Digital-Finite Difference Method and its Application to 2D Simulations of Flow around Three Circular Cylinders

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Abstract. In this article we propose a numerical method using pixels in a digital image and show a numerical simulation for an incompressible viscous flow around three circular cylinders. The characteristic we present here combines the digital image, the fictitious domain and the finite difference method. Since the digital image consists of so many pixels and each of pixels forms a small square, the finite difference method under the regular mesh is applied to Navier-Stokes equations in complex shaped domains. Hence the numerical algorithm is easily constructed and it will be widely applied to various problems in the environmental mathematics.

Key–Words: Finite difference method, Digital image, Pixel, Fictitious domain method, Navier-Stokes equations

1 Introduction

Supposed that $\Omega = (0,2) \times (0,1) \subset R^2$ containing three circular cylinders Ω_2 and that $\Omega_1 = \Omega \setminus \overline{\Omega_2}$, as shown in Figure 1, below; we denote by $\Gamma_i (i = 1,2,3,4)$ and γ the boundaries of Ω and Ω_2, respectively. Here Γ_1 is the primary inflow boundary, Γ_2 is the outflow boundary, and $\Gamma_3 \cup \Gamma_4$ is the wall.

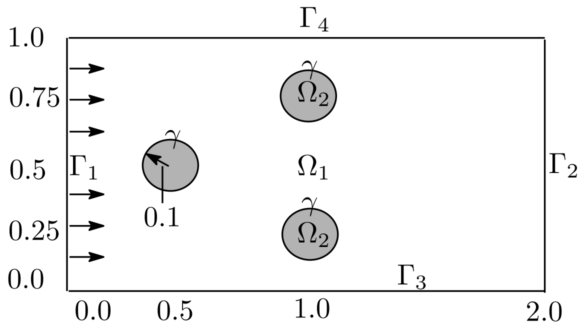

Figure 1 The domain of analysis.

The geometrical situation being like in Figure 1, we consider for T ≥ 0 the solution of the following system of Navier-Stokes equations in $Q_T \equiv (0,T) \times \Omega_1$;

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \frac{1}{Re}\Delta \boldsymbol{v}, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.2}$$

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boundary conditions on γ ;

$$\frac{\partial p}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad (0,T) \times \gamma, \tag{1.3}$$

$$\boldsymbol{v} = 0 \quad \text{on} \quad (0,T) \times \gamma, \tag{1.4}$$

boundary conditions on $\Gamma_i (i = 1, 2, 3, 4)$;

$$\boldsymbol{v} = (0.2, 0.0) \quad \text{on} \quad (0,T) \times \Gamma_1, \tag{1.5}$$

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad (0,T) \times \Gamma_2, \tag{1.6}$$

$$\boldsymbol{v} = 0 \quad \text{on} \quad (0,T) \times (\Gamma_3 \cup \Gamma_4), \tag{1.7}$$

$$p = 0 \quad \text{on} \quad (0,T) \times \Gamma_2, \tag{1.8}$$

$$\frac{\partial p}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad (0,T) \times (\partial\Omega_1 \setminus \Gamma_2), \tag{1.9}$$

and initial conditions:

$$\boldsymbol{v} = 0, \quad p = 0 \quad \text{in} \quad (0,T) \times \Omega_1 \tag{1.10}$$

where $\boldsymbol{v}$ is the velocity vector, p is the pressure and Re is the Reynolds number.
In order to construct numerical solution of (1)-(10), we first use a digital image and approximate the domain Ω_1 by pixels, each of which is the smallest unit of image and forms a small square. These points are the characteristics of our article. Hence The purpose of this article is that one is to present a computational method using the digital image and pixels, and that two is to show its application to numerical simulations of flow around three circular cylinders. The content of this article is as follows:
In Section 2, we shall introduce a concept of cell defined by pixels of digital image, and show digital approximations of Ω_1 and Ω_2. Fictitious domain formulation of the above problem will be given in Section 3. And finally we shall present numerical algorithm by use of cells and show numerical simulations.

