Jacobi type formula for Matsumoto hyperelliptic curves

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On the occasion of the final lecture at Chiba university

1 Introduction

This is a continuation of our previous work " A variant of Jacobi type formula for Picard curves "([M-S]). Here we start from the algebraic curve $w^4 = z^2(z-1)^2(z-\lambda_1)(z-\lambda_2)$ with two complex parameters λ_1, λ_2. Generically it is a hyperelliptic curve of genus three. This curve has been studied by K. Matsumoto [M] (and see also [K-S]) in detail. We call it Matsumoto hyperelliptic curve. In this article we show a Jacobi type formula of two variables that connects the theta function and the hypergeometric function via the modular function for the Matsumoto hyperelliptic curves defined on the two dimensional hyperball. Our main theorem is stated in (2.6). As an application, we give a new proof for the extended Gauss arithmetic-geometric mean theorem discovered in [K-S]. We consider this is one aspect of various natures of the $K3$ modular function that was initiated by the author in 1977 (see [S1], [S2]). And he believes that this result will be understood in a more general context in the future.

2 Statement of Main Theorem

First recall the classical Jacobi formula. Set

$$\lambda(\tau) = \frac{\vartheta_{01}^4(\tau)}{\vartheta_{00}^4(\tau)} \tag{2.1}$$

for $\tau \in \boldsymbol{H} = \{\mathrm{Im}\ \tau > 0\}$, where ϑ_{jk} indicates the Jacobi theta constant

$$\vartheta_{jk}(\tau) = \sum_{n\in\mathbb{Z}} \exp[\pi i(n+\frac{j}{2})^2\tau + 2\pi i(n+\frac{j}{2})\frac{k}{2}].$$

Theorem 2.1. (Classical Jacobi Formula, see [J] p.235) *Under the relation (2.1) we have*

$$\vartheta_{00}^2(\tau) = F\left(\frac{1}{2},\frac{1}{2},1;1-\lambda\right) = \frac{1}{\pi}\int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-(1-\lambda))}}. \tag{2.2}$$

We have a two dimensional counterpart of this theorem.

Set

$$\boldsymbol{B} = \{(u,v) \in \boldsymbol{C}^2 :\ 2\,\mathrm{Im}\ v\ -\ |u|^2 > 0\}, \tag{2.3}$$

and set

$$\Omega(u,v) = \begin{pmatrix} v+iu^2/2 & -u^2/2 & -iu \\ -u^2/2 & v-iu^2/2 & u \\ -iu & u & i \end{pmatrix}.$$

Recall the Riemann theta constant

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\Omega) = \sum_{n\in\mathbb{Z}^3} \exp[\pi i(n+a)\Omega^t(n+a) + 2\pi i(n+a)^t b],$$

here a and b are row vectors in $\boldsymbol{Q}^3$, and Ω is a variable on the Siegel upper half space of degree 3. Let us denote

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega(u,v)) = \Theta \begin{bmatrix} a \\ b \end{bmatrix} (u,v) \text{ for } (u,v) \in \boldsymbol{B}.$$

Set

$$(\lambda_1, \lambda_2) = \left(\frac{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} (u,v)}{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u,v)}, \frac{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} (u,v)}{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u,v)} \right). \tag{2.4}$$

Theorem 2.2. (Main Theorem: Extended Jacobi formula)

For the Appell hypergeometric function

$$F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1; x, y) = \sum_{m,n \geq 0} \frac{\left(\frac{1}{2}, m+n\right) \left(\frac{1}{4}, m\right) \left(\frac{1}{4}, n\right)}{(1, m+n)\, m!n!} x^m y^n, \tag{2.5}$$

where $(a,n) = a(a+1)\cdots(a+n-1)$ *for* $n \geq 1$ *and* $(a,0)=1$, *we have*

$$\Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (u,v) = \vartheta_{00}(i)\ F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1; 1-\lambda_1, 1-\lambda_2), \tag{2.6}$$

with (λ_1, λ_2) *given by (2.4).*

Remark 2.1. *Here we set* (λ_1, λ_2) *in the real triangle* $\{(\lambda_1,\lambda_2) \in (0,1) \times (0,1) : \lambda_1 < \lambda_2\}$. *Then we can find the variables* $(u,v) \in \boldsymbol{B}$ *with the relation (2.4). At first we have an exact equality only for these variables, next we can make the analytic continuations of both sides on the whole domain* $\boldsymbol{B}$.

