

AN UPPER BOUND ON THE REDUCTION NUMBER OF AN IDEAL

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ABSTRACT. Let A be a commutative ring and I an ideal of A with a reduction Q . In this paper we give an upper bound on the reduction number of I with respect to Q , when a suitable family of ideals in A is given. As a corollary it follows that if some ideal J containing I satisfies $J^2 = QJ$, then $I^{v+2} = QI^{v+1}$, where v denotes the number of generators of J/I as an A -module.

1. INTRODUCTION

Let Q , I and J be ideals of a commutative ring A such that $Q \subseteq I \subseteq J$. As is noted in [1, 2.6], if J/I is cyclic as an A -module and $J^2 = QJ$, then we have $I^3 = QI^2$. The purpose of this paper is to generalize this fact. We will show that if J/I is generated by v elements as an A -module and $J^2 = QJ$, then $I^{v+2} = QI^{v+1}$. We get this result as a corollary of the following theorem, which generalizes Rossi's assertion stated in the proof of [7, 1.3].

Theorem 1.1. *Let A be a commutative ring and $\{F_n\}_{n \geq 0}$ a family of ideals in A such that $F_0 = A$ and $IF_n \subseteq F_{n+1}$ for any $n \geq 0$. Let $k \geq 0$ be a non-negative integer and \mathfrak{a} an ideal in A such that $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1}$. Suppose that $F_n/(QF_{n-1} + I^n)$ is generated by v_n elements for any $n \geq 0$ and $v_n = 0$ for $n \gg 0$. We put $v = \sum_{n \geq 0} v_n$. Then we have*

$$I^{v+k+1} = QI^{v+k} + \mathfrak{a}I^{v+k+1}.$$

If a family $\{F_n\}_{n \geq 0}$ of ideals in A satisfies all of the conditions required in 1.1 in the case where $\mathfrak{a} = (0)$, we have $F_n = QF_{n-1}$ for $n \gg 0$. As a typical example of such $\{F_n\}_{n \geq 0}$, we find $\{\tilde{I}^n\}_{n \geq 0}$ when I contains a non-zero-divisor, where \tilde{I}^n denotes the Ratliff-Rush closure of I^n (cf. [9]). If A is an analytically unramified local ring, then $\{\overline{I}^n\}_{n \geq 0}$ is also an important example, where \overline{I}^n denotes the integral closure of I^n . It is obvious that $\{J^n\}_{n \geq 0}$ always satisfies the required condition on $\{F_n\}_{n \geq 0}$ for any ideal J with $I \subseteq J \subseteq \overline{I}$.

We prove 1.1 following Rossi's argument in the proof of [7, 1.3]. However we do not assume that A/I has finite length. And furthermore we can deduce the following corollary which gives an upper bound on the reduction number $r_Q(I)$ of I with respect to Q using numbers of gerators of certain A -modules.

Corollary 1.2. *Let (A, \mathfrak{m}) be a Noetherian local ring and $\{F_n\}_{n \geq 0}$ a family of ideals in A such that $F_0 = A$ and $IF_n \subseteq F_{n+1}$ for any $n \geq 0$. Then, if $I^{k+1} \subseteq QF_k + \mathfrak{m}F_{k+1}$ for*

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some $k \geq 0$, we have

$$r_Q(I) \leq k + \sum_{n \geq 1} \mu_A(F_n / (QF_{n-1} + I^n)),$$

and furthermore it follows that

$$r_Q(I) \leq 1 + \mu_A(F_1/I) + \sum_{n \geq 2} \mu_A(F_n/QF_{n-1}).$$

Throughout this paper A denotes a commutative ring. We do not assume that A is Noetherian unless otherwise specified. Furthermore I and Q denote ideals of A such that $Q \subseteq I$. We set $r_Q(I) = \inf\{n \geq 0 \mid I^{n+1} = QI^n\}$. Of course, $r_Q(I) = \infty$ if Q is not a reduction of I . For a finitely generated A -module M , we denote by $\mu_A(M)$ the minimal number of generators of M . If (A, \mathfrak{m}) is a Noetherian local ring and M is annihilated by some power of \mathfrak{m} , the length of M is denoted by $\ell_A(M)$.

2. PROOF OF THEOREM 1.1

In order to prove 1.1 we prepare the following lemma, which generalizes [4, 2.3].

