Definition of subdifferential) The subdifferential $\partial\Phi$ of the convex function Φ is defined as a multi-valued operator $\partial\Phi : H \longrightarrow 2^H$, such that any $v \in H$ is assigned to a set $\partial\Phi(v)$ of all elements $w \in H$, satisfying the following variational inequality:

$$(w, z - v)_H + \Phi(v) \leq \Phi(z), \quad \text{for any } z \in D(\Phi). \tag{ap.1}$$

Then, the domain of $\partial\Phi$ is denoted by $D(\partial\Phi)$, with the definition:

$$D(\partial\Phi) := \left\{ \ \tilde{v} \in H \ \middle| \ \partial\Phi(\tilde{v}) \neq \emptyset \ \right\};$$

and for any $v \in D(\partial\Phi)$, each $w \in \partial\Phi(v)$ is called a subgradient of Φ at v.

Remark Ap.1 In general, multi-valued mappings are often identified with its graph. Hence, in the case of the subdifferential $\partial\Phi$ in Definition Ap.1, the phrase "$v \in D(\partial\Phi)$ and $w \in \partial\Phi(v)$ in H" is often described as "$[v, w] \in \partial\Phi$ in $H \times H$".

Remark Ap.2 One of advantages of using subdifferential is in the point that this notion can respond to nonsmooth situations of convex functions. Indeed, in Definition Ap.1, the multi-valued situations of $\partial\Phi$ will appear at the nonsmooth points of the graph of Φ. Additionally, the variational inequality (ap.1) implies that the hyper-plane of the slope w supports the epigraph of Φ, and also it is tangential at the point $[v, \Phi(v)] \in H \times \mathbb{R}$. Hence, when the convex function Φ is sufficiently smooth, the subdifferential $\partial\Phi$ turns out to be a single-valued mapping, that can be identified with some usual differential, such as Gâteaux differential and Fréchet differential, e.t.c..

Ap.2 Subdifferential of weighted total variation

Let us fix any $\rho \in C(\overline{\Omega})$, such that $\rho \geq \delta_\rho$ on $\overline{\Omega}$ for some constant $\delta_\rho > 0$, and let us define a functional $V_\rho : L^2 \longrightarrow [0, \infty]$ on the Hilbert space L^2, by putting:

$$v \in L^2 \mapsto V_\rho(v) := \begin{cases} \displaystyle\int_\Omega \rho |Dv|, & \text{if } v \in BV, \\ \infty, & \text{otherwise.} \end{cases}$$

In this paper, the functionals, kindred to the above V_ρ, are collectively called weighted total variation functional, or simply weighted total variation.

The focus in this section is on the representation of the subdifferential ∂V_ρ of the weighted total variation V_ρ, in the topology of L^2. Nowadays, this topic has been studied by a number of mathematicians, from various viewpoints (e.g. [2, 3, 4, 12, 13, 19, 22, 25, 26]). In such previous studies, we here quote [4, Lemma 6.4], in the form of the following lemma.

Lemma Ap.1 *(Representation lemma for ∂V_ρ) For a pair $[v, w] \in L^2 \times L^2$ of functions:*

$$[v, w] \in \partial V_\rho \quad \text{in } L^2 \times L^2;$$

if and only if there exists a function $\varpi^* \in H^1$, such that:

(a) $|\varpi^*(x)| \leq \rho(x)$ *for any* $x \in \overline{\Omega}$;

(b) $w = -(\varpi^*)_x$ *in* $\mathcal{D}'(\Omega)$ *(in the distribution sense);*

(c) $\displaystyle\int_\Omega w(z - v)\, dx \leq \int_\Omega \varpi^* Dz - \int_\Omega \rho |Dv|,$ *for any* $z \in BV$.

Remark Ap.3 In [4, Lemma 6.4], the term $\int_\Omega \varpi^* Dz$, in (c), is described by an integral form $\int_\Omega (\varpi^*, Dz)$ with respect to the pairing measure (ϖ^*, Dz), proposed by Anzellotti [5]. However, since we here have $\varpi^* \in H^1 \subset C(\overline{\Omega})$, we infer from [5, Proposition 2.3] that $(\varpi^*, Dz) = \varpi^* Dz$ in $\mathcal{M}$. On that basis, let us put:

$$\omega^*(x) := \frac{1}{\rho(x)}\, \varpi^*(x), \quad \text{for any } x \in \overline{\Omega}.$$

Here, if $\rho \in H^1$, then it is immediately checked that:

$$\omega^* \in H^1 \subset C(\overline{\Omega}), \ |\omega^*| \leq 1 \text{ on } \overline{\Omega}, \ \text{ and } \ w = -(\rho\,\omega^*)_x \text{ in } \mathcal{D}'(\Omega). \tag{ap.2}$$

