An isomorphic correspondence between the Hilbert Modular Group and the Unimodular Group of a Domain of Type IV

Toshimasa Ishige
Inst. of Math. and Physics
Chiba University

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1 Introduction

In the present paper, we construct an injective isomorphism ρ' from the projective Hilbert modular group into the integral orthogonal group $PO(A,\mathbb{Z})$ for a real Hermitian matrix A of signature $(2,2)$. For the precise statement we use the following notation.

K : a real quadratic field with the discriminant d,
$h(K)$: the class number of K,
ε : the fundamental unit of K with $\varepsilon > 1$,
$\mathcal{O}$: the ring of integers in K,
$\{w_1, w_2\}$: a basis of $\mathcal{O}$,
$U = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,
$W = \left[\begin{smallmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{smallmatrix}\right]$, where $^-$ denotes the automorphism of K with $\overline{\sqrt{d}} = -\sqrt{d}$,
$A = U \oplus (-WU^tW)$,
$PO(A,\mathbb{R}) = \{P \in PGL(4,\mathbb{R}) \mid {}^tPAP = A\}$,
$PO(A,\mathbb{R})_0$: the connected component of the identity element in $PO(A,\mathbb{R})$,
$PO(A,\mathbb{Z}) = PGL(4,\mathbb{Z}) \cap PO(A,\mathbb{R})$,
$PO(A,\mathbb{Z})_0 = PGL(4,\mathbb{Z}) \cap PO(A,\mathbb{R})_0$,
$\mathbb{H}(\mathbb{H}_-)$: the upper half plane (the lower half plane, resp.).
$\mathcal{D}(A) = \{\eta \in P^3(\mathbb{C}) \mid \eta A\,{}^t\eta = 0, \overline{\eta} A\,{}^t\eta > 0\}$,
$\mathcal{D}(A)_0 = \{\eta = [\eta_0, \cdots, \eta_3] \in P^3(\mathbb{C}) \mid \eta A\,{}^t\eta = 0, \overline{\eta} A\,{}^t\eta > 0, \eta_1^{-1}(\eta_2,\eta_3)W \in \mathbb{H} \times \mathbb{H}_-\}$.

We define an action of the Hilbert modular group $PSL(2,\mathcal{O})$ on $\mathbb{H} \times \mathbb{H}_-$. And we construct a natural biholomorphic isomorphism $\iota' : \mathbb{H} \times \mathbb{H}_- \to \mathcal{D}(A)_0$. It becomes a modular isomorphism in a certain sense. So we obtain a required isomorphism ρ' between two modular groups. In other words, with ρ' defining an action of $PSL(2,O)$ on $\mathcal{D}(A)_0$, ι' becomes a $PSL(2,O)$-equivariant map. That is our first main theorem (Theorem 3.2).

To speak exactly, we have the following diagram:

$$\begin{array}{ccc}
\Gamma & \xrightarrow[\sim]{\rho'} & PO(A,\mathbb{Z})_0 \\
\uparrow & & \uparrow \\
PSL(2,\mathcal{O}) & \xrightarrow[\sim]{\rho'} & \rho'(PSL(2,\mathcal{O}))
\end{array}$$

,where Γ is an extension of $PSL(2, \mathcal{O})$ defined in Theorem 3.2. In other words, we have

$$\mathbb{H} \times \mathbb{H}_{-}/\Gamma \cong \mathcal{D}(A)_0/PO(A, \mathbb{Z})_0.$$

Moreover we obtain the following: If $h(K) = 1$ and $N_{K/\mathbb{Q}} = -1$, then we have

$$\Gamma = PSL(2, \mathcal{O}).$$

And if $h(K) = 1$ and $N_{K/\mathbb{Q}} = 1$, then we have

$$[\Gamma : PSL(2, \mathcal{O})] = 2.$$

These results are stated in Theorem 3.3.

As a biproduct of our study, we obtain a condition for $PSL(2, \mathcal{O})$ to have a modular embedding into the paramodular symplectic group $Sp(4, \mathbb{Z}, D)$, where $D = \left[\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right]$ $(d_1|d_2)$ indicates a fixed polarization. The result is stated in Theorem 5.1.

Also we obtained an explicit system of generators of $PO(A, \mathbb{Z})$ for $d = 5, 8$. It is stated as Proposition 4.5.

Our present study comes from the following background.

For the case $d = 8$, $\mathcal{D}(A)$ becomes the space of moduli of a family $\mathscr{F}$ of $K3$ surfaces with the transcendental lattice A for a generic member. The member of $\mathscr{F}$ has a representation

$$S(\lambda, \mu) : xyz(x + y + z + 1) + \lambda xy + \mu = 0,$$

with 2 parameters $\lambda, \mu \in P^1(\mathbb{C})$. The situation is almost the same for the case $d = 5$ (see Narumiya [6]). So it is interesting to study the period map, the period differential equation and the modular map that is obtained as the inverse of the period map. The study in this direction will be published elsewhere.

Our work is greatly owing to Narumiya for his suggestion made years ago while he was working under Hironori Shiga that there must be an injective isomorphism described here. The author owes great thanks to Professor Shiga for giving the opportunity to work on this conjecture.

2 A Symmetric Bounded Domain of Type IV

Let A be a real Hermitian matrix with signature $(2, n)$. A represents a bilinear form $\zeta A^t \eta$ for $\zeta = [\zeta_0, \zeta_1, .., \zeta_{n+1}]$, $\eta = [\eta_0, \eta_1, .., \eta_{n+1}] \in \mathbb{P}^{n+1}(\mathbb{C})$.

Definition 2.1. *For such matrix A the subspace of $\mathbb{P}^{n+1}(\mathbb{C})$ consisting of the points $\eta = [\eta_0, \eta_1, .., \eta_{n+1}] \in \mathbb{P}^{n+1}(\mathbb{C})$ which satisfy the quadratic relations*

$$\begin{aligned} \eta A^t \eta &= 0, \\ \overline{\eta} A^t \eta &> 0, \end{aligned} \tag{2.1}$$

is called a n dimensional symmetric bounded domain of type IV associated to A and denoted by $\mathcal{D}^n(A)$.

Note that these relations are well defined on $\mathbb{P}^{n+1}(\mathbb{C})$. Henceforth we only deal with the case $n = 2$ where A is a real Hermitian matrix with signature (2,2) and abuse the notation $\mathcal{D}^2(A)$ by $\mathcal{D}(A)$. We also use the notation $O(A, \mathbb{R}) = \{P \in GL(4, \mathbb{R})|^t PAP = A\}$ $(PO(A, \mathbb{R}) = O(A, \mathbb{R})/\{\pm I_4\})$ and call it the orthogonal group (the projective orthogonal group resp.) associated to A. $PO(A, \mathbb{R})$ is a subgroup of the projective transformation group on $\mathbb{P}^3(\mathbb{C})$ and stabilizes the relations (2.1). So it becomes the projective transfomation group on $\mathcal{D}(A)$.

Let $\mathcal{G}$ denote the group consisting of all the general linear transformations which preserve the indefinite quadratic form

$$x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 \cdots - x_{p+q}^2 \quad (p > 0, q > 0, p + q = n).$$

Let $E_{p,q} = I_p \oplus -I_q$. Then $\mathcal{G} = \{P \in GL(n, \mathbb{R}) \mid {}^tPE_{p,q}P = E_{p,q}\}$. Here we review the basic facts about the group $\mathcal{G}$. We have Fact 2.1 through 2.7 as known (Yamauchi-Sugiura [11],p.145 Problem 4).

Fact 2.1. *Let $\mathfrak{g}$ be the Lie ring of $\mathcal{G}$. Then*

$$\mathfrak{g} = \{X \in M(n, \mathbb{R}) \mid {}^tXE_{p,q} + E_{p,q}X = 0\}.$$

Fact 2.2. *$\mathcal{G}$ is self conjugate,i.e., if $P \in \mathcal{G}$, then ${}^tP \in \mathcal{G}$.*

Fact 2.3. *For a real symmetric matrix $X({}^tX = X)$ if $\exp(X) \in \mathcal{G}$, then $X \in \mathfrak{g}$. In particular for any real number s, $\exp(sX) \in \mathcal{G}$.*

We denote by $O(n, \mathbb{R})$ the n dimensional real orthogonal group.

Fact 2.4. *Any matrix $M \in GL(n, \mathbb{R})$ can be wriitten uniquely as $M = US$ where $U \in O(n, \mathbb{R})$ and S is a positive definite real symmetric matrix. This is called the polar representation of M.*

Fact 2.5. *For any $P \in \mathcal{G}$, let $P = US$ be its polar representation. Then $U \in \mathcal{G}$ and $S \in \mathcal{G}$.*

Fact 2.6. *We have $\mathcal{G} \cap O(n, \mathbb{R}) = O(p, \mathbb{R}) \oplus O(q, \mathbb{R})$. In other words any $P \in \mathcal{G} \cap O(n, \mathbb{R})$ can be written as*

$$P = P_1 \oplus P_2 \quad \text{where} \;\; P_1 \in O(p, \mathbb{R}), \;\; P_2 \in O(q, \mathbb{R}).$$

Conversely any element of $\mathcal{G}$ in this form is an element of $\mathcal{G} \cap O(n)$.

Fact 2.7. *$\mathcal{G}$ is decomposed into four connected components, $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, where*

$$I_{p+q} \in \mathcal{G}_0, \quad I_p \oplus E_{q-1,1} \in \mathcal{G}_1, \quad E_{p-1,1} \oplus I_q \in \mathcal{G}_2, \quad E_{p-1,1} \oplus E_{q-1,1} \in \mathcal{G}_3.$$

From now on we only consider the case $n = 4, p = 2, q = 2$.
Take an element P of $\mathcal{G}$ and let $P = US$ be the polar representation . From Fact 2.1 and 2.3, we have

$$S = \exp\left(\begin{bmatrix} 0 & S_1 \\ {}^tS_1 & 0 \end{bmatrix}\right) \quad \text{with} \;\; S_1 \in M(2, \mathbb{R}).$$

This and Fact 2.6 give a system of genarators for $\mathcal{G}$.