Lemma 2.1. *Let I_1, I_2, \dots, I_N be finite number of ideals of A . For any $1 \leq n \leq N$, we assume that I_n is generated by v_n elements and*

$$I \cdot I_n \subseteq I^{n+1} + \sum_{\ell=1}^N Q^{n+1-\ell} I_\ell.$$

Let $v := v_1 + v_2 + \dots + v_N > 0$. Then, for any v elements a_1, a_2, \dots, a_v in I , there exists $\sigma \in QI^{v-1}$ such that

$$a_1 a_2 \cdots a_v - \sigma \in \bigcap_{n=1}^N [I^{n+v} : I_n].$$

Proof. We put $w_0 = 0$ and $w_n = v_1 + \dots + v_n$ for $1 \leq n \leq N$. Then $0 = w_0 \leq w_1 \leq w_2 \leq \dots \leq w_N = v$. Hence, if $1 \leq i \leq v$, we have $w_{n-1} < i \leq w_n$ for some $1 \leq n \leq N$, and we denote this number n by n_i . Now we choose elements x_1, x_2, \dots, x_v of A so that I_n is generated by $\{x_i \mid w_{n-1} < i \leq w_n\}$ for any $1 \leq n \leq N$ with $v_n \neq 0$. Then $x_i \in I_{n_i}$ and

$$a_i x_i \in I \cdot I_{n_i} \subseteq I^{n_i+1} + \sum_{\ell=1}^N Q^{n_i+1-\ell} I_\ell$$

for any $1 \leq i \leq v$. Hence there exists a family $\{c_{ij}\}_{1 \leq i, j \leq v}$ of elements in A such that

$$a_i x_i \equiv \sum_{j=1}^v c_{ij} x_j \pmod{I^{n_i+1}} \quad \text{and} \quad c_{ij} \in Q^{n_i+1-n_j}$$

for any $1 \leq i, j \leq v$. Let $R = A[It, t^{-1}]$ and $T = A[t, t^{-1}]$, where t is an indeterminate. We regard T/R as a graded R -module, and for any $f \in T$ we denote by \bar{f} the class of f

in T/R . Then we have

$$a_i t \cdot \overline{x_i t^{n_i}} = \sum_{j=1}^v c_{ij} t^{n_i - n_j + 1} \cdot \overline{x_j t^{n_j}}$$

for any $1 \leq i, j \leq v$. Here we put

$$\begin{aligned} b_{ij} &= \begin{cases} a_i - c_{ii} & \text{if } i = j \\ -c_{ij} & \text{if } i \neq j \end{cases}, \\ m_{ij} &= b_{ij} t^{n_i - n_j + 1} \in R, \text{ and} \\ e_i &= \overline{x_i t^{n_i}} \in T/R \end{aligned}$$

for any $1 \leq i, j \leq v$. Let us consider the $v \times v$ matrix $M = (m_{ij})$ with entries in R . Because we have

$$M \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

it follows that $\Delta e_i = 0$ for any $1 \leq i \leq v$, where $\Delta = \det M$. Then we get

$$(*) \quad \Delta \cdot x_i t^{n_i} \in R$$

for any $1 \leq i \leq v$. On the other hand, by the definition of determinant, we have

$$\Delta = \sum_{(p_1, p_2, \dots, p_v) \in S_v} \text{sgn}(p_1, p_2, \dots, p_v) m_{1p_1} m_{2p_2} \cdots m_{vp_v},$$

where S_v denotes the set of permutations of $1, 2, \dots, v$ and $\text{sgn}(p_1, p_2, \dots, p_v)$ denotes the signature of $(p_1, p_2, \dots, p_v) \in S_v$. Because

$$\deg\left(\prod_{i=1}^v m_{ip_i}\right) = \sum_{i=1}^v (n_i - n_{p_i} + 1) = \sum_{i=1}^v n_i - \sum_{i=1}^v n_{p_i} + v = v,$$