Remark 2.2. *If we put* $u = 0$, *then by using (2.4) we have* $\lambda_1 = \lambda_2$. *In this case, our main theorem coincides with the classical Jacobi formula (2.2).*

3 Geometric back ground

3.1 Matsumoto hyperelliptic modular function

Set

$$C(\lambda) = C(\lambda_1, \lambda_2) : w^4 = z^2(z-1)^2(z-\lambda_1)(z-\lambda_2) \tag{3.1}$$

for $\lambda = (\lambda_1, \lambda_2) \in \Lambda = \{(\lambda_1,\lambda_2) \in \boldsymbol{C}^2 : \lambda_1\lambda_2(\lambda_1-1)(\lambda_2-1)(\lambda_1-\lambda_2) \neq 0\}$. It is a hyperelliptic curve of genus 3. The differentials

$$dz/w, z(z-1)dz/w^3, z^2(z-1)dz/w^3$$

form a basis of holomorphic 1-forms on $C(\lambda)$.

For the moment suppose that λ_1 and λ_2 are real and satisfy $0 \leq \lambda_1 < \lambda_2 < 1$. Let ρ be the automorphism $(z,w) \mapsto (z, \sqrt{-1}w)$ of $C(\lambda)$.

Set

$$a_1(\lambda_1,\lambda_2) = 2\int_{-\infty}^{0} \frac{dz}{w},\ a_2(\lambda_1,\lambda_2) = 2i\int_{1}^{+\infty} \frac{dz}{w},\ a_5(\lambda_1,\lambda_2) = (1-i)\int_{\lambda_1}^{\lambda_2} \frac{dz}{w},$$

here we used the real positive branch of w for a_1, a_2 and $-iw > 0$ for a_5. They are periods of $C(\lambda)$.

[Fact 1] $a_1(\lambda_1,\lambda_2)$, $a_2(\lambda_1,\lambda_2)$, $a_5(\lambda_1,\lambda_2)$ satisfy the differential equation $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1)$ of the Appell hypergeometric function in (2.5), and they form a basis of the space of solutions.

We define the Schwarz map $\Phi : \Lambda \to \boldsymbol{C}^2$ for $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1)$ by

$$\Phi : (\lambda_1, \lambda_2) \mapsto (u,v) = (a_5/a_1, a_2/a_1) \in \boldsymbol{C}^2.$$

[Fact 2] The image of Φ is contained in the hyperball $\boldsymbol{B}$ of (2.3), and it is open dense in $\boldsymbol{B}$. The map Φ has a continuation on $\boldsymbol{P}^1 \times \boldsymbol{P}^1 - \{(0,0),(1,1),(\infty,\infty)\}$ and the image is equal to $\boldsymbol{B}$.
[Fact 3] The fundamental group $\pi_1(\Lambda, *)$ induces the monodromy group G of Φ acting on $\boldsymbol{B}$. The following five transformations give a generator system of G:

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1+i & 1 & 1-i \\ -1-i & 0 & i \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 2+i & -1-i & -1-i \\ 1+i & -i & -1-i \\ 1-i & -1+i & i \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} i & 1-i & 1-i \\ 0 & i & 0 \\ 0 & -1-i & -1 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 2+i & -1-i & 1-i \\ 1+i & -i & 1-i \\ -1-i & 1+i & i \end{pmatrix},$$

where g_i $(i = 1,2,3,4,5)$ acts on ${}^t(1, v, u)$ from left. Their orders are $4, 4, 2, 4, 4$ and eigen values are

$$\{1,1,i\}, \{1,1,i\}, \{1,1,-1\}, \{-1,i,i\}, \{1,1,i\},$$

respectively (the original article contains a printing error for g_5).
[Fact 4] The Schawarz map Φ induces a biholomorphic correspondence