Proposition 2.1. *We have a system of generators for $\mathcal{G}$ with a parameter $\alpha \in \mathbb{R}$:*

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$G_3(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_4(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

$$G_5(\alpha) = \exp\left(\begin{bmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cosh\alpha & 0 & \sinh\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\alpha & 0 & \cosh\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_6(\alpha) = \exp\left(\begin{bmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\alpha & 0 & \sinh\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sinh\alpha & 0 & \cosh\alpha \end{bmatrix},$$

$$G_7(\alpha) = \exp\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{bmatrix},$$

$$G_8(\alpha) = \exp\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha & 0 \\ 0 & \sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $U \oplus -U$ is of signature $(2,2)$, $O(U \oplus -U, \mathbb{R})$ is isomophic to $\mathcal{G}$. From Proposition 2.1 , we obtain

Corollary 2.1. *We have a system of generators for $PO(U \oplus -U, \mathbb{R})$ with a parameter $\alpha \in \mathbb{R}$:*

$$H_1 = I_2 \oplus U, \quad H_2 = U \oplus I_2,$$

$$H_3(\alpha) = \begin{bmatrix} \cos^2\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & -\cos\alpha\sin\alpha \\ -\sin^2\alpha & \cos^2\alpha & \cos\alpha\sin\alpha & -\cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \sin^2\alpha \\ \cos\alpha\sin\alpha & \cos\alpha\sin\alpha & \sin^2\alpha & \cos^2\alpha \end{bmatrix},$$

$$H_4(\alpha) = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\cos\alpha\sin\alpha \\ \sin^2\alpha & \cos^2\alpha & \cos\alpha\sin\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & -\sin^2\alpha \\ \cos\alpha\sin\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos^2\alpha \end{bmatrix},$$

$$H_5(\alpha) = \begin{bmatrix} \cosh^2\alpha & \sinh^2\alpha & \cosh\alpha\sinh\alpha & \cosh\alpha\sinh\alpha \\ \sinh^2\alpha & \cosh^2\alpha & \cosh\alpha\sinh\alpha & \cosh\alpha\sinh\alpha \\ \cosh\alpha\sinh\alpha & \cosh\alpha\sinh\alpha & \cosh^2\alpha & \sinh^2\alpha \\ \cosh\alpha\sinh\alpha & \cosh\alpha\sinh\alpha & \sinh^2\alpha & \cosh^2\alpha \end{bmatrix},$$

$$H_6(\alpha) = \begin{bmatrix} \cosh^2\alpha & -\sinh^2\alpha & \cosh\alpha\sinh\alpha & -\cosh\alpha\sinh\alpha \\ -\sinh^2\alpha & \cosh^2\alpha & -\cosh\alpha\sinh\alpha & \cosh\alpha\sinh\alpha \\ \cosh\alpha\sinh\alpha & -\cosh\alpha\sinh\alpha & \cosh^2\alpha & -\sinh^2\alpha \\ -\cosh\alpha\sinh\alpha & \cosh\alpha\sinh\alpha & -\sinh^2\alpha & \cosh^2\alpha \end{bmatrix},$$

$$H_7(\alpha) = \begin{bmatrix} \cosh\alpha - \sinh\alpha & 0 & 0 & 0 \\ 0 & \cosh\alpha + \sinh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$H_8(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\alpha - \sinh\alpha & 0 \\ 0 & 0 & 0 & \cosh\alpha + \sinh\alpha \end{bmatrix}.$$

[Proof]. Pick a matrix $C \in M(4, \mathbb{R})$ satisfying ${}^tC(U \oplus -U)C = E_{2,2}$. We abuse the notation $G_i(\alpha)$ by G_i, $H_i(\alpha)$ by H_i for $i = 3, \cdots, 8$. If we set $H_i = CG_iC^{-1}$ for $i = 1, \cdots, 8$, then we have

$$\begin{aligned} {}^tH_i(U \oplus -U)H_i &= {}^tC^{-1}{}^tG_i{}^tC(U \oplus -U)CG_iC^{-1} \\ &= {}^tC^{-1}{}^tG_iE_{2,2}G_iC^{-1} \\ &= {}^tC^{-1}E_{2,2}C^{-1} \\ &= U \oplus -U. \end{aligned}$$

So $H_i \in PO(U \oplus -U, \mathbb{R})$. We set here $C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and obtain the generators above.

q.e.d.

From Fact 2.7, we have

Corollary 2.2. *$PO(U \oplus -U, \mathbb{R})$ is decomposed into four connected components as follows:*

$$PO(U, \mathbb{R}) = PO(U, \mathbb{R})_0 \cup PO(M, \mathbb{R})_0 H_1 \cup PO(M, \mathbb{R})_0 H_2 \cup PO(M, \mathbb{R})_0 H_1 H_2,$$

where $PO(U, \mathbb{R})_0$ is the connected component of $PO(U, \mathbb{R})$ containing the identity element.

We now construct the Segre map.

Definition 2.2. *The Segre map ι is defined by*

$$\begin{aligned} \iota : \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H} &\longrightarrow \mathbb{P}^3 \\ (z, w) &\longmapsto [zw, 1, z, w]. \end{aligned}$$

Proposition 2.2. *$\iota : \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H} \to \mathcal{D}(U \oplus -U)$ is biholomorphic.*

[Proof]. Let $\eta = [\eta_0, \eta_1, \eta_2, \eta_3] \in \mathcal{D}(U \oplus -U)$. By the definition of $\mathcal{D}(U \oplus -U)$ we have

$$\begin{aligned} &\eta_0\eta_1 - \eta_2\eta_3 = 0, \\ &\overline{\eta_0}\eta_1 + \eta_0\overline{\eta_1} - \overline{\eta_2}\eta_3 - \eta_2\overline{\eta_3} = \Re(\overline{\eta_0}\eta_1 - \overline{\eta_2}\eta_3) > 0. \end{aligned} \tag{2.2}$$

So $\eta_i \neq 0 (i = 0, \cdots, 3)$, and η can be written as

$$\eta = [zw, 1, z, w] \quad \text{where } z = \frac{\eta_2}{\eta_1}, w = \frac{\eta_3}{\eta_1}.$$

Substituting $[zw, 1, z, w]$ into the inequality of (2.2), we have

$$\overline{zw} + zw - \overline{z}w - z\overline{w} = (z - \overline{z})(w - \overline{w}) = -\Im(z)\Im(w) > 0.$$

Hence $(z, w) \in \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}$. Conversely for $(z, w) \in \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}$, $[zw, 1, z, w] \in \mathcal{D}(U \oplus -U)$.

q.e.d.

Corollary 2.3. *The restriction $\iota : \mathbb{H} \times \mathbb{H}_- \to \mathcal{D}(U \oplus -U)_0$ is biholomorphic, where $\mathcal{D}(U \oplus -U)_0 = \{\eta = [\eta_0, \cdots, \eta_3] \in \mathcal{D}(A) \mid \eta_1^{-1}(\eta_2, \eta_3) \in \mathbb{H} \times \mathbb{H}_-\}$.*

3 Hilbert Modular Group

We first define an extention of $PSL(2, K)$ to construct an isomorphism to $PO(A, \mathbb{R})_0$ and next determine the subgroup which is isomophic to $PO(A, \mathbb{Z})_0$.

Let σ_1, σ_2 be the two isomorphisms from K into $\mathbb{R}$ as follows:

$$\begin{aligned} &\sigma_1 : p + q\sqrt{d} \mapsto p + q\sqrt{d}, \\ &\sigma_2 : p + q\sqrt{d} \mapsto p - q\sqrt{d}, \qquad p, q \in \mathbb{Q}. \end{aligned}$$

We define a distance between $\alpha, \beta \in K$ by $\max\{|\sigma_1(\alpha - \beta)|, |\sigma_2(\alpha - \beta)|\}$. The axioms of distance are obviously satisfied. We denote by $K_\mathbb{R}$ the completion of K in this distance.

Proposition 3.1. (Shimizu [9],p.225) *$K_\mathbb{R}$ is isomorphic to $\mathbb{R} \times \mathbb{R}$.*

[proof]. For any $(\alpha^{(1)}, \alpha^{(2)}) \in \mathbb{R} \times \mathbb{R}$ and any real number $\epsilon > 0$, there exits $\alpha \in K$ such that

$$|\alpha^{(1)} - \sigma_1(\alpha)| < \epsilon, \quad |\alpha^{(2)} - \sigma_2(\alpha)| < \epsilon.$$

For instance, set $\alpha = p + q\sqrt{d}$ with $p, q \in \mathbb{Q}$ such that

$$|p - \frac{\alpha^{(1)} + \alpha^{(2)}}{2}| < \frac{\epsilon}{2}, \quad |q\sqrt{d} - \frac{\alpha^{(1)} - \alpha^{(2)}}{2}| < \frac{\epsilon}{2}.$$

So we can take a Cauchy sequence in K such that the image by isomorphism (σ_1, σ_2) converges to $(\alpha^{(1)}, \alpha^{(2)})$. This means $K_{\mathbb{R}} \cong \mathbb{R} \times \mathbb{R}$.

q.e.d

For any $x \in K$ we denote $\sigma_2(x)$ by $\overline{x}$ and define the trace and the norm by

$$Tr_{K/\mathbb{Q}}(x) = x + \overline{x}, \quad N_{K/\mathbb{Q}}(x) = x\overline{x}.$$

Henceforth we identify $K_{\mathbb{R}}$ with $\mathbb{R} \times \mathbb{R}$ and define the special linear transformation group over $K_{\mathbb{R}}$ as follows:

Definition 3.1. *The special linear transformation group over $K_{\mathbb{R}}$ is defined as*

$$SL(2, K_{\mathbb{R}}) = \left\{ \begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix} \times \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix} \mid \begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix}, \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix} \in SL(2, \mathbb{R}) \right\}.$$

The action of the group on $\mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}$ is defined by

$$\begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix} \times \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix} [(z, w)] = \left(\frac{\alpha_1^{(1)} z + \alpha_2^{(1)}}{\alpha_3^{(1)} z + \alpha_4^{(1)}}, \frac{\alpha_1^{(2)} w + \alpha_2^{(2)}}{\alpha_3^{(2)} w + \alpha_4^{(2)}} \right).$$

Since the actions of $\pm \begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix} \times \pm \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix}$ *are the same, we can define the action of the projective special linear transformation group $PSL(2, K_{\mathbb{R}}) = SL(2, K_{\mathbb{R}})/\{\pm I_2 \times \pm I_2\}$ on $\mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}$.*

We define an action of $P \in PGL(4, \mathbb{R})$ on $\eta \in P^3(\mathbb{C})$ as $P[\eta] = {}^t(P^t\eta)$.
The Segre map ι induces a group homomorphism $\rho : PSL(2, K_{\mathbb{R}}) \to PO(U \oplus -U, \mathbb{R})$ such that $\iota(g[(z, w)]) = \rho(g)[\iota((z, w))]$ for $g \in PSL(2, K_{\mathbb{R}})$ and $(z, w) \in \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}$.