we have $\prod_{i=1}^v m_{ip_i} = (\prod_{i=1}^v b_{ip_i}) t^v$. Therefore $\Delta = \delta t^v$, where δ denotes the determinant of the $v \times v$ matrix (b_{ij}) with entries in A . Hence, by $(*)$ we have $\delta x_i \in I^{v+n_i}$ for any $1 \leq i \leq v$. This means $\delta I_n \subseteq I^{v+n}$ for any $1 \leq n \leq N$, and so $\delta \in \bigcap_{n=1}^N [I^{v+n} : I_n]$. If $(p_1, p_2, \dots, p_v) \neq (1, 2, \dots, v)$, then $j > p_j$ for some $1 \leq j \leq v$, which means $n_j \geq n_{p_j}$, and so $b_{jp_j} = -c_{jp_j} \in Q^{n_j - n_{p_j} + 1} \subseteq QI^{n_j - n_{p_j}}$. As a consequence, if $(1, 2, \dots, v) \neq (p_1, p_2, \dots, p_v) \in S_v$, we get

$$\prod_{i=1}^v b_{ip_i} = b_{jp_j} \cdot \prod_{i \neq j} b_{ip_i} \in QI^{n_j - n_{p_j}} \cdot \prod_{i \neq j} I^{n_i - n_{p_i} + 1} \subseteq Q \cdot I^{n_j - n_{p_j} + \sum_{i \neq j} (n_i - n_{p_i} + 1)} = QI^{v-1}.$$

Furthermore, as $a_i \in I$ and $c_{ii} \in Q$ for any $1 \leq i \leq v$, we have

$$\prod_{i=1}^v b_{ii} = \prod_{i=1}^v (a_i - c_{ii}) = a_1 a_2 \cdots a_v - d$$

for some $d \in QI^{v-1}$. Therefore, there exists $\sigma \in QI^{v-1}$ such that $\delta = a_1 a_2 \cdots a_v - \sigma$, and the proof is complete.

Proof of Theorem 1.1. If $v = 0$, then we have $F_n = I^n$ for any $n \geq 0$, and so $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1} = QI^k + \mathfrak{a}I^{k+1} \subseteq I^{k+1}$, which means $I^{k+1} = QI^k + \mathfrak{a}I^{k+1}$. Hence we may assume $v > 0$. For any $n \geq 0$, let us take an ideal I_n generated by v_n elements so that $F_n = QF_{n-1} + I^n + I_n$. We can easily show that

$$(\#) \quad F_n = I^n + \sum_{\ell=0}^n Q^{n-\ell} I_\ell$$

for any $n \geq 0$ by induction on n . Now we choose an integer N so that $N > k$ and $I_n = 0$ for any $n > N$. Then by $(\#)$ it follows that

$$I \cdot I_n \subseteq F_{n+1} = I^{n+1} + \sum_{\ell=0}^N Q^{n+1-\ell} I_\ell$$

for any $0 \leq n \leq N$. Let a_1, a_2, \dots, a_v be any elements of I . Then, by 2.1 there exists $\sigma \in QI^{v-1}$ such that

$$a_1 a_2 \cdots a_v - \sigma \in \bigcap_{n=0}^N [I^{n+v} : I_n].$$

We put $\xi = a_1 a_2 \cdots a_v - \sigma$. Then by $(\#)$ we get

$$\xi F_n = \xi I^n + \sum_{\ell=0}^n Q^{n-\ell} \cdot \xi I_\ell \subseteq I^v \cdot I^n + \sum_{\ell=0}^n Q^{n-\ell} \cdot I^{\ell+v} \subseteq I^{v+n}$$

for any $0 \leq n \leq N$. Now the assumption that $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1}$ implies

$$\xi I^{k+1} \subseteq Q \cdot \xi F_k + \mathfrak{a} \cdot \xi F_{k+1} \subseteq Q \cdot I^{v+k} + \mathfrak{a} \cdot I^{v+k+1}.$$

Therefore we get

$$a_1 a_2 \cdots a_v \cdot I^{k+1} = (\xi + \sigma) I^{k+1} \subseteq QI^{v+k} + \mathfrak{a}I^{v+k+1}.$$

Then, as the elements a_1, a_2, \dots, a_v are chosen arbitrarily from I , it follows that $I^v \cdot I^{k+1} \subseteq QI^{v+k} + \mathfrak{a}I^{v+k+1} \subseteq I^{v+k+1}$. Thus we get $I^{v+k+1} = QI^{v+k} + \mathfrak{a}I^{v+k+1}$.