$$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \cong \overline{\boldsymbol{B}/G},$$

and three points $(0,0), (1,1), (\infty,\infty)$ correspond to the boundary $\overline{\boldsymbol{B}/G} - \boldsymbol{B}/G$. We have the representatives

$$\Phi(0,0) = (0,0), \Phi(1,1) = (0,-1), \Phi(\infty,\infty) = (0, i\infty)$$

on $\partial \boldsymbol{B}$.
[Fact 5] We can realize $\boldsymbol{B}$ in the form:

$$\{\xi = [\xi_0, \xi_1, \xi_2] \in \boldsymbol{P}^2 : \ \xi H {}^t\overline{\xi} < 0\},$$
$$H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map

$$\Omega(u,v) = \begin{pmatrix} v + iu^2/2 & -u^2/2 & -iu \\ -u^2/2 & v - iu^2/2 & u \\ -iu & u & i \end{pmatrix}$$

gives an modular embedding of $\boldsymbol{B}$ into the Siegel upper half space of degree 3. Set

$$G_0 = U(2,1,\boldsymbol{Z}[i]) = \{g \in GL(3, \boldsymbol{Z}[i]) : \ gH {}^t\overline{g} = H\}.$$

G_0 becomes to be a restriction of $Sp(6, \boldsymbol{Z})$ on $\Omega(\boldsymbol{B})$.

[Fact 6] Set

$$g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

And let G be the monodromy group of $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Then the group generated by G and g_6 together with $i = i \cdot \mathrm{id}$ becomes to be the congruence subgroup

$$G(1+i) := \{g \in G_0 : \ g \equiv \mathrm{id} \pmod{(1+i)}\}.$$

[Fact 7] We set the following theta constants:

$$\begin{aligned} \Theta_{xN} &= \Theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} (u,v), \quad \Theta_{xD} = \Theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u,v), \\ \Theta_{yN} &= \Theta \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} (u,v), \quad \Theta_{yD} = \Theta \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u,v), \\ \Theta_{h1} &= \Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (u,v), \quad \Theta_{Z} = \Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} (u,v), \quad \Theta_{h3} = \Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} (u,v), \end{aligned} \tag{3.2}$$

here we used the same notation for the Riemann theta constant defined in the previous section. We have the following algebraic relations (see [K-S]):

$$\begin{aligned} \Theta_{h_3}(u,v) &= \Theta_{h_1}(u,v) \frac{\Theta_{xD}(u,v)\Theta_{yD}(u,v)}{\Theta_{xN}(u,v)\Theta_{yN}(u,v)} \\ \Theta_{Z}(u,v) &= \Theta_{h_1}(u,v) \frac{1}{\Theta_{xN}(u,v)\Theta_{yN}(u,v)} \sqrt{\frac{1}{2}\left(\Theta_{xD}^2(u,v)\Theta_{yN}^2(u,v) + \Theta_{yD}^2(u,v)\Theta_{xN}^2(u,v)\right)}. \end{aligned}$$

[Fact 8] ::Matsumoto hyperelliptic theta formula (see [M] and [K-S])::

It holds

$$(\lambda_1, \lambda_2) = \left(\frac{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} (u,v)}{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u,v)}, \ \frac{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} (u,v)}{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u,v)} \right). \tag{3.3}$$

3.2 modular behavior of the theta constants

Proposition 3.1. *Every fourth power of the theta constants listed in (3.2), except Θ_Z, is a modular form of weight 4 with respect to G, namely it satisfies*

$$\Theta^4 \begin{bmatrix} a \\ b \end{bmatrix} (g \circ (u,v)) = (p_1 + p_2 v + p_3 u)^4 \Theta^4 \begin{bmatrix} a \\ b \end{bmatrix} (u,v), \quad g = \begin{pmatrix} p_1 & p_2 & p_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G.$$

Proposition 3.2. (see Prop. 2.4 in [K-S])

The vector space of modular forms of weight 4 with respect to G is generated by $\Theta_{xN}^4, \Theta_{yN}^4, \Theta_{h1}^4, \Theta_{h3}^4$.