Example 3.1. *(i) We have*

$$g_1 = \begin{bmatrix} \alpha^{(1)} & \beta^{(1)} \\ \gamma^{(1)} & \delta^{(1)} \end{bmatrix} \times I_2 \overset{\rho}{\longmapsto} \begin{bmatrix} \alpha^{(1)} & 0 & 0 & \beta^{(1)} \\ 0 & \delta^{(1)} & \gamma^{(1)} & 0 \\ 0 & \beta^{(1)} & \alpha^{(1)} & 0 \\ \gamma^{(1)} & 0 & 0 & \delta^{(1)} \end{bmatrix},$$

i.e., the diagram

$$\begin{array}{lcl} & (z, w) \overset{\iota}{\longmapsto} [zw, 1, z, w] & \\ g_1 \downarrow & & \downarrow \ \rho(g_1) \\ & (z', w) \overset{\iota}{\longmapsto} [z'w, 1, z', w] = [(\alpha^{(1)}z + \beta^{(1)})w, \gamma^{(1)}z + \delta^{(1)}, \alpha^{(1)}z + \beta^{(1)}, (\gamma^{(1)}z + \delta^{(1)})w] & \end{array}$$

is commutative.

(ii) Also we have

$$g_2 = I_2 \times \begin{bmatrix} \alpha^{(2)} & \beta^{(2)} \\ \gamma^{(2)} & \delta^{(2)} \end{bmatrix} \overset{\rho}{\longmapsto} \begin{bmatrix} \alpha^{(2)} & 0 & \beta^{(2)} & 0 \\ 0 & \delta^{(2)} & 0 & \gamma^{(2)} \\ \gamma^{(2)} & 0 & \delta^{(2)} & 0 \\ 0 & \beta^{(2)} & 0 & \alpha^{(2)} \end{bmatrix},$$

i.e., the diagram

$$\begin{array}{lcl} & (z, w) \overset{\iota}{\longmapsto} [zw, 1, z, w] & \\ g_2 \downarrow & & \downarrow \ \rho(g_2) \\ & (z, w') \overset{\iota}{\longmapsto} [zw', 1, z, w'] = [z(\alpha^{(2)}w + \beta), \gamma^{(2)}w + \delta^{(2)}, z(\gamma^{(2)}w + \delta^{(2)}), \alpha^{(2)}w + \beta^{(2)}] & \end{array}$$

is commutative.

(iii) By taking the product of g_1 and g_2, we obtain

$$g_1g_2 = g_2g_1 = \left[\begin{smallmatrix}\alpha^{(1)} & \beta^{(1)}\\ \gamma^{(1)} & \delta^{(1)}\end{smallmatrix}\right] \times \left[\begin{smallmatrix}\alpha^{(2)} & \beta^{(2)}\\ \gamma^{(2)} & \delta^{(2)}\end{smallmatrix}\right] \overset{\rho}{\longmapsto} \begin{bmatrix}\alpha^{(1)}\alpha^{(2)} & \beta^{(1)}\beta^{(2)} & \alpha^{(1)}\beta^{(2)} & \beta^{(1)}\alpha^{(2)}\\ \gamma^{(1)}\gamma^{(2)} & \delta^{(1)}\delta^{(2)} & \gamma^{(1)}\delta^{(2)} & \delta^{(1)}\gamma^{(2)}\\ \alpha^{(1)}\gamma^{(2)} & \beta^{(1)}\delta^{(2)} & \alpha^{(1)}\delta^{(2)} & \beta^{(1)}\gamma^{(2)}\\ \gamma^{(1)}\alpha^{(2)} & \delta^{(1)}\beta^{(2)} & \gamma^{(1)}\beta^{(2)} & \delta^{(1)}\alpha^{(2)}\end{bmatrix}.$$

Lemma 3.1. *$\rho : PSL(2, K_{\mathbb{R}}) \to PO(U \oplus -U, \mathbb{R})_0$ is an isomorphism.*

[proof]. For surgectivity it is sufficient to show that the generators $H_i(\alpha), i = 3, \cdots, 8$ for $PO(U \oplus -U, \mathbb{R})_0$ are in the image of ρ of $PSL(2, K_{\mathbb{R}})$.

$$\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} \times \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix} \overset{\rho}{\longmapsto} H_3(\alpha),$$

$$\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} \times \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} \overset{\rho}{\longmapsto} H_4(\alpha),$$

$$\begin{bmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{bmatrix} \times \begin{bmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{bmatrix} \overset{\rho}{\longmapsto} H_5(\alpha),$$

$$\begin{bmatrix}\cosh\alpha & -\sinh\alpha\\ -\sinh\alpha & \cosh\alpha\end{bmatrix} \times \begin{bmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{bmatrix} \overset{\rho}{\longmapsto} H_6(\alpha),$$

$$\begin{bmatrix}\cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2} & 0\\ 0 & \cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2}\end{bmatrix} \times \begin{bmatrix}\cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2} & 0\\ 0 & \cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2}\end{bmatrix} \overset{\rho}{\longmapsto} H_7(\alpha),$$

$$\begin{bmatrix}\cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2} & 0\\ 0 & \cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2}\end{bmatrix} \times \begin{bmatrix}\cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2} & 0\\ 0 & \cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2}\end{bmatrix} \overset{\rho}{\longmapsto} H_8(\alpha).$$

Injectivity is immediate since the kernel of ρ is the identity of $PSL(2, K_{\mathbb{R}})$ from Example 3.1, (iii).

q.e.d.

Remark 3.1. *From Example 3.1, (iii), the restriction $\rho : PSL(2, \mathbb{Z}) \times PSL(2, \mathbb{Z}) \to PO(U \oplus -U, \mathbb{Z})_0$ is an isomorphism.*

In order to extend ρ to be surjective on entire $PO(U \oplus -U, \mathbb{R})$, we need the following proposition:

Proposition 3.2. *H_1(H_2) in $PO(U \oplus -U, \mathbb{R})$ is the image by ρ of the following h_1(h_2 resp.) which is an involutive automorphism on $\mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}$:*

$$\begin{array}{rcl} h_1 : \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H} & \longrightarrow & \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}\\ (z, w) & \mapsto & (w, z)\\ h_2 : \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H} & \longrightarrow & \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}\\ (z, w) & \mapsto & (\frac{1}{w}, \frac{1}{z}). \end{array}$$

[proof]. For h_1 the diagram

$$\begin{array}{ccc} (z, w) & \overset{\iota}{\longmapsto} & [zw, 1, z, w]\\ h_1 \downarrow & & \downarrow H_1.\\ (w, z) & \overset{\iota}{\longmapsto} & [zw, 1, w, z] \end{array}$$

is commutative. For h_2 the diagram

$$\begin{array}{ccc} (z, w) & \overset{\iota}{\longmapsto} & [zw, 1, z, w]\\ h_2 \downarrow & & \downarrow H_2.\\ (\frac{1}{w}, \frac{1}{z}) & \overset{\iota}{\longmapsto} & [\frac{1}{zw}, 1, \frac{1}{w}, \frac{1}{z}] = [1, zw, z, w] \end{array}$$

is commutative.

q.e.d.

Note that $h_1, h_2 \notin PSL(2, K)$.

Corollary 3.1. *(i)* $\rho : \langle PSL(2,K_{\mathbb{R}}), h_1, h_2\rangle \to PO(U\oplus -U,\mathbb{R})$ *is an isomorphism.*

(ii) $PO(U\oplus -U),\mathbb{R})$ *is decomposed into four connected components.*

$$PO(U\oplus -U,\mathbb{R}) = \rho(PSL(2,K_{\mathbb{R}}))\cup \rho(PSL(2,K_{\mathbb{R}}))H_1 \\ \cup \rho(PSL(2,K_{\mathbb{R}}))H_2 \cup \rho(PSL(2,K_{\mathbb{R}}))H_1H_2.$$

Proposition 3.3. *For* $\eta \in \mathcal{D}(U\oplus -U)$*, we have* $(I_2\oplus {}^tW)^{-1}[\eta]\in \mathcal{D}(A)$ *and the map* $\kappa : \mathcal{D}(U\oplus -U)\to \mathcal{D}(A)$ *defined by* $\kappa : \eta \mapsto (I_2\oplus {}^tW)^{-1}[\eta]$ *is biholomorphic.*

[proof]. This is immediate from the direct calculation.

q.e.d.

We define $\iota' : \mathbb{H}\times\mathbb{H}_-\cup\mathbb{H}_-\times\mathbb{H}\to\mathcal{D}(A)$ by $\iota' = \kappa\circ\iota$. Then ι' induces a group homomorphism $\rho' : PSL(2,K_{\mathbb{R}})\to PO(A,\mathbb{R})$. We now obtain

Theorem 3.1. $\rho' : PSL(2,K_{\mathbb{R}})\to PO(A,\mathbb{R})_0$ *is an isomorphism.*

Set $H_1' = I_2\oplus({}^tW^{-1}UW),\quad H_2' = U\oplus I_2$.

Corollary 3.2. $PO(A,\mathbb{R})$ *is decomposed into four connected components.*

$$PO(A,\mathbb{R}) = \rho'(PSL(2,K_{\mathbb{R}}))\cup \rho'(PSL(2,K_{\mathbb{R}}))H_1' \\ \cup \rho'(PSL(2,K_{\mathbb{R}}))H_2' \cup \rho'(PSL(2,K_{\mathbb{R}}))H_1'H_2'.$$

Let $\mathcal{O}^*$ be the dual ideal of $\mathcal{O}$ w.r.t. the trace : the maximal subset of K such that $Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*)\subset\mathbb{Z}$. Let $\{w_1', w_2'\}$ be the basis of $\mathcal{O}^*$ dual to $\{w_1, w_2\}$. Put $W' = \begin{bmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'}\end{bmatrix}$. Then we have $W'{}^tW = I_2$.
The condition for the image of ρ' being in $PO(A,\mathbb{Z})_0$ is given in the following theorem.