Also, setting $z = v$ in (c) of Lemma Ap.1, it is deduced that:

$$\int_\Omega \rho |Dv| \leq \int_\Omega \rho\, \omega^* Dv, \text{ and hence } \int_\Omega \rho \left(1 - \omega^* \cdot \tfrac{Dv}{|Dv|}\right) |Dv| \leq 0,$$

$$\text{while } \ \rho > 0 \text{ on } \overline{\Omega}, \ \text{ and } \ 1 - \omega^* \cdot \tfrac{Dv}{|Dv|} \geq 0, \ |Dv|\text{-a.e. in } \Omega;$$

where $\frac{Dv}{|Dv|}$ is the Radon-Nikodým density of the variation measure Dv for its total variation $|Dv|$. It implies that:

$$\omega^*(x) = \tfrac{Dv}{|Dv|}, \quad |Dv|\text{-a.e. in } \Omega. \tag{ap.3}$$

On account of (ap.2)-(ap.3), we can roughly summarize that the function $w = -(\rho\,\omega^*)_x$ somehow give a certain expression of the singular diffusion $-(\rho\frac{Dv}{|Dv|})_x$ for $v \in BV$.

Ap.3 Evolution equations governed by time-dependent subdifferentials

Let H be an abstract Hilbert space. Let $u_0 \in H$ be a given element, let $f : (0, T) \longrightarrow H$ be a given measurable function, and let $\{\Phi^t \,|\, 0 \leq t \leq T\}$ be a class of time-dependent convex functions $\Phi^t : D(\Phi^t) \subset H \longrightarrow (-\infty, \infty]$, $0 \leq t \leq T$, on the Hilbert space H.

In this section, we denote by (E; u_0, f) the following Cauchy problem of evolution equation:

$$(\mathrm{E}; u_0, f) \qquad \begin{cases} u_t(t) + \partial\Phi^t(u(t)) \ni f(t) \text{ in } H, \ \ 0 < t \leq T, \\ u(0) = u_0 \text{ in } H; \end{cases}$$

governed by the time-dependent subdifferentials $\partial\Phi^t$, $0 \leq t \leq T$.

With regard to this problem, we can quote the general theory of Kenmochi [17], as one of representative studies for its mathematical treatment.

According to [17, Chapter 1], the solution (strong solution) of the problem (E; u_0, f) is prescribed as follows.

Definition Ap.2 A function $u : [0,T] \longrightarrow H$ is called a solution of $(E; u_0, f)$, if and only if $u \in W^{1,2}(0,T;H)$ with $u(0) = u_0$ in H, and:

$$(f - u_t)(t) \in \partial\Phi^t(u(t)) \text{ in } H, \text{ for a.e. } t \in (0,T);$$

equivalently:

$$((u_t - f)(t), u(t) - z)_H + \Phi^t(u(t)) \leq \Phi^t(z), \text{ for any } z \in D(\Phi^t) \text{ and a.e. } t \in (0,T).$$

Now, the following lemma is the summary of the solvability part of the general theory, which is obtained as a consequence of [17, Theorems 1.1.1, 1.1.2 and 1.5.1].

Lemma Ap.2 *(Solvability of the Cauchy problem* $(\mathrm{E}; u_0, f)$*) Let us assume that* $u_0 \in D(\Phi^0)$ *and* $f \in L^2(0,T;H)$*. Also, let us assume that the class* $\{\Phi^t \,|\, 0 \leq t \leq T\}$ *of time-dependent convex functions fulfills the following condition.*

(Φ) There exist two functions $a \in W^{1,2}(0,T)$ *and* $b \in W^{1,1}(0,T)$*, and for arbitrary* $s, t \in [0,T]$ *and arbitrary* $z \in D(\Phi^s)$*, there exists* $\tilde{z} \in D(\Phi^t)$*, such that:*

$$\begin{cases} |\tilde{z} - z|_H \leq |a(t) - a(s)|(1 + |\Phi^s(z)|^{1/2}), \\ \Phi^t(\tilde{z}) - \Phi^s(z) \leq |b(t) - b(s)|(1 + |\Phi^s(z)|). \end{cases}$$

Then, the Cauchy problem $(\mathrm{E}; u_0, f)$ *admits a unique solution* u*.*

Remark Ap.4 Let $\{\Psi^t \,|\, 0 \leq t \leq T\}$ be a class of the time-dependent convex functions Ψ^t, $0 \leq t \leq T$, given in (3.1). Then, we easily check the compatibility of $\{\Psi^t\}$ with the condition (Φ) in Lemma Ap.2. In fact, since:

$$D(\Psi^t) = H^1, \text{ for all } 0 \leq t \leq T;$$

namely, since the domain $D(\Psi^t)$ of each Ψ^t is actually independent of time, we can take just the same element $z \in D(\Psi^s)$ as the function $\tilde{z} \in D(\Psi^t)$, required in (Φ). From this viewpoint, the inequality (3.3) implies that the setting:

$$a(t) := 0 \text{ and } b(t) := A_1 \left(1 + \frac{1}{\nu}\right) \int_0^t (1 + |\bar{\eta}_t(\tau)|_{L^2})\, d\tau, \text{ for } 0 \leq t \leq T;$$

provides a certain possible choice of the functions a and b, as in (Φ).