Remark 3.1. *More precisely we have*

$$\begin{aligned} \Theta_{h_1}(g_2(u,v)) &= i(2 + i + (-1-i)v + (-1-i)u)\Theta_{h_1}(u,v), \\ \Theta_{h_1}(g_5(u,v)) &= i(2 + i + (-1-i)v + (1-i)u)\Theta_{h_1}(u,v). \end{aligned}$$

We shall use these formula in our proof of Main Theorem.

4 Proof of Main Theorem

Lemma 4.1. *We have*

$$F_1\left(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;1-\lambda_1,1-\lambda_2\right)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{dz}{w}=\frac{1}{2\pi}a_1(\lambda_1,\lambda_2)\quad(0<\lambda_1,\lambda_2<1). \tag{4.1}$$

Proof]. According to [App] we have

$$F_1\left(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;s,t\right)=\frac{1}{\pi}\int_{1}^{\infty}z'^{-1/2}(z'-1)^{-1/2}(z'-s)^{-1/4}(z'-t)^{-1/4}dz'\quad(|s|,|t|<1).$$

By changing the variable $z'=1-z$ we obtain the required equality.

q.e.d.

Lemma 4.2. $\Theta^4\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}(u,v)$ *and* $F_1(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;1-\lambda_1,1-\lambda_2)^4$ *are modular forms of weight 4 with respective to* G.

Proof]. The first statement is a direct consequence of Prop. 3.1. Using Lemma 4.1 the second statement is derived from the monodromy action

$$\begin{pmatrix}a_1(g(u,v))\\a_2(g(u,v))\\a_5(g(u,v))\end{pmatrix}=\begin{pmatrix}p_1&p_2&p_3\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}a_1(u,v)\\a_2(u,v)\\a_5(u,v)\end{pmatrix}\quad\text{with } g=\begin{pmatrix}p_1&p_2&p_3\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{pmatrix}\in G.$$

q.e.d.

Lemma 4.3. *The modular form* $a_1(u,v)$ *vanishes along the divisor* V_1 *on* $\boldsymbol{B}/G$ *represented by* $\{u+v=1\}+\{iu+v=1\}$.

Proof of Main Theorem]. The differential $\frac{dz}{w}$ on $C(\lambda_1,\lambda_2)$ is a holomorphic differential for every point (λ_1,λ_2) on the affine part $\boldsymbol{C}\times\boldsymbol{C}$ except two points $(0,0)$ and $(1,1)$. So $a_1(u,v)$ does not vanish on the affine part. And we can see that $D_\lambda=\{\lambda_1=\infty\}+\{\lambda_2=\infty\}$ is the zero devisor of $a_1(\lambda_1,\lambda_2)$. By the direct calculation of the integrals a_1,a_2,a_5 on D_λ, we know that $\Phi(D_\lambda)=V_1$. Hence the zero divisor of $a_1(u,v)$ is equal to V_1.

According to Remark 3.1, we have

$$\Theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(g_2(u,v))=i(2+i+(-1-i)u+(-1-i)v)\Theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(u,v),$$

here

$$g_2(u,v)=\left(\frac{(1-i)+i\,u-(1-i)\ v}{(2+i)-(1+i)\ u-(1+i)\ v},\frac{(1+i)-(1+i)\ u-i\,v}{(2+i)-(1+i)\ u-(1+i)\ v}\right).$$

And we have $g_2(u,v)=(u,v)$ if and only if $u=1-v$. So it holds

$$\Theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(u,v)=i\Theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(u,v)$$

for $(u,v)\in\boldsymbol{B}$ with $u+v=1$. Namely we have $\Theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(u,v)=0$ on the divisor $\{u=1-v\}\subset\boldsymbol{B}$. By the same argument for g_5, we have $\Theta\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}(u,v)=0$ on the divisor $\{u=-i(1-v)\}\subset\boldsymbol{B}$. Hence the zero divisor of $\Theta_{h_1}(u,v)$ is greater than or equal to V_1.