Theorem 3.2. *Let* Γ *be a subgroup of* $PSL(2,K_{\mathbb{R}})$ *consisting of all elements* $\begin{bmatrix}\alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)}\end{bmatrix}\times\begin{bmatrix}\alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)}\end{bmatrix}$ *satisfying*

(i) $\alpha_k^{(1)}\alpha_k^{(2)}\in\mathbb{Z}\qquad k=1,\cdots,4$ *and*

(ii) $\alpha_k^{(1)}\alpha_l^{(2)} = \overline{\alpha_k^{(2)}\alpha_l^{(1)}}\in\mathcal{O}\qquad k,l=1,\cdots,4\qquad k\neq l.$

Then $\rho' : \Gamma\to PO(A,\mathbb{Z})_0$ *is an isomorphism.*

[proof]. Let

$$g = \begin{bmatrix}\alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)}\end{bmatrix}\times\begin{bmatrix}\alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)}\end{bmatrix}\in PSL(2,K_{\mathbb{R}}).$$

We show that the necessary and sufficient condition for $\iota'_*g\in PO(A,\mathbb{Z})$ is $g\in\Gamma$.

[Necessity]. Examine the condition for

$$\rho(g) = \begin{bmatrix}\alpha_1^{(1)}\alpha_1^{(2)} & \alpha_2^{(1)}\alpha_2^{(2)} & \alpha_1^{(1)}\alpha_2^{(2)} & \alpha_2^{(1)}\alpha_1^{(2)} \\ \alpha_3^{(1)}\alpha_3^{(2)} & \alpha_4^{(1)}\alpha_4^{(2)} & \alpha_3^{(1)}\alpha_4^{(2)} & \alpha_4^{(1)}\alpha_3^{(2)} \\ \alpha_1^{(1)}\alpha_3^{(2)} & \alpha_2^{(1)}\alpha_4^{(2)} & \alpha_1^{(1)}\alpha_4^{(2)} & \alpha_2^{(1)}\alpha_3^{(2)} \\ \alpha_3^{(1)}\alpha_1^{(2)} & \alpha_4^{(1)}\alpha_2^{(2)} & \alpha_3^{(1)}\alpha_2^{(2)} & \alpha_4^{(1)}\alpha_1^{(2)}\end{bmatrix}$$

to satisfy

$$\rho'(g) = \begin{bmatrix} I_2 & 0 \\ 0 & {}^tW\end{bmatrix}^{-1}\rho(g)\begin{bmatrix} I_2 & 0 \\ 0 & {}^tW\end{bmatrix} = [n_{ij}]_{i,j=1,\cdots,4}\in M(4,\mathbb{Z}).$$

We divide $\rho(g)$ into four 2×2 blocks.
(1) For the left upper block, we have

$$\begin{bmatrix}\alpha_1^{(1)}\alpha_1^{(2)} & \alpha_2^{(1)}\alpha_2^{(2)} \\ \alpha_3^{(1)}\alpha_3^{(2)} & \alpha_4^{(1)}\alpha_4^{(2)}\end{bmatrix}\in M(2,\mathbb{Z}).$$

(2) For the right upper block, we have

$$\begin{bmatrix} \alpha_1^{(1)}\alpha_2^{(2)} & \alpha_1^{(2)}\alpha_2^{(1)} \\ \alpha_3^{(1)}\alpha_4^{(2)} & \alpha_3^{(2)}\alpha_4^{(1)} \end{bmatrix} = \begin{bmatrix} n_{13} & n_{14} \\ n_{23} & n_{24} \end{bmatrix} W'$$
$$= \begin{bmatrix} n_{13}w_1' + n_{14}w_2' & n_{13}\overline{w_1'} + n_{14}\overline{w_2'} \\ n_{23}w_1' + n_{24}w_2' & n_{23}\overline{w_1'} + n_{24}\overline{w_2'} \end{bmatrix}.$$

Hence $\alpha_1^{(1)}\alpha_2^{(2)} \in \mathcal{O}^*, \alpha_1^{(2)}\alpha_2^{(1)} \in \mathcal{O}^*$, and $\alpha_1^{(1)}\alpha_2^{(2)} = \overline{\alpha_1^{(2)}\alpha_2^{(1)}}$.
So $Tr_{K/\mathbb{Q}}(\alpha_1^{(1)}\alpha_2^{(2)}) = \alpha_1^{(1)}\alpha_2^{(2)} + \alpha_1^{(2)}\alpha_2^{(1)} \in \mathbb{Z}$, and from (1), $N_{K/\mathbb{Q}}(\alpha_1^{(1)}\alpha_2^{(2)}) = \alpha_1^{(1)}\alpha_2^{(2)}\alpha_1^{(2)}\alpha_2^{(1)} \in \mathbb{Z}$.
Therefore we have

$$\alpha_1^{(1)}\alpha_2^{(2)} = \overline{\alpha_1^{(2)}\alpha_2^{(1)}} \in \mathcal{O}, \quad \text{and likewise} \quad \alpha_3^{(1)}\alpha_4^{(2)} = \overline{\alpha_3^{(2)}\alpha_4^{(1)}} \in \mathcal{O}.$$

(3) For the left lower block, we have

$$\begin{bmatrix} \alpha_1^{(1)}\alpha_3^{(2)} & \alpha_2^{(1)}\alpha_4^{(2)} \\ \alpha_1^{(2)}\alpha_3^{(1)} & \alpha_2^{(2)}\alpha_4^{(1)} \end{bmatrix} = {}^tW \begin{bmatrix} n_{31} & n_{32} \\ n_{41} & n_{42} \end{bmatrix}$$
$$= \begin{bmatrix} n_{31}w_1 + n_{41}w_2 & n_{32}w_1 + n_{42}w_2 \\ n_{31}\overline{w_1} + n_{41}\overline{w_2} & n_{32}\overline{w_1} + n_{42}\overline{w_2} \end{bmatrix}$$

Hence

$$\alpha_1^{(1)}\alpha_3^{(2)} = \overline{\alpha_1^{(2)}\alpha_3^{(1)}} \in \mathcal{O}, \quad \alpha_2^{(1)}\alpha_4^{(2)} = \overline{\alpha_2^{(2)}\alpha_4^{(1)}} \in \mathcal{O}.$$

(4)For the right lower block, we have

$$\begin{bmatrix} \alpha_1^{(1)}\alpha_4^{(2)} & \alpha_2^{(1)}\alpha_3^{(2)} \\ \alpha_2^{(2)}\alpha_3^{(1)} & \alpha_1^{(2)}\alpha_4^{(1)} \end{bmatrix} = {}^tW \begin{bmatrix} n_{33} & n_{34} \\ n_{43} & n_{44} \end{bmatrix} W'$$
$$= \begin{bmatrix} n_{33}w_1w_1' + n_{43}w_2w_1' + n_{34}w_1w_2' + n_{44}w_2w_2' & n_{33}w_1\overline{w_1'} + n_{43}w_2\overline{w_1'} + n_{34}w_1\overline{w_2'} + n_{44}w_2\overline{w_2'} \\ n_{33}\overline{w_1}w_1' + n_{43}\overline{w_2}w_1' + n_{34}\overline{w_1}w_2' + n_{44}\overline{w_2}w_2' & n_{33}\overline{w_1}\overline{w_1'} + n_{43}\overline{w_2}\overline{w_1'} + n_{34}\overline{w_1}\overline{w_2'} + n_{44}\overline{w_2}\overline{w_2'} \end{bmatrix}.$$

Hence $\alpha_1^{(1)}\alpha_4^{(2)} = \overline{\alpha_1^{(2)}\alpha_4^{(1)}} \in \mathcal{O}^*$, $Tr_{K/\mathbb{Q}}(\alpha_1^{(1)}\alpha_4^{(2)}) \in \mathbb{Z}$, and from (1), $N_{K/\mathbb{Q}}(\alpha_1^{(1)}\alpha_4^{(2)}) \in \mathbb{Z}$.
So

$$\alpha_1^{(1)}\alpha_4^{(2)} = \overline{\alpha_1^{(2)}\alpha_4^{(1)}} \in \mathcal{O}, \quad \text{and likewise} \quad \alpha_2^{(1)}\alpha_3^{(2)} = \overline{\alpha_2^{(2)}\alpha_3^{(1)}} \in \mathcal{O}.$$

[Sufficiency]. It is obvious as follows:

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} \alpha_1^{(1)}\alpha_1^{(2)} & \alpha_2^{(1)}\alpha_2^{(2)} \\ \alpha_3^{(1)}\alpha_3^{(2)} & \alpha_4^{(1)}\alpha_4^{(2)} \end{bmatrix} \in M(2,\mathbb{Z}),$$
$$\begin{bmatrix} n_{13} & n_{14} \\ n_{23} & n_{24} \end{bmatrix} = \begin{bmatrix} \alpha_1^{(1)}\alpha_2^{(2)}w_1 + \alpha_1^{(2)}\alpha_2^{(1)}\overline{w_1} & \alpha_1^{(1)}\alpha_2^{(2)}w_2 + \alpha_1^{(2)}\alpha_2^{(1)}\overline{w_2} \\ \alpha_3^{(1)}\alpha_4^{(2)}w_1 + \alpha_3^{(2)}\alpha_4^{(1)}\overline{w_1} & \alpha_3^{(1)}\alpha_4^{(2)}w_2 + \alpha_3^{(2)}\alpha_4^{(1)}\overline{w_2} \end{bmatrix} \in M(2, Tr_{K/\mathbb{Q}}(\mathcal{O})),$$
$$\begin{bmatrix} n_{31} & n_{32} \\ n_{41} & n_{42} \end{bmatrix} = \begin{bmatrix} \alpha_1^{(1)}\alpha_3^{(2)}w_1' + \alpha_1^{(2)}\alpha_3^{(1)}\overline{w_1'} & \alpha_2^{(1)}\alpha_4^{(2)}w_1' + \alpha_2^{(2)}\alpha_4^{(1)}\overline{w_1'} \\ \alpha_1^{(1)}\alpha_3^{(2)}w_2' + \alpha_1^{(2)}\alpha_3^{(1)}\overline{w_2'} & \alpha_2^{(1)}\alpha_4^{(2)}w_2' + \alpha_2^{(2)}\alpha_4^{(1)}\overline{w_2'} \end{bmatrix} \in M(2, Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*)),$$
$$n_{33} = \alpha_1^{(1)}\alpha_4^{(2)}w_1'w_1 + \alpha_1^{(2)}\alpha_4^{(1)}\overline{w_1'}\overline{w_1} + \alpha_2^{(1)}\alpha_3^{(2)}w_1'\overline{w_1} + \alpha_2^{(2)}\alpha_3^{(1)}\overline{w_1'}w_1 \in Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*),$$
$$n_{34} = \alpha_1^{(1)}\alpha_4^{(2)}w_2'w_1 + \alpha_1^{(2)}\alpha_4^{(1)}\overline{w_2'}\overline{w_1} + \alpha_2^{(1)}\alpha_3^{(2)}w_2'\overline{w_1} + \alpha_2^{(2)}\alpha_3^{(1)}\overline{w_2'}w_1 \in Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*),$$
$$n_{43} = \alpha_1^{(1)}\alpha_4^{(2)}w_1'w_2 + \alpha_1^{(2)}\alpha_4^{(1)}\overline{w_1'}\overline{w_2} + \alpha_2^{(1)}\alpha_3^{(2)}w_1'\overline{w_2} + \alpha_2^{(2)}\alpha_3^{(1)}\overline{w_1'}w_2 \in Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*),$$
$$n_{44} = \alpha_1^{(1)}\alpha_4^{(2)}w_2'w_2 + \alpha_1^{(2)}\alpha_4^{(1)}\overline{w_2'}\overline{w_2} + \alpha_2^{(1)}\alpha_3^{(2)}w_2'\overline{w_2} + \alpha_2^{(2)}\alpha_3^{(1)}\overline{w_2'}w_2 \in Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*).$$

q.e.d.

Corollary 3.3. *We have*

$$\begin{aligned} \mathcal{D}(A)/PO(A,\mathbb{Z}) &\cong \mathbb{H}\times\mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}/\langle \Gamma, h_1, h_2\rangle \\ &\cong \mathbb{H}\times\mathbb{H}_-/\langle\Gamma, h_2\rangle, \\ \mathcal{D}(A)/PO(A,\mathbb{Z})_0 &\cong \mathbb{H}\times\mathbb{H}_- \cup \mathbb{H}_-\times\mathbb{H}/\Gamma, \\ \mathcal{D}(A)_0/PO(A,\mathbb{Z})_0 &\cong \mathbb{H}\times\mathbb{H}_-/\Gamma. \end{aligned}$$

Definition 3.2. *We define the following groups as subgroups of $PSL(2, K_{\mathbb{R}})$:*

$$PSL(2,K) = \left\{ \begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix} \times \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix} \in PSL(2,K_{\mathbb{R}}) \mid \alpha_k^{(1)} = \overline{\alpha_k^{(2)}} \in K \qquad k=1,\cdots,4 \right\},$$
$$PSL(2,\mathcal{O}) = \left\{ \begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix} \times \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix} \in PSL(2,K_{\mathbb{R}}) \mid \alpha_k^{(1)} = \overline{\alpha_k^{(2)}} \in \mathcal{O} \qquad k=1,\cdots,4 \right\}.$$

Note that the above definitions are equivalent to the conventional ones.
The condition to be a member of Γ is restated for the case $h(K) = 1$ where $\mathcal{O}$ is a PID and so a UFD. Thus any $\alpha \in K$ is factored as

$$\alpha = \pm\varepsilon^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s},$$

where $p_1, .., p_s$ are distinct primes in $\mathcal{O}$ and $n_t \in \mathbb{Z}(0 \le t \le s)$. The factorization is unique up to the order of the primes and the redundance of $p_t^{n_t}$ with $n_t = 0$.

Definition 3.3. *Let n be a non zero integer. We use the notation $p_t^n|\alpha$ for any $\alpha \in K$, if and only if $p_t^{n_t}$ occurs in the above factorization with $0 < n \le n_t$ or with $n_t \le n < 0$. We call α square free if and only if $|n_t| \le 1(0 \le t \le s)$.*

Lemma 3.2. *Assume $h(K) = 1$. Then $PSL(2, O) = PSL(2, K) \cap \Gamma$.*

[proof]. Suppose $g = \left[\begin{smallmatrix}\alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4\end{smallmatrix}\right] \times \left[\begin{smallmatrix}\overline{\alpha_1} & \overline{\alpha_2} \\ \overline{\alpha_3} & \overline{\alpha_4}\end{smallmatrix}\right] \in PSL(2,K)$ is in Γ ,i.e.

(i) $\alpha_k\overline{\alpha_k} \in \mathbb{Z} \qquad k = 1, \cdots, 4 \quad$ *and*

(ii) $\alpha_k\overline{\alpha_l} \in \mathcal{O} \qquad k, l = 1, \cdots, 4 \qquad k \ne l.$

Then we only need to show that g is in $PSL(2, \mathcal{O})$. The converse is apparent.

(i) Take a prime $p \in \mathbb{Z}$. Assume $\alpha_1 = p^{-n_1}\alpha_1'$ with n_1 a positive integer and $p \not|\alpha_1'$. Then we have a contradiction:

$$\mathbb{Z} \ni \alpha_1\overline{\alpha_1} = p^{-2n_1}\alpha_1'\overline{\alpha_1'} \notin \mathbb{Z}.$$

(ii) Take a prime $p \notin \mathbb{Z}$. Assume $\alpha_1 = p^{-n_1}\alpha_1$ with n_1 a positive integer, $\quad p \not|\alpha_1'$. Then

$$\begin{aligned} &\mathbb{Z} \ni \alpha_1\overline{\alpha_1} = p^{-n_1}\overline{p}^{-n_1}\alpha_1'\overline{\alpha_1'}, \quad \text{so } \ \alpha_1' = \overline{p}^{n_1}\alpha_1'' \ \text{ with } \ \overline{p}^{-1} \not|\alpha_1'', \\ &\mathcal{O} \ni \alpha_1\overline{\alpha_2} = p^{-n_1}\alpha_1'\overline{\alpha_2}, \quad \text{so } \ \alpha_2 = \overline{p}^{n_1}\alpha_2' \ \text{ with } \ \overline{p}^{-1} \not|\alpha_2', \\ &\mathcal{O} \ni \alpha_1\overline{\alpha_3} = p^{-n_1}\alpha_1'\overline{\alpha_3}, \quad \text{so } \ \alpha_3 = \overline{p}^{n_1}\alpha_3' \ \text{ with } \ \overline{p}^{-1} \not|\alpha_3', \\ &\mathcal{O} \ni \alpha_1\overline{\alpha_4} = p^{-n_1}\alpha_1'\overline{\alpha_4} \ \text{ and} \\ &\mathcal{O} \ni \alpha_1\alpha_4 = p^{-n_1}\alpha_1'\alpha_4, \quad \text{so } \ \alpha_4 = p^{n_1}\overline{p}^{n_1}\alpha_4' \ \text{ with } \ p^{-1}\overline{p}^{-1} \not|\alpha_4'. \end{aligned}$$

Moreover

$$1 = \alpha_1\alpha_4 - \alpha_2\alpha_3 = (p^{-n_1}\overline{p}^{n_1}\alpha_1'')(p^{n_1}\overline{p}^{n_1}\alpha_4') - (\overline{p}^{n_1}\alpha_2')(\overline{p}^{n_1}\alpha_3') = \overline{p}^{2n_1}(\alpha_1''\alpha_4' - \alpha_2'\alpha_3').$$

Since $\overline{p}^{-1} \not|\alpha_1''\alpha_4' - \alpha_2'\alpha_3', \quad$ we have $n_1 = 0$ which is a contradiction.

From 1 and 2, $\alpha_1 \in \mathcal{O}$. The situation is the same for $\alpha_k (k = 2, 3, 4)$.

q.e.d.

When $N_{K/\mathbb{Q}}(\varepsilon) = 1$, we put

$$g_\varepsilon = \begin{bmatrix} \sqrt{\varepsilon} & 0 \\ 0 & \sqrt{\varepsilon} \end{bmatrix} \times \begin{bmatrix} \sqrt{\bar{\varepsilon}} & 0 \\ 0 & \sqrt{\bar{\varepsilon}} \end{bmatrix}.$$

Note that $g_\varepsilon \in \Gamma$.

Lemma 3.3. *Assume* $h(K) = 1$.

(i) If $N_{K/\mathbb{Q}}(\varepsilon) = -1$, *then* $\Gamma = PSL(2, O)$*;*

(ii) If $N_{K/\mathbb{Q}}(\varepsilon) = 1$, *then* $\Gamma = \langle PSL(2, O), g_\varepsilon \rangle$.

[proof]. Let $g = \begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix} \times \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix} \in SL(2, K_\mathbb{R})$. We only need to show that the necessary condition for g being in Γ is

(i) when $N_{K/\mathbb{Q}}(\varepsilon) = -1$, $\quad g \in PSL(2, \mathcal{O})$,

(ii) when $N_{K/\mathbb{Q}}(\varepsilon) = 1$, $\quad g \in \langle PSL(2, \mathcal{O}), g_\varepsilon \rangle$.