Remark Ap.5 In rigorous, the general theory, summarized in Lemma Ap.2 cannot guarantee the solvability of the Cauchy problem (3.2), because this problem includes a time-dependent perturbation:

$$(t, v) \in [0,T] \times L^2 \mapsto -\left(\frac{(\bar{\alpha}_0)_t(t,\cdot)}{2\bar{\alpha}_0(t,\cdot)} + A_0\right) v \in L^2.$$

However, we now find several papers, e.g. [24, 27], which report general theories, to handle such situation. In particular, the theory, obtained in [27], is one of enhanced studies, originating from [17].

Finally, we mention a key-lemma, which is related to the measurability and absolute continuity of the composite function $t \in (0,T) \mapsto \Phi^t(\zeta(t))$, for $\zeta \in L^2(0,T;H)$.

Lemma Ap.3 *(Measurability and absolute continuity of composite functions) Let $I \subset (0,T)$ be any open interval. Then, under the condition (Φ) as in Lemma Ap.2, the following two items hold.*

(A) For any $\zeta \in L^2(I;H)$, the composite function $t \in I \mapsto \Phi^t(\zeta(t)) \in (-\infty,\infty]$ is measurable on I.

(B) If $\zeta \in W^{1,2}(I;H)$, if $\zeta^* \in L^2(I;H)$, and if:

$$\zeta^*(t) \in \partial\Phi^t(\zeta(t)) \text{ in } H, \text{ for a.e. } t \in I;$$

then the composite function $t \in I \mapsto \Phi^t(\zeta(t)) \in \mathbb{R}$ turns out to be absolutely continuous on $\overline{I}$.

Remark Ap.6 Items (A) and (B) in Lemma Ap.3 are obtained by taking into account of [17, Lemma 1.2.2] and [17, Theorem 1.5.1 and Corollaries of Lemmas 1.2.5 and 2.1.1], respectively.

On that basis, let us take any $\bar{\eta} \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \subset C(\overline{Q_T})$, satisfying $0 \leq \bar{\eta} \leq 1$ on $\overline{Q_T}$, and let us consider the class $\{\Phi_{\alpha,\nu}(\bar{\eta}(t);\,\cdot\,)\,|\,0 \leq t \leq T\}$ of time-dependent convex functions, that appears in the auxiliary Cauchy problem (P2; $\bar{\alpha}_0, \bar{\eta})_\nu$, with a given constant $\nu > 0$ and a given function $\bar{\alpha}_0 \in L^\infty(Q_T)$. Then, by using Hölder's inequality:

$$\begin{aligned}
&|\Phi_{\alpha,\nu}(\bar{\eta}(t);z) - \Phi_{\alpha,\nu}(\bar{\eta}(s);z)| \\
&\leq \int_\Omega |\alpha(\bar{\eta}(t)) - \alpha(\bar{\eta}(s))||z_x|\,dx \leq \alpha'(1)|\bar{\eta}(t) - \bar{\eta}(s)|_{L^2}|z_x|_{L^2} \\
&\leq \alpha'(1)\left(1 + \frac{1}{2\nu}\right)\left|\int_s^t |\bar{\eta}_t(\tau)|_{L^2}\,d\tau\right|(1 + \Phi_{\alpha,\nu}(\bar{\eta}(s);z)); \\
&\text{for all } s,t \in [0,T] \text{ and all } z \in D(\Phi_{\alpha,\nu}(\bar{\eta}(s);\,\cdot\,)) = D(\Phi_{\alpha,\nu}(\bar{\eta}(t);\,\cdot\,)) = H^1.
\end{aligned} \tag{ap.4}$$

So, just as in Remark Ap.4, we verify that the class $\{\Phi_{\alpha,\nu}(\bar{\eta}(t);\,\cdot\,)\,|\,0 \leq t \leq T\}$ is compatible with the condition (Φ) in Lemma Ap.2.

To conclude, if $\nu > 0$, then we immediately have the measurability of the function:

$$t \in I \mapsto \Phi_{\alpha,\nu}(\bar{\eta}(t);\zeta(t)) \in [0,\infty], \quad \text{for any } \zeta \in L^2(I;L^2);$$

and hence we also have the lower semi-continuity and convexity of the functionals $\hat{\Phi}_n(\,\cdot\,)_I$, for $n = 1,2,3,\cdots$, defined in (2.20). Furthermore, if $\nu > 0$, and if we somehow find the solution θ_ν of the auxiliary Cauchy problem (P2; $\bar{\alpha}_0, \bar{\eta})_\nu$, then we can immediately conclude the absolute continuity of the function $t \in [0,T] \mapsto \Phi_{\alpha,\nu}(\bar{\eta}(t);\theta_\nu(t))$.

But, we cannot apply similar arguments for the class $\{\Phi_\alpha(\bar{\eta}(t);\,\cdot\,)\,|\,0 \leq t \leq T\}$ of convex functions, with $\bar{\eta} \in C(\overline{Q_T})$, that is associated with our second Cauchy problem (1.4), because the use of Hölder's inequality, as in (ap.4), is available only when $\nu > 0$.

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