Because $a_1(u,v)^4$ and $\Theta_{h_1}^4$ are modular forms with the same automorphic behavior, their zero divisors are linearly equivalent. So the zero divisor of $\Theta_{h_1}(u,v)$ is equal to V_1. Hence we have

$$\Theta_{h_1}(u,v)=c\cdot a_1(u,v)$$

with some constant c. We can determine it by the calculation on $u=0$. Consequently, we obtain the required equality.

q.e.d.

5 Application to the Extended Gauss AGM

It holds the following CM-isogeny formula (see [K-S]):

Theorem 5.1.

$$\begin{cases} \Theta_{h1}((1+i)u,2v)) = \frac{1}{2}\left(\Theta_{h1}(u,v)+\Theta_{h3}(u,v)\right), \\ \Theta_{h3}((1+i)u,2v)) = \Theta_Z(u,v), \\ \Theta_Z((1+i)u,2v)) = \sqrt[4]{\frac{1}{4}\left(\Theta_Z^2+\Theta_{h1}\Theta_{h3})(\Theta_Z^2+\frac{\Theta_{h1}^2+\Theta_{h3}^2}{2}\right)}. \end{cases} \tag{5.1}$$

Let (a,b,c) be a triple of positive real numbers. Set an AGM system:

$$(a',b',c') = \psi(a,b,c) = \left(\frac{1}{2}(a+b), c, \sqrt[4]{\frac{1}{4}(c^2+ab)(c^2+(a^2+b^2)/2)}\right). \tag{5.2}$$

We can define the common limit $\lim_{n\to\infty} \psi^n(a,b,c)$. And we denote it by $M_{ext}(a,b,c)$.

Corollary 5.1. (Extended Gauss AGM formula [K-S]) *Assume* $0<x,y<1$ *and set* $\lambda_1 = 1-x^2$, $\lambda_2 = 1-y^2$. *Then we have :*

$$\frac{1}{M_{ext}(1,\sqrt{xy},\sqrt{(x+y)/2})} = \frac{1}{\pi}\int_1^\infty \frac{du}{\sqrt[4]{u^2(u-1)^2(u-\lambda_1)(u-\lambda_2)}} = F_1(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;\lambda_1,\lambda_2). \tag{5.3}$$

[Proof]. Put

$$x = \frac{\Theta_{xD}^2(u,v)}{\Theta_{xN}^2(u,v)}, y = \frac{\Theta_{yD}^2(u,v)}{\Theta_{yN}^2(u,v)}.$$

According to Fact 7, we have

$$(a,b,c) = \Theta_{h_1}(u,v)(1,\sqrt{xy},\sqrt{\frac{1}{2}(x+y)}) = (\Theta_{h_1}(u,v),\Theta_{h_3}(u,v),\Theta_Z(u,v))$$

with

$$\begin{cases} b = \Theta_{h_1}(u,v)\frac{\Theta_{xD}(u,v)\Theta_{yD}(u,v)}{\Theta_{xN}(u,v)\Theta_{yN}(u,v)}, \\ c = \Theta_{h_1}(u,v)\frac{1}{\Theta_{xN}(u,v)\Theta_{yN}(u,v)}\sqrt{\frac{1}{2}\left(\Theta_{xD}^2(u,v)\Theta_{yN}^2(u,v)+\Theta_{yD}^2(u,v)\Theta_{xN}^2(u,v)\right)}. \end{cases}$$

By using the isogeny formula (5.1) we have

$$\psi(a,b,c) = (\Theta_{h_1}((1+i)u,2v),\Theta_{h_3}((1+i)u,2v),\Theta_Z((1+i)u,2v)).$$

Consequently we have

$$\Theta_{h_1}(u,v)M_{ext}(1,\sqrt{xy},\sqrt{(x+y)/2}) = \lim_{n\to\infty}\psi^n(a,b,c) = \vartheta_{00}(i)(1,1,1).$$

According to our Main Theorem, we have

$$\vartheta_{00}(i) = \frac{\Theta_{h_1}(u,v)}{F_1(\frac{1}{2},\frac{1}{4},\frac{1}{4},1;1-x^2,1-y^2)}.$$

By conjunction of these two equalities we obtain the required equality (5.3).

q.e.d.

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