(Sufficiency is evident since $PSL(2, \mathcal{O}) \subset \Gamma$ and $g_\varepsilon \in \Gamma$.)
Let $i, j \in \{1, 2\}$ with $i \neq j$, $\quad k, l \in \{1, \cdots, 4\}$ with $k \neq l$.
From Theorem 3.1, $\alpha_k^{(1)} \alpha_k^{(2)} = r_k \in \mathbb{Z}$ and $\alpha_k^{(1)} \alpha_l^{(2)} \in K$.
If $\alpha_k^{(j)} \neq 0$, then $K \ni \alpha_k^{(i)} \alpha_l^{(j)} = \frac{r_k}{\alpha_k^{(j)}} \alpha_l^{(j)}$, so

$$\frac{\alpha_l^{(j)}}{\alpha_k^{(j)}} \in K. \tag{3.1}$$

Assume here $\alpha_1^{(j)} \neq 0$. Then there exists $\gamma_l^{(j)} \in K$ such that

$$\alpha_l^{(j)} = \gamma_l^{(j)} \alpha_1^{(j)}, \quad \text{for} \quad l = 2, 3, 4.$$

So we have

$$1 = \alpha_1^{(j)} \alpha_4^{(j)} - \alpha_2^{(j)} \alpha_3^{(j)} = (\gamma_4^{(j)} - \gamma_2^{(j)} \gamma_3^{(j)})(\alpha_1^{(j)})^2.$$

Hence $(\alpha_1^{(j)})^2 \in K$. Thus for any $k \in \{1, \cdots, 4\}$

$$(\alpha_k^{(j)})^2 \in K. \tag{3.2}$$

From (3.1) and (3.2) we have

$$\alpha_k^{(j)} \alpha_l^{(j)} \in K. \tag{3.3}$$

Since $\alpha_1^{(j)} \alpha_4^{(j)} - \alpha_2^{(j)} \alpha_3^{(j)} = 1$ and from the hypothesis, we have

$$\begin{aligned} \mathbb{Z} \ni \alpha_2^{(i)} \alpha_2^{(j)} \alpha_3^{(i)} \alpha_3^{(j)} &= (\alpha_1^{(i)} \alpha_4^{(i)} - 1)(\alpha_1^{(j)} \alpha_4^{(j)} - 1) \\ &= \alpha_1^{(i)} \alpha_1^{(j)} \alpha_4^{(i)} \alpha_4^{(j)} + 1 - (\alpha_1^{(i)} \alpha_4^{(i)} + \alpha_1^{(j)} \alpha_4^{(j)}). \end{aligned}$$

And from the hypothesis

$$N_{K/\mathbb{Q}}(\alpha_1^{(i)} \alpha_4^{(i)}) = \alpha_1^{(i)} \alpha_4^{(i)} \alpha_1^{(j)} \alpha_4^{(j)} \in \mathbb{Z}. \tag{3.4}$$

Hence

$$Tr_{k/\mathbb{Q}}(\alpha_1^{(i)} \alpha_4^{(i)}) = \alpha_1^{(i)} \alpha_4^{(i)} + \alpha_1^{(j)} \alpha_4^{(j)} \in \mathbb{Z}. \tag{3.5}$$

From (3.3),(3.4), and (3.5), we have

$$\alpha_1^{(i)}\alpha_4^{(i)} \in \mathcal{O}, \quad \text{and likewise} \;\; \alpha_2^{(i)}\alpha_3^{(i)} \in \mathcal{O}.$$

Since $(\alpha_k^{(i)})^2 \in K$, we may write

$$\alpha_k^{(i)} = \sqrt{\beta_k^{(i)}}\alpha_k'^{(i)} \quad \text{where} \;\; \beta_k^{(i)}, \alpha_k'^{(i)} \in K, \;\; \text{with} \;\; \beta_k^{(i)} > 0.$$

Summarise what we obtained so far as follows:

product	$\sqrt{\beta_1^{(2)}}\alpha_1'^{(2)}$	$\sqrt{\beta_2^{(2)}}\alpha_2'^{(2)}$	$\sqrt{\beta_3^{(2)}}\alpha_3'^{(2)}$	$\sqrt{\beta_4^{(2)}}\alpha_4'^{(2)}$	$\sqrt{\beta_1^{(1)}}\alpha_1'^{(1)}$	$\sqrt{\beta_2^{(1)}}\alpha_2'^{(1)}$	$\sqrt{\beta_3^{(1)}}\alpha_3'^{(1)}$	$\sqrt{\beta_4^{(1)}}\alpha_4'^{(1)}$
$\sqrt{\beta_1^{(1)}}\alpha_1'^{(1)}$	$\mathbb{Z}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	K	K	K	$\mathcal{O}$
$\sqrt{\beta_2^{(1)}}\alpha_2'^{(1)}$	$\mathcal{O}$	$\mathbb{Z}$	$\mathcal{O}$	$\mathcal{O}$	K	K	$\mathcal{O}$	K
$\sqrt{\beta_3^{(1)}}\alpha_3'^{(1)}$	$\mathcal{O}$	$\mathcal{O}$	$\mathbb{Z}$	$\mathcal{O}$	K	$\mathcal{O}$	K	K
$\sqrt{\beta_4^{(1)}}\alpha_4'^{(1)}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	$\mathbb{Z}$	$\mathcal{O}$	K	K	K

As seen from the table, we may assume without loss of generality

$$\beta_k^{(i)} = \beta^{(i)} \in K, \;\; \beta^{(j)} = (\beta^{(i)})^{-1}.$$

Then $\alpha_k'^{(i)}\alpha_k'^{(j)} = \alpha_k^{(i)}\alpha_k^{(j)} = r_k \in \mathbb{Z}$ and

$$\alpha_k'^{(j)} = \frac{r_k}{N_{K/\mathbb{Q}}(\alpha_k'^{(i)})}\overline{\alpha_k'^{(i)}}.$$

Since $\alpha_k'^{(i)}\alpha_l'^{(j)} = \overline{\alpha_k'^{(j)}\alpha_l'^{(i)}}$, we have $\frac{r_l}{N_{K/\mathbb{Q}}(\alpha_l'^{(i)})}\alpha_k'^{(i)}\overline{\alpha_l'^{(i)}} = \frac{r_k}{N_{K/\mathbb{Q}}(\alpha_k'^{(i)})}\alpha_k'^{(i)}\overline{\alpha_l'^{(i)}}$. So we can write

$$\frac{r_k}{N_{K/\mathbb{Q}}(\alpha_k'^{(i)})} = c^{(i)} \in \mathbb{Q}.$$

Since $\alpha_1^{(i)}\alpha_4^{(i)} - \alpha_2^{(i)}\alpha_3^{(i)} = 1$ for $i \in \{1,2\}$, we have

$$\beta^{(i)}(\alpha_1'^{(i)}\alpha_4'^{(i)} - \alpha_2'^{(i)}\alpha_3'^{(i)}) = 1,$$
$$(\beta^{(i)})^{-1}(c^{(i)})^2(\overline{\alpha_1'^{(i)}\alpha_4'^{(i)}} - \overline{\alpha_2'^{(i)}\alpha_3'^{(i)}}) = 1.$$

Taking the conjugate of the latter expression and comparing with the former we have

$$(c^{(i)})^2 = \beta^{(i)}\overline{\beta^{(i)}} > 0.$$

So we have

$$\overline{\beta^{(i)}} > 0 \quad \text{and} \quad c^{(i)} = \pm\sqrt{\beta^{(i)}\overline{\beta^{(i)}}}.$$

Then we have by definition

$$\alpha_k^{(i)} = \sqrt{\beta^{(i)}}\alpha_k'^{(i)},$$
$$\alpha_k^{(j)} = (\sqrt{\beta^{(i)}})^{-1}c^{(i)}\overline{\alpha_k'^{(i)}} = \pm\sqrt{\overline{\beta^{(i)}}}\overline{\alpha_k'^{(i)}}.$$

So we may write

$$\alpha_k^{(1)} = \sqrt{\beta}\alpha_k', \quad \alpha_k^{(2)} = \pm\sqrt{\overline{\beta}}\overline{\alpha_k'} \quad \text{where} \;\; \beta, \alpha_k' \in K, \;\; \beta, \overline{\beta} > 0 \;\; \text{and} \;\; \beta \;\; \text{is square free.} \tag{3.6}$$

We may assume β and α_k' are factorized as follows:

$$\begin{aligned} \beta &= \varepsilon^{\delta_0} p_1^{\delta_1} \cdots p_s^{\delta_s} \overline{p_1}^{\delta_1'} \cdots \overline{p_s}^{\delta_s'}, \\ \alpha_k' &= \pm\varepsilon^{n_{k0}} p_1^{n_{k1}} \cdots p_s^{n_{ks}} \overline{p_1}^{n_{k1}'} \cdots \overline{p_s}^{n_{ks}'}, \\ &\text{where} \;\; p_1, \cdots, p_s \;\; \text{are distinct primes,} \;\; \delta_t, \delta_t' = 0,1 \;\; (0 \le t \le s), \;\; n_{kt}, n_{kt}' \in \mathbb{Z} \;\; (0 \le t \le s), \\ &\delta_t' = n_{kt}' = 0 \;\; \text{for} \;\; p_t \in \mathbb{Z}. \end{aligned} \tag{3.7}$$

Since $\overline{\beta} > 0$, $\delta_0 = 0$ if $N_{K/\mathbb{Q}}(\varepsilon) = 1$. Since $\alpha_k^{(1)}\alpha_k^{(2)} \in \mathbb{Z}$, we have

$$\sqrt{\beta\overline{\beta}}\alpha'_k\overline{\alpha'_k} = \sqrt{(p_1\overline{p_1})^{\delta_1+\delta'_1}\cdots(p_s\overline{p_s})^{\delta_s+\delta'_s}}(p_1\overline{p_1})^{n_{k1}+n'_{k1}}\cdots(p_s\overline{p_s})^{n_{ks}+n'_{ks}} \in \mathbb{Z}. \tag{3.8}$$

Therefore for any t $(1 \leq t \leq s)$ we have the following cases:

(1) If $p_t \notin \mathbb{Z}$ and $\delta_t\delta'_t = 0$, then since $p_t\overline{p_t}$ is square free in $\mathbb{Z}$, $\delta_t = \delta'_t = 0$;

(2) If $p_t \notin \mathbb{Z}$ and $\delta_t = \delta'_t = 1$, then

$$1 = \alpha_1^{(1)}\alpha_4^{(1)} - \alpha_2^{(1)}\alpha_3^{(1)} = (p_t\overline{p_t})\cdots(\alpha_1'^{(1)}\alpha_4'^{(1)} - \alpha_2'^{(1)}\alpha_3'^{(1)})$$

which give rise to one of the following three cases

(A) $(p_t\overline{p_t})^{-1}|\alpha_1'^{(1)}\alpha_4'^{(1)}$ or $(p_t\overline{p_t})^{-1}|\alpha_2'^{(1)}\alpha_3'^{(1)}$,
(B) $p_t^{-1}|\alpha_1'^{(1)}\alpha_4'^{(1)}$ and $\overline{p_t}^{-1}|\alpha_2'^{(1)}\alpha_3'^{(1)}$,
(C) $\overline{p_t}^{-1}|\alpha_1'^{(1)}\alpha_4'^{(1)}$ and $p_t^{-1}|\alpha_2'^{(1)}\alpha_3'^{(1)}$.