Proof of Corollary 1.2. We put $v = \sum_{n \geq 1} \mu_A(F_n / (QF_{n-1} + I^n))$. We may assume $v < \infty$. Then, setting $\mathfrak{a} = \mathfrak{m}$ in 1.1, it follows that $I^{v+k+1} = QI^{v+k} + \mathfrak{m}I^{v+k+1}$. Hence we get $I^{v+k+1} = QI^{v+k}$ by Nakayama's lemma, and so $\text{r}_Q(I) \leq v + k$. In order to prove the second inequality, we choose k as small as possible. If $k \leq 1$, we have

$$\text{r}_Q(I) \leq k + v \leq 1 + \mu_A(F_1/I) + \sum_{n \geq 2} \mu_A(F_n/QF_{n-1}).$$

So, we assume $k \geq 2$ in the rest of this proof. In this case we have

$$(\ddagger) \quad \text{r}_Q(I) \leq k + \mu_A(F_1/I) + \sum_{n=2}^k \mu_A(F_n / (QF_{n-1} + I^n)) + \sum_{n \geq k+1} \mu_A(F_n/QF_{n-1}).$$

If $2 \leq n \leq k$, then $I^n \not\subseteq QF_{n-1} + \mathfrak{m}F_n$, and so the canonical surjection

$$F_n / (QF_{n-1} + \mathfrak{m}F_n) \longrightarrow F_n / (QF_{n-1} + I^n + \mathfrak{m}F_n)$$

is not injective, which means

$$\mu_A(F_n/QF_{n-1} + I^n) \leq \mu_A(F_n/QF_{n-1}) - 1.$$

Thus we get

$$\sum_{n=2}^k \mu_A(F_n/QF_{n-1} + I^n) \leq \left\{ \sum_{n=2}^k \mu_A(F_n/QF_{n-1}) \right\} - (k-1).$$

Therefore the required inequality follows from (‡).

3. COROLLARIES

In this section we collect some results deduced from 1.1 and 1.2.

Corollary 3.1. *Let J be an ideal of A such that $J \supseteq I$ and $J^2 = QJ$. If J/I is finitely generated as an A -module, then $r_Q(I) \leq \mu_A(J/I) + 1$.*

Proof. We apply 1.1 setting $F_n = J^n$ for any $n \geq 0$ and $\mathfrak{a} = (0)$. Because $I^2 \subseteq J^2 = QJ$, we may put $k = 1$, and hence we get $I^{v+2} = QI^{v+1}$, where $v = \mu_A(J/I)$. Then $r_Q(I) \leq v + 1$.

Corollary 3.2. *Let (A, \mathfrak{m}) be a two-dimensional regular local ring (or, more generally, a two-dimensional pseudo-rational local ring) such that A/\mathfrak{m} is infinite. If I is an \mathfrak{m} -primary ideal with a minimal reduction Q , then $r_Q(I) \leq \mu_A(\bar{I}/I) + 1$.*

Proof. This follows from 3.1 since $(\bar{I})^2 = Q\bar{I}$ by [5, 5.1] (or [6, 5.4]).

Corollary 3.3. *Let \mathfrak{p} be a prime ideal of A with $\text{ht } \mathfrak{p} = g \geq 2$. Let $Q = (a_1, a_2, \dots, a_g)$ be an ideal generated by a regular sequence contained in the k -th symbolic power $\mathfrak{p}^{(k)}$ of \mathfrak{p} for some $k \geq 2$. Then we have $r_Q(I) \leq \mu_A((Q : \mathfrak{p}^{(k)})/Q) + 1$ for any ideal I with $Q \subseteq I \subseteq Q : \mathfrak{p}^{(k)}$, if one of the following three conditions holds ; (i) $A_{\mathfrak{p}}$ is not a regular local ring, (ii) $A_{\mathfrak{p}}$ is a regular local ring and $g \geq 3$, (iii) $A_{\mathfrak{p}}$ is a regular local ring, $g = 2$, and $a_i \in \mathfrak{p}^{(k+1)}$ for any $1 \leq i \leq g$.*

Proof. This follows from 3.1 since $(Q : \mathfrak{p}^{(k)})^2 = Q(Q : \mathfrak{p}^{(k)})$ by [10, 3.1].