2 Digital approximation of domain

The digital image is composed of pixels which are arranged in two dimensional grigs. Each pixel is addressable and has a intensity. In order to use this fact, we first convert the original profile (Fig.1) into a digital image, below Figure 2 which consists of 800 pixel width and 400 pixel height. From this digital image, we construct the digital approximation of domain $\{\Omega_1, \Omega_2\}$. We proceed in two steps.
Step 1: Cell value at each cell
First we introduce a concept of a cell, which forms a 10 pixels width and 10 pixels height. Each cell consists of 100 pixels. Figure 2 is a binary format image and the intensity 0 corresponds to black and 255 to white.

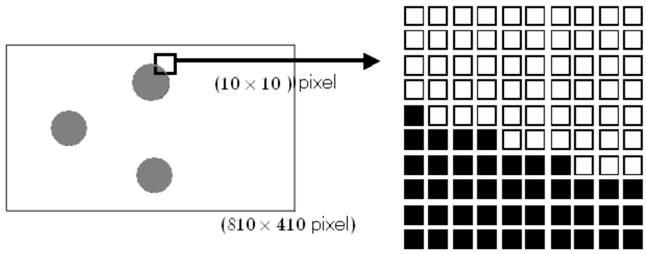

Figure 2　The image expanded partly

In our computation, we equate a regular grid point with a cell as follows:

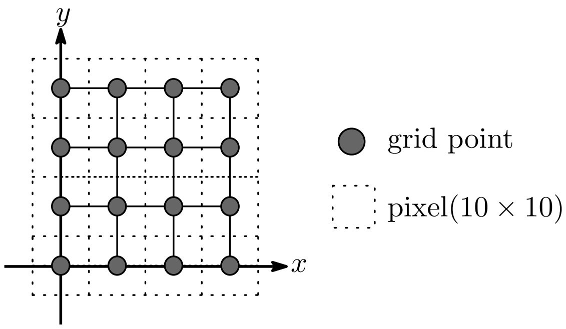

Figure 3　The relation between grid point and pixel.

The cell value y of each cell is defined by

$$y = \frac{x}{255 \times 100} \quad (0.0 \leq y \leq 1.0) \tag{2.1}$$

where x is a total of intensity for each pixel included in one cell.

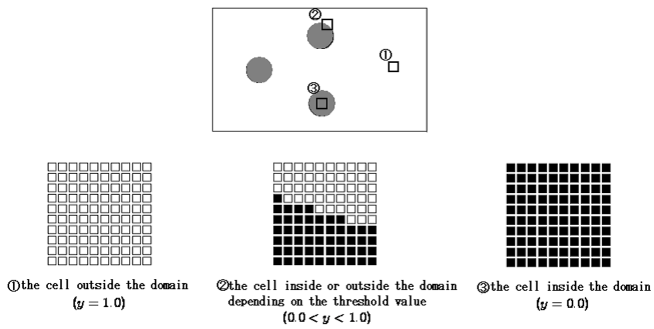

Figure 4　The decision on the domain by the formula (11).

Using cell values, we define the digital approximations $\widetilde{\Omega_1}$ and $\widetilde{\Omega_2}$ of Ω_1 and Ω_2, respectively ;

$$\widetilde{\Omega_1} = \{\text{cells} : \text{y} > \text{z}\}, \tag{2.2}$$

$$\widetilde{\Omega_2} = \{\text{cells} : \text{y} \leq \text{z}\} \tag{2.3}$$

where z is a given threshold value.

3　Numerical algorithm by use of digital image

In this section, we present a numerical algorithm coupled with the digital image for the mathematical model (1)-(10) .