These can be reduced to the following cases without loss of generality.

(a) $p_t^{-1}\overline{p_t}^{-1}|\alpha'_1$,
(b) $p_t^{-1}|\alpha'_1$ and $\overline{p_t}^{-1}|\alpha'_4$,
(c) $p_t^{-1}|\alpha'_1$ and $\overline{p_t}^{-1}|\alpha'_2$.

In case (a), we may write $\alpha_1 = p_t^{-1}\overline{p_t}^{-1}\alpha''_1$ where $\alpha''_1 \in K$ with $p_t\overline{p_t} \nmid \alpha''_1$ and have

$$\alpha_1^{(1)} = \sqrt{p_t\overline{p_t}\cdots}p_t^{-1}\overline{p_t}^{-1}\alpha''_1.$$

Then we have a contradiction:

$$\mathbb{Z} \ni \alpha_1^{(1)}\alpha_1^{(2)} = \pm p_t\overline{p_t}\cdots(p_t^{-1}\overline{p_t}^{-1})^2\alpha''_1\overline{\alpha''_1} \in\notin \mathbb{Z}.$$

In case (b), we may write $\alpha'_1 = \overline{p_t}^{-1}\alpha''_1, \alpha'_4 = \overline{p_t}^{-1}\alpha''_4$ where $\alpha''_1, \alpha''_4 \in K$ with $p_t \nmid \alpha''_1, \overline{p_t} \nmid \alpha''_4$ and have

$$\alpha_1^{(1)} = \sqrt{p_t\overline{p_t}\cdots}p_t^{-1}\alpha''_1,$$
$$\alpha_4^{(2)} = \pm\sqrt{p_t\overline{p_t}\cdots}p_t^{-1}\overline{\alpha''_4}.$$

Then we have a contradiction:

$$\mathcal{O} \ni \alpha_1^{(1)}\alpha_4^{(2)} = \pm(p_t\overline{p_t})\cdots p_t^{-2}\alpha''_1\overline{\alpha''_4} \notin \mathcal{O}.$$

In case (c), we may write $\alpha'_2 = \overline{p_t}^{-1}\alpha''_2$ where $\alpha''_2 \in K$ with $\overline{p_t}|\alpha''_2$ and have

$$\alpha_1^{(1)} = \sqrt{p_t\overline{p_t}\cdots}p_t^{-1}\alpha''_1,$$
$$\alpha_2^{(2)} = \pm\sqrt{p_t\overline{p_t}\cdots}p_t^{-1}\overline{\alpha''_2}.$$

Then we have a contradiction:

$$\mathcal{O} \ni \alpha_1^{(1)}\alpha_2^{(2)} = \pm(p_t\overline{p_t})\cdots p_t^{-2}\alpha''_1\overline{\alpha''_2} \notin \mathcal{O}.$$

Therefore the case $p_t \notin \mathbb{Z}$ and $\delta_t = \delta'_t = 1$ is impossible.

(3) If $p_t \in \mathbb{Z}$, then $\alpha_k^{(1)} = \sqrt{p_t \cdots} \alpha'_k$ and

$$1 = \alpha_1^{(1)}\alpha_4^{(1)} - \alpha_2^{(1)}\alpha_3^{(1)} = p_t \cdots (\alpha'_1\alpha'_4 - \alpha'_2\alpha'_3).$$

Hence $p_t^{-1} | \alpha'_1\alpha'_4$ or $p_t^{-1} | \alpha'_2\alpha'_3$. It is sufficient to examine the case $p_t^{-1} | \alpha'_1$. In this case we have $\alpha'_1 = p_t^{-1}\alpha''_1$, with $p_t \not| \alpha''_1$. So we have a contradiction:

$$\mathbb{Z} \ni \alpha_1^{(1)}\alpha_1^{(2)} = \pm p_t \cdots p_t^{-2}\alpha''_1\overline{\alpha''_1} \notin \mathbb{Z}.$$

Therefore $p_t \notin \mathbb{Z}$.

From (1),(2) and (3), we now know $\beta = \varepsilon^{\delta_0}$. From (3.6), we have

$$\begin{aligned} g &= \begin{bmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} \\ \alpha_3^{(1)} & \alpha_4^{(1)} \end{bmatrix} \times \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} \\ \alpha_3^{(2)} & \alpha_4^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} (\sqrt{\varepsilon})^{\delta_0}\alpha'_1 & (\sqrt{\varepsilon})^{\delta_0}\alpha'_2 \\ \sqrt{\varepsilon})^{\delta_0}\alpha'_3 & (\sqrt{\varepsilon})^{\delta_0}\alpha'_4 \end{bmatrix} \times \begin{bmatrix} \pm(\sqrt{\overline{\varepsilon}})^{\delta_0}\overline{\alpha'_1} & \pm(\sqrt{\varepsilon\overline{\varepsilon}})^{\delta_0}\overline{\alpha'_2} \\ \pm(\sqrt{\overline{\varepsilon}})^{\delta_0}\overline{\alpha'_3} & \pm(\sqrt{\varepsilon\overline{\varepsilon}})^{\delta_0}\overline{\alpha'_4} \end{bmatrix} \\ &= g_1 g_\varepsilon^{\delta_0} g_c, \end{aligned}$$

where

$$g_1 = \begin{bmatrix} \alpha'_1 & \varepsilon^{\delta_0}\alpha'_2 \\ \alpha'_3 & \varepsilon^{\delta_0}\alpha'_4 \end{bmatrix} \times \begin{bmatrix} \overline{\alpha'_1} & \overline{\varepsilon}^{\delta_0}\overline{\alpha'_2} \\ \overline{\alpha'_3} & \overline{\varepsilon}^{\delta_0}\overline{\alpha'_4} \end{bmatrix}, \quad g_\varepsilon = \begin{bmatrix} \sqrt{\varepsilon} & 0 \\ 0 & \sqrt{\overline{\varepsilon}} \end{bmatrix} \times \begin{bmatrix} \sqrt{\overline{\varepsilon}} & 0 \\ 0 & \sqrt{\varepsilon} \end{bmatrix},$$

$$g_c = I_2 \times (\pm I_2).$$

Since $g, g_\varepsilon, g_c \in \Gamma$, we have $g_1 \in \Gamma$. So from Lemma 3.2, $g_1 \in PSL(2, \mathcal{O})$. By definition if $N_{K/\mathbb{Q}}(\varepsilon) = -1$, $\delta_0 = 0$. g_c is the identity in $PSL(2, K_\mathbb{R})$.

q.e.d.

Remark 3.2. *When $h(K) = 1$ and $N_{K/\mathbb{Q}}(\varepsilon) = 1$, it would be enticing to use the notation $PSL(2, \mathcal{O}(\sqrt{\varepsilon}))$ for Γ. However it is inappropriate:*
Assume $h = \begin{bmatrix} 1+\sqrt{\varepsilon} & 1+2\sqrt{\varepsilon} \\ 4+3\sqrt{\varepsilon} & 5+6\sqrt{\varepsilon} \end{bmatrix} \times \begin{bmatrix} \alpha+\beta\sqrt{\varepsilon} & \gamma+\delta\sqrt{\varepsilon} \\ * & * \end{bmatrix} \in \Gamma.$
Since $h \in \Gamma$, we have

$$\begin{aligned} &\mathbb{Z} \ni (1+\sqrt{\varepsilon})(\alpha+\beta\sqrt{\varepsilon}) = * + (\alpha+\beta)\sqrt{\varepsilon}, \quad \textit{so} \quad \beta = -\alpha, \\ &\mathcal{O} \ni (1+2\sqrt{\varepsilon})(\alpha-\alpha\sqrt{\varepsilon}) = * + \alpha\sqrt{\varepsilon}, \quad \textit{so} \quad \alpha = 0, \\ &\mathbb{Z} \ni (1+2\sqrt{\varepsilon})(\gamma+\delta\sqrt{\varepsilon}) = * + (2\gamma+\delta)\sqrt{\varepsilon}, \quad \textit{so} \quad \delta = -2\gamma, \\ &\mathcal{O} \ni (1+\sqrt{\varepsilon})(\gamma-2\gamma\sqrt{\varepsilon}) = * - \gamma\sqrt{\varepsilon}, \quad \textit{so} \quad \gamma = 0. \end{aligned}$$

Thus we have a contradiction:

$$h = \begin{bmatrix} 1+\sqrt{\varepsilon} & 1+2\sqrt{\varepsilon} \\ 4+3\sqrt{\varepsilon} & 5+6\sqrt{\varepsilon} \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} \notin \Gamma.$$

Theorem3.2 and Lemma 3.3 yield the main theorem:

Theorem 3.3. *Assume $h(K) = 1$.*

(i) If $N_{K/\mathbb{Q}}(\varepsilon) = -1$, then $\rho' : PSL(2, \mathcal{O}) \to PO(A, \mathbb{Z})_0$ is an isomorphism;

(ii) If $N_{K/\mathbb{Q}}(\varepsilon) = 1$, then $\rho' : \langle PSL(2, \mathcal{O}), g_\varepsilon \rangle \to PO(A, \mathbb{Z})_0$ is an isomorphism.