Corollary 3.4. *Let (A, \mathfrak{m}) be a Buchsbaum local ring. Assume that the multiplicity of A with respect to \mathfrak{m} is 2 and $\text{depth } A > 0$. Then, for any parameter ideal Q in A and an ideal I with $Q \subseteq I \subseteq Q : \mathfrak{m}$, we have $r_Q(I) \leq \mu_A((Q : \mathfrak{m})/Q) + 1$.*

Proof. This follows from 3.1 since $(Q : \mathfrak{m})^2 = Q(Q : \mathfrak{m})$ by [3, 1.1].

In order to state the last corollary, let us recall the definition of Hilbert coefficients. Let (A, \mathfrak{m}) be a d -dimensional Noetherian local ring and I an \mathfrak{m} -primary ideal. Then there exists a family $\{e_i(I)\}_{0 \leq i \leq d}$ of integers such that

$$\ell_A(A/I^{n+1}) = \sum_{i=0}^d (-1)^i e_i(I) \binom{n+d-i}{d-i}$$

for $n \gg 0$. We call $e_i(I)$ the i -th Hilbert coefficient of I . On the other hand, if A is an analytically unramified local ring, then $\{\overline{I}^n\}_{n \geq 0}$ is a Hilbert filtration (cf. [2]), and so there exists a family $\{\bar{e}_i(I)\}_{0 \leq i \leq d}$ of integers such that

$$\ell_A(A/\overline{I}^{n+1}) = \sum_{i=0}^d (-1)^i \bar{e}_i(I) \binom{n+d-i}{d-i}$$

for $n \gg 0$. As is proved in [7, 1.5], if A is a two-dimensional Cohen-Macaulay local ring, then we have

$$r_Q(I) \leq e_1(I) - e_0(I) + \ell_A(A/I) + 1$$

for any minimal reduction Q of I . We can generalize this result as follows.

Corollary 3.5. *Let (A, \mathfrak{m}) be a two-dimensional Cohen-Macaulay local ring with infinite residue field and I an \mathfrak{m} -primary ideal with a minimal reduction Q . Then we have the following inequalities.*

- (1) $r_Q(I) \leq e_1(J) - e_0(J) + \ell_A(A/I) + 1$ for any ideal J such that $I \subseteq J \subseteq \bar{I}$.
- (2) $r_Q(I) \leq \bar{e}_1(I) - \bar{e}_0(I) + \ell_A(A/I) + 1$, if A is analytically unramified.

Proof. (1) Setting $F_n = \widetilde{J}^n$ for any $n \geq 0$ in 1.2, we get

$$\begin{aligned} r_Q(I) &\leq 1 + \mu_A(\widetilde{J}/I) + \sum_{n \geq 2} \mu_A(\widetilde{J}^n/Q\widetilde{J}^{n-1}) \\ &\leq 1 + \ell_A(\widetilde{J}/I) + \sum_{n \geq 2} \ell_A(\widetilde{J}^n/Q\widetilde{J}^{n-1}) \\ &= \sum_{n \geq 1} \ell_A(\widetilde{J}^n/Q\widetilde{J}^{n-1}) - \ell_A(I/Q) + 1. \end{aligned}$$

Because $e_1(J) = \sum_{n \geq 1} \ell_A(\widetilde{J}^n/Q\widetilde{J}^{n-1})$ by [2, 1.10] and

$$\ell_A(I/Q) = \ell_A(A/Q) - \ell_A(A/I) = e_0(J) - \ell_A(A/I),$$

the required inequality follows.

(2) Similarly as the proof of (1), setting $F_n = \overline{I}^n$ for any $n \geq 0$ in 1.2, we get

$$r_Q(I) \leq \sum_{n \geq 1} \ell_A(\overline{I}^n/Q\overline{I}^{n-1}) - \ell_A(I/Q) + 1.$$

Because the depth of the associated graded ring of the filtration $\{\overline{I}^n\}_{n \geq 0}$ is positive, we have $\bar{e}_1(I) = \sum_{n \geq 1} \ell_A(\overline{I}^n/Q\overline{I}^{n-1})$ by [2, 1.9]. Hence we get the required inequality as $\ell_A(I/Q) = \bar{e}_0(I) - \ell_A(A/I)$.

4. EXAMPLE

In this section we give an example which shows that the maximum value stated in 3.1 can be reached. It provides an example in the case where $\dim A/I > 0$.