3.1 Navier-Stokes equation, Poisson equation and fictitious domain formulation

Before proceeding to the numerical algorithm by use of digital image, we recall the fictitious domain method (also called domain embedding method) (cf. [2],[4],[6]).
It is well known that from Navier-Stokes equations in $(0,T)\times\Omega_1$:

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\nabla p+\frac{1}{Re}\Delta\boldsymbol{v},$$
$$\operatorname{div}\boldsymbol{v}=0,$$

the following Poisson equation is derived:

$$-\Delta p=\frac{\partial(\nabla\cdot\boldsymbol{v})}{\partial t}+\nabla\cdot\big((v\cdot\nabla)v\big)-\frac{1}{Re}\Delta(\nabla\cdot v).$$

Then one usually solves the avove equation with the following boundary conditions;

$$\begin{aligned}\frac{\partial p}{\partial \boldsymbol{n}}&=0 \quad \text{on} \quad (0,T)\times\gamma,\\ p&=0 \quad \text{on} \quad (0,T)\times\Gamma_2,\\ \frac{\partial p}{\partial \boldsymbol{n}}&=0 \quad \text{on} \quad (0,T)\times(\Gamma_1\cup\Gamma_3\cup\Gamma_4).\end{aligned} \tag{3.1}$$

However, there is a useful theory which approximates the Neuman boundary condition (14) of Poisson equation. That is called the fictitious domain method (cf. [1],[3],[5]). Hence we apply this method to the above Poisson equation; i.e., introducing the function $a(x,y)$;

$$a(x,y)=\begin{cases}1 & \text{on} \quad \Omega_1\\ \varepsilon & \text{on} \quad \Omega_2\end{cases}$$

we set up the fictitious domain formulation as follows:
find p_ε satisfying

$$-\nabla\cdot(a(x,y)\nabla p_\varepsilon)=f \quad \text{in } \Omega, \tag{3.2}$$

and the boundary conditions

$$\begin{aligned}p_\varepsilon&=0 \quad \text{on} \quad (0,T)\times\Gamma_2,\\ \frac{\partial p_\varepsilon}{\partial \boldsymbol{n}}&=0 \quad \text{on} \quad (0,T)\times(\partial\Omega\setminus\Gamma_2).\end{aligned}$$

Here

$$f(x,y)=\begin{cases}\frac{\partial(\nabla\cdot\boldsymbol{v})}{\partial t}+\nabla\cdot((\boldsymbol{v}\cdot\nabla)\boldsymbol{v})\\ -\frac{1}{Re}\Delta(\nabla\cdot\boldsymbol{v}) \quad \text{in} \quad \Omega_1,\\ 0 \quad \text{in} \quad \Omega_2.\end{cases}$$

Remark Let $\varepsilon\to 0$. Then p_ε converges the solution p such that

$$\begin{aligned}-\Delta p&=f \quad \text{in } \Omega_1,\\ \frac{\partial p}{\partial \boldsymbol{n}}&=0 \quad \text{on } \gamma,\\ p&=0 \quad \text{on} \quad (0,T)\times\Gamma_2,\\ \frac{\partial p}{\partial \boldsymbol{n}}&=0 \quad \text{on} \quad (0,T)\times(\partial\Omega\setminus\Gamma_2).\end{aligned}$$

(cf. [3], [5]).

3.2 Numerical algorithm by digital image

In the previous section, we introduced the cell values and decided the digital approximation $\widetilde{\Omega_1}$ and $\widetilde{\Omega_2}$ of Ω_1 and Ω_2, respectively. Using the digital approximations and the fictitious domain formulation in 3.1, we propose the following numerical algorithm .

Step 1: We define the function on cells;

$$\widetilde{a}(x,y) = \begin{cases} 1 & \text{on } \widetilde{\Omega_1} \\ \varepsilon & \text{on } \widetilde{\Omega_2}. \end{cases}$$

Step 2: We solve the Poisson equation on cells in Ω;

$$\begin{aligned} -\nabla\cdot(\widetilde{a}(x,y)\nabla p) = f \quad & \text{in } \ (0,T)\times\Omega \\ p = 0 \quad & \text{on } \ (0,T)\times\Gamma_2, \\ \frac{\partial p}{\partial \boldsymbol{n}} = 0 \quad & \text{on } \ (0,T)\times(\partial\Omega\setminus\Gamma_2). \end{aligned}$$

One can easily get the numerical solution of this problem since we equated a regular grid point with the cell in Section 2.