As for the entire group $PO(A, \mathbb{Z})$, we have

Corollary 3.4. *Assume $h(K) = 1$.*
$PO(A, \mathbb{Z})$ is decomposed into four components. Each of them is the intersection of a connected component of $PO(A, \mathbb{R})$ and $M(4, \mathbb{Z})$ as follows:

(i) If $N_{K/\mathbb{Q}}(\varepsilon) = -1$, *then*

$$\begin{aligned} PO(A,\mathbb{Z}) &= \rho'\left(PSL(2,\mathcal{O})\right) \cup \rho'\left(PSL(2,\mathcal{O})\right) H_1' \\ &\cup \rho'\left(PSL(2,\mathcal{O})\right) H_2' \cup \rho'\left(PSL(2,\mathcal{O})\right) H_1' H_2'; \end{aligned}$$

(ii) If $N_{K/\mathbb{Q}}(\varepsilon) = 1$, *then*

$$\begin{aligned} PO(A,\mathbb{Z}) &= \rho'\left(PSL(2,\mathcal{O})\right) \cup \rho'\left(PSL(2,\mathcal{O})g_\varepsilon\right) \\ &\cup (\rho'\left(PSL(2,\mathcal{O})\right) \cup \rho'\left(PSL(2,\mathcal{O})g_\varepsilon\right))H_1' \\ &\cup (\rho'\left(PSL(2,\mathcal{O})\right) \cup \rho'\left(PSL(2,\mathcal{O})g_\varepsilon\right))H_2' \\ &\cup (\rho'\left(PSL(2,\mathcal{O})\right) \cup \rho'\left(PSL(2,\mathcal{O})g_\varepsilon\right))H_1' H_2'. \end{aligned}$$

Corollary 3.5. *Assume* $h(K) = 1$.

(i) If $N_{K/\mathbb{Q}}(\varepsilon) = -1$, *then*

$$\begin{aligned} \mathcal{D}(A)/PO(A,\mathbb{Z}) &\cong \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}/\left\langle PSL(2,\mathcal{O}), h_1, h_2 \right\rangle \\ &\cong \mathbb{H} \times \mathbb{H}_-/\left\langle PSL(2,\mathcal{O}), h_2 \right\rangle, \\ \mathcal{D}(A)/PO(A,\mathbb{Z})_0 &\cong \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}/PSL(2,\mathcal{O}), \\ \mathcal{D}(A)_0/PO(A,\mathbb{Z})_0 &\cong \mathbb{H} \times \mathbb{H}_-/PSL(2,\mathcal{O}); \end{aligned}$$

(ii) If $N_{K/\mathbb{Q}}(\varepsilon) = 1$, *then*

$$\begin{aligned} \mathcal{D}(A)/PO(A,\mathbb{Z}) &\cong \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}/\left\langle PSL(2,\mathcal{O}), g_\varepsilon, h_1, h_2 \right\rangle \\ &\cong \mathbb{H} \times \mathbb{H}_-/\left\langle PSL(2,\mathcal{O}), g_\varepsilon, h_2 \right\rangle, \\ \mathcal{D}(A)/PO(A,\mathbb{Z})_0 &\cong \mathbb{H} \times \mathbb{H}_- \cup \mathbb{H}_- \times \mathbb{H}/\left\langle PSL(2,\mathcal{O}), g_\varepsilon \right\rangle, \\ \mathcal{D}(A)_0/PO(A,\mathbb{Z})_0 &\cong \mathbb{H} \times \mathbb{H}_-/\left\langle PSL(2,\mathcal{O}), g_\varepsilon \right\rangle. \end{aligned}$$

Remark 3.3. *It is well known that the following three conditions are equivalent:*

(i) $N_{K/\mathbb{Q}}(\varepsilon) = -1$,

(ii) For any prime number p *with* $p|d$, $p = 2$ *or* $p \equiv 1 \bmod 4$,

(iii) d *is a sum of two squares.*

4 Generators of $PO(A,\mathbb{Z})_0$

We first give a system of generators of $PSL(2,\mathcal{O})$ for the case $d = 5$ or 8.

Proposition 4.1. *Take* $g = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \in PSL(2,\mathcal{O})$ *such that* $\alpha_1\alpha_2\alpha_3\alpha_4 = 0$. *Let* $\{1,\omega\}$ *be a basis of* $\mathcal{O}$. *Then:*

(i) When $N_{K/\mathbb{Q}}(\varepsilon) = -1$, $g \in \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \varepsilon & 1 \\ 0 & -\overline{\varepsilon} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$;

(ii) When $N_{K/\mathbb{Q}}(\varepsilon) = 1$, $g \in \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \varepsilon & 1 \\ 0 & \overline{\varepsilon} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$.

[Proof]. Note that for any $\alpha = n_1 + n_2\ \omega \in \mathcal{O}$ with $n_1, n_2 \in \mathbb{Z}$,we have

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{n_1} \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}^{n_2},$$

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1},$$

$$\begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}.$$

Assume $N_{K/\mathbb{Q}}(\varepsilon) = -1$. Then we have

$$\begin{bmatrix} \varepsilon & \alpha \\ 0 & -\overline{\varepsilon} \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\overline{\varepsilon} \end{bmatrix} \begin{bmatrix} 1 & -\alpha\overline{\varepsilon} \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon & 0 \\ \alpha & -\overline{\varepsilon} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\alpha\overline{\varepsilon} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & -\overline{\varepsilon} \end{bmatrix},$$

$$\begin{bmatrix} \alpha & \varepsilon \\ \overline{\varepsilon} & 0 \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\overline{\varepsilon} \end{bmatrix} \begin{bmatrix} -\alpha\overline{\varepsilon} & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \varepsilon \\ \overline{\varepsilon} & \alpha \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\overline{\varepsilon} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & \alpha\varepsilon \end{bmatrix}.$$

Assume $N_{K/\mathbb{Q}}(\varepsilon) = 1$. Then we have

$$\begin{bmatrix} \varepsilon & \alpha \\ 0 & \overline{\varepsilon} \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \overline{\varepsilon} \end{bmatrix} \begin{bmatrix} 1 & \alpha\overline{\varepsilon} \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon & 0 \\ \alpha & \overline{\varepsilon} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha\overline{\varepsilon} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & -\overline{\varepsilon} \end{bmatrix},$$

$$\begin{bmatrix} \alpha & \varepsilon \\ -\overline{\varepsilon} & 0 \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \overline{\varepsilon} \end{bmatrix} \begin{bmatrix} \alpha\overline{\varepsilon} & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \varepsilon \\ -\overline{\varepsilon} & \alpha \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & \overline{\varepsilon} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & \alpha\varepsilon \end{bmatrix}.$$

q.e.d.

Proposition 4.2. *Assume K is a Euclidian domain. Then we have*

(i) when $N_{K/\mathbb{Q}}(\varepsilon) = -1, \quad PSL(2,\mathcal{O}) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \varepsilon & 1 \\ 0 & -\overline{\varepsilon} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$;

(ii) when $N_{K/\mathbb{Q}}(\varepsilon) = 1, \ PSL(2,\mathcal{O}) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \varepsilon & 1 \\ 0 & \overline{\varepsilon} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$.

[Proof]. Let $g = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \in SL(2,\mathcal{O})$ with $\alpha_1\alpha_2\alpha_3\alpha_4 \neq 0$. Then g can be transformed to $g' = \begin{bmatrix} \alpha_1' & \alpha_2' \\ \alpha_3' & \alpha_4' \end{bmatrix}$ with $\alpha_1' = 0$ or $\alpha_2' = 0$ using the Euclidian algorithm for α_1 and α_2. Each step of the algorithm is performed by multiplying from right a suitable matrix in the form $\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$ with $\beta, \gamma \in \mathcal{O}$.

q.e.d.

We now show that when $d = 5$ or 8, K is a Euclidian domain.

Proposition 4.3. *When $d = 8$, K is a Euclidian domain w.r.t. the absolute value of the norm.*

[Proof]. Take $\{1, \sqrt{2}\}$ as a basis of $\mathcal{O}$. For $a, b \in \mathbb{Q}$, we have $N_{K/\mathbb{Q}}(a + b\sqrt{2}) = a^2 - 2b^2$.
It is sufficient to show that for any $\alpha, \beta \in \mathcal{O}$ with $\beta \neq 0$ there exists $\gamma \in \mathcal{O}$ such that $|N(\alpha - \beta\gamma)| < |N(\beta)|$.
Assume $\frac{\alpha}{\beta} = a + b\sqrt{2}$, with $a, b \in \mathbb{Q}$.
Take $\gamma \in \mathcal{O}$ such that $\gamma = m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$ satisfying $|a - m| \leq \frac{1}{2}$ and $|b - n| \leq \frac{1}{2}$.
Then we have

$$N_{K/\mathbb{Q}}(\frac{\alpha}{\beta} - \gamma) = N_{K/\mathbb{Q}}((a + b\sqrt{2}) - (m + n\sqrt{2})) = (a - m)^2 - 2(b - n)^2, \quad -\frac{1}{2} \leq N_{K/\mathbb{Q}}(\frac{\alpha}{\beta} - \gamma) \leq \frac{1}{4}.$$

Hence we have

$$|N_{K/\mathbb{Q}}(\alpha - \beta\gamma)| = |N_{K/\mathbb{Q}}(\beta)||N_{K/\mathbb{Q}}(\frac{\alpha}{\beta} - \gamma)| \leq \frac{1}{2}|N_{K/\mathbb{Q}}(\beta)| < |N_{K/\mathbb{Q}}(\beta)|.$$

q.e.d.

Proposition 4.4. *When d=5, K is a Euclidian domain w.r.t. the absolute value of the norm.*

[Proof]. Take $\{1, \frac{1+\sqrt{5}}{2}\}$ as a basis of $\mathcal{O}$. For $a, b \in \mathbb{Q}$, we have $N_{K/\mathbb{Q}}(a + b\frac{1+\sqrt{5}}{2}) = (a + \frac{b}{2})^2 - \frac{5}{4}b^2$. It is sufficient to show that for any $\alpha, \beta \in \mathcal{O}$ with $\beta \neq 0$ there exists $\gamma \in \mathcal{O}$ such that $|N(\alpha - \beta\gamma)| < |N(\beta)|$. Assume $\frac{\alpha}{\beta} = a + b\frac{1+\sqrt{5}}{2}$, with $a, b \in \mathbb{Q}$.
Take $\gamma \in \mathcal{O}$ such that $\gamma = m + n\frac{1+\sqrt{5}}{2}$ with $m, n \in \mathbb{Z}$ satisfying $|a - m| \leq \frac{1}{2}$ and $|b - n| \leq \frac{1}{2}$.
Then we have

$$N_{K/\mathbb{Q}}(\frac{\alpha}{\beta} - \gamma) = N_{K/\mathbb{Q}}((