Example 4.1. Let $n \geq 3$ be an integer and $S = k[X_0, X_1, \dots, X_n]$ be the polynomial ring with $n + 1$ variables over a field k . Let $A = S/\mathfrak{a}$, where \mathfrak{a} is the ideal of S generated by the maximal minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_{n-1} \\ X_1 & X_2 & \cdots & X_n \end{pmatrix}.$$

We denote the image of X_i in A by x_i for $0 \leq i \leq n$. It is well known that A is a two-dimensional Cohen-Macaulay graded ring with the graded maximal ideal $\mathfrak{m} = (x_0, x_1, \dots, x_n)$.

- (1) Let $I = (x_0, x_1, x_n)$ and $Q = (x_0, x_n)$. Then we have $\mathfrak{m}^2 = Q\mathfrak{m}$, $\mu_A(\mathfrak{m}/I) = n - 2$, and $r_Q(I) = n - 1$.
- (2) Let $I = (x_0, x_1, x_{n-1})$, $J = (x_0, x_1, \dots, x_{n-1})$, and $Q = (x_0, x_{n-1})$. Then we have $\dim A/I = 1$, $J^2 = QJ$, $\mu_A(J/I) = n - 3$, and $r_Q(I) = n - 2$.

Proof. (1) Let $0 \leq i \leq j \leq n$. If $i = 0$ or $j = n$, then $x_i x_j \in Q\mathfrak{m}$. On the other hand, if $i > 0$ and $j < n$, then the determinant of the matrix

$$\begin{pmatrix} X_{i-1} & X_j \\ X_i & X_{j+1} \end{pmatrix}$$

is contained in \mathfrak{a} , and so $x_i x_j = x_{i-1} x_{j+1}$. Hence we can show that $x_i x_j \in Q\mathfrak{m}$ for any $0 \leq i \leq j \leq n$ by descending induction on $j - i$. Thus we get $\mathfrak{m}^2 = Q\mathfrak{m}$. It is obvious that $\mu_A(\mathfrak{m}/I) = n - 2$. Therefore $I^n = QI^{n-1}$ by 3.1 (In fact, we have $x_1^n = x_1^{n-2} \cdot x_1^2 = x_1^{n-2} \cdot x_0 x_2 = x_0 x_1^{n-3} \cdot x_1 x_2 = x_0 x_1^{n-3} \cdot x_0 x_3 = x_0^2 x_1^{n-4} \cdot x_1 x_3 = \cdots = x_0^{n-2} \cdot x_1 x_{n-1} = x_0^{n-2} \cdot x_0 x_n = x_0^{n-1} x_n \in Q^n \subseteq QI^{n-1}$). In order to prove $r_Q(I) = n - 1$, we show $x_1^{n-1} \notin QI^{n-2}$. For that purpose we use the isomorphism

$$\varphi : A \longrightarrow k[\{s^{n-i}t^i\}_{0 \leq i \leq n}]$$

of k -algebras such that $\varphi(x_i) = s^{n-i}t^i$ for $0 \leq i \leq n$, where s and t are indeterminates. We have to show $\varphi(x_1)^{n-1} \notin \varphi(Q)\varphi(I)^{n-2}$. Because $\varphi(I) = (s^n, s^{n-1}t, t^n)$, we get

$$\varphi(I)^\ell \subseteq (\{s^{\alpha n - \beta} t^{(\ell - \alpha)n + \beta} \mid 0 \leq \alpha \leq \ell, 0 \leq \beta \leq \alpha\})$$

for any $\ell \geq 1$ by induction on ℓ , and so

$$\varphi(Q)\varphi(I)^{n-2} \subseteq (\{s^{(\alpha+1)n - \beta} t^{(n-2-\alpha)n + \beta}, s^{\alpha n - \beta} t^{(n-1-\alpha)n + \beta} \mid 0 \leq \alpha \leq n-2, 0 \leq \beta \leq \alpha\}).$$

Therefore, if $\varphi(x_1)^{n-1} = (s^{n-1}t)^{n-1} = s^{(n-1)^2}t^{n-1} \in \varphi(Q)\varphi(I)^{n-2}$, one of the following two cases