Step 3: Using the numerical solution p in Step 2, we solve the fluid velocity $\boldsymbol{v}$ satisfying

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\widetilde{a}(x,y)\nabla p + \frac{1}{Re}\Delta\boldsymbol{v} \quad \text{in } \ (0,T)\times\Omega,$$

the boundary conditions;

$$\begin{aligned} \boldsymbol{v} = (0.2, 0.0) \quad & \text{on } \ (0,T)\times\Gamma_1 \\ \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = 0 \quad & \text{on } \ (0,T)\times\Gamma_2 \\ \boldsymbol{v} = 0 \quad & \text{on } \ (0,T)\times(\Gamma_3\cup\Gamma_4) \end{aligned}$$

and the equation

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{in } \ (0,T)\times\widetilde{\Omega_2}. \tag{3.3}$$

Our numerical program depends on the implicit and the third-order upwind scheme, and we develop the projection program satisfying (16).

4 Numerical simulation

We now show the numerical simulations of flow around three circular cylinders in the following case:

the digital image of Figure 1 consists of 332,100 pixels with 810 pixel width and 410 pixel heigh.

the Reynolds number Re=3,000.

the time step=0.01.

the threshold value z=0.0.

Then the following figure 5 represents the streamlines of the fluid vector at time t=0.001, 2.50, 25.0 and 50.0.

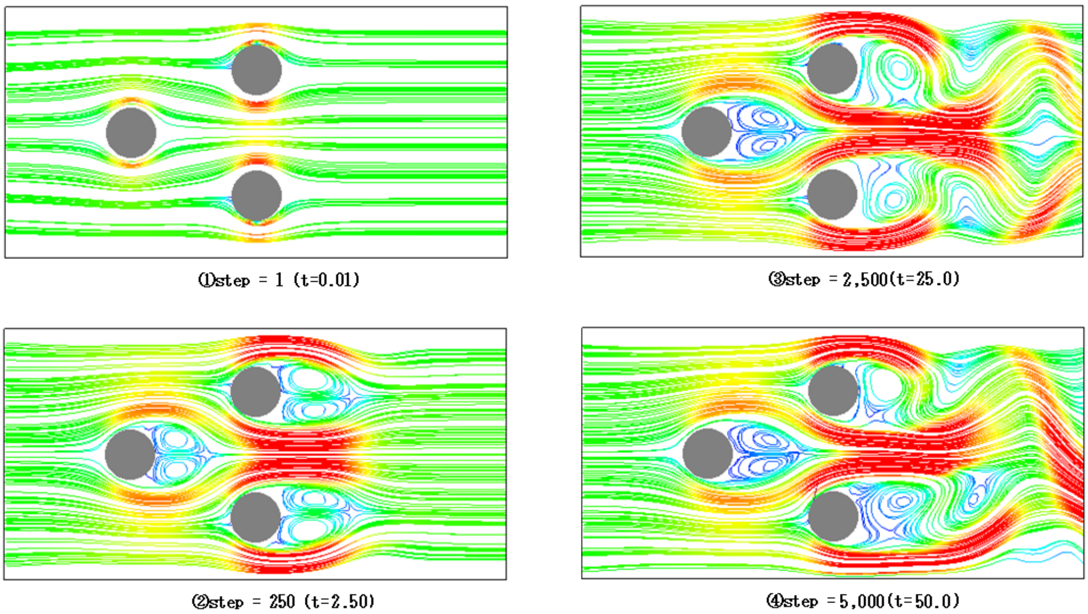

Figure 5. The result of numerical computation with the stream line by time step.

5 Conclusion

We showed the numerical method coupled with the digital image, the finite difference method and the fictitious domain method. This is widely applied to the various fields since the numerical algorithm is clear and is based on the finite difference method. In fact, we applied this method to analyse the groundwater flow through industrial waste ([7]).

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