- (i) $(\alpha + 1)n - \beta \leq (n - 1)^2$ and $(n - 2 - \alpha)n + \beta \leq n - 1$, or
- (ii) $\alpha n - \beta \leq (n - 1)^2$ and $(n - 1 - \alpha)n + \beta \leq n - 1$

must occur for some α and β with $0 \leq \alpha \leq n - 2$ and $0 \leq \beta \leq \alpha$. Suppose that the case (i) occurred. Then we have

$$(\alpha + 1)n - \beta \leq (n - 1)n - (n - 1) \text{ and } (n - 2 - \alpha)n \leq n - 1 - \beta.$$

As the first inequality implies

$$n - 1 - \beta \leq (n - 1)n - (\alpha + 1)n = (n - 2 - \alpha)n,$$

it follows that

$$n - 1 - \beta = (n - 1)n - (\alpha + 1)n,$$

and so

$$\alpha n - \beta = n^2 - 3n + 1.$$

Then, as $\alpha n > n^2 - 3n = (n - 3)n$, we have $n - 3 < \alpha \leq n - 2$, which implies $\alpha = n - 2$.

Thus we get

$$(n - 2)n - \beta = n^2 - 3n + 1,$$

and so $\beta = n - 1$, which contradicts to $\beta \leq \alpha$. Therefore the case (ii) must occur. Then we have

$$\alpha n - \beta \leq (n - 1)n - (n - 1) \text{ and } (n - 1 - \alpha)n \leq n - 1 - \beta.$$

As the first inequality implies

$$n - 1 - \beta \leq (n - 1)n - \alpha n = (n - 1 - \alpha)n,$$

it follows that

$$n - 1 - \beta = (n - 1)n - \alpha n,$$

and so

$$\alpha n - \beta = n^2 - 2n + 1.$$

Then, as $\alpha n > n^2 - 2n = (n - 2)n$, we get $\alpha > n - 2$, which contradicts to $\alpha \leq n - 2$. Thus we have seen that $x_1^{n-1} \notin QI^{n-2}$.

(2) Let $\mathfrak{b} = (X_0, X_1, \dots, X_{n-1})S$. Then $\mathfrak{a} \subseteq \mathfrak{b}$, and so \mathfrak{b} is the kernel of the canonical surjection $S \rightarrow A/J$. Hence $A/J \cong k[X_n]$, which implies $\dim A/J = 1$. Let $0 \leq i \leq j \leq n - 1$. If $i = 0$ or $j = n - 1$, then $x_i x_j \in QJ$. On the other hand, if $i > 0$ and $j < n$, then $x_i x_j = x_{i-1} x_{j+1}$. Hence we can show that $x_i x_j \in QJ$ for any $0 \leq i \leq j \leq n - 1$ by descending induction on $j - i$. Thus we get $J^2 = QJ$. It is obvious that $\mu_A(J/I) = n - 3$. Therefore $I^{n-1} = QI^{n-2}$ by 3.1. This means $\dim A/I = \dim A/Q = \dim A/J = 1$. In order to prove $\text{r}_Q(I) = n - 2$, we show $x_1^{n-2} \notin QI^{n-3}$. For that purpose we use again the isomorphism φ stated in the proof of (1). Although we have to prove $\varphi(x_1)^{n-2} \notin \varphi(Q)\varphi(I)^{n-3}$, it is enough to show

$$(s^{n-1}t)^{n-2} \notin (s^n, st^{n-1})(s^n, s^{n-1}t, st^{n-1})^{n-3}B,$$

where $B = k[s, t]$. Because

$$(s^{n-1}t)^{n-2} = s^{n-2} \cdot (s^{n-2}t)^{n-2}$$

in B and

$$(s^n, st^{n-1})(s^n, s^{n-1}t, st^{n-1})^{n-3}B = s^{n-2} \cdot (s^{n-1}, t^{n-1})(s^{n-1}, s^{n-2}t, t^{n-1})^{n-3}B,$$

we would like to show

$$(s^{n-2}t)^{n-2} \notin (s^{n-1}, t^{n-1})(s^{n-1}, s^{n-2}t, t^{n-1})^{n-3}B.$$

However, it can be done by the same argument as the proof of

$$(s^{n-1}t)^{n-1} \notin (s^n, t^n)(s^n, s^{n-1}t, t^n)^{n-2},$$

and hence we have proved (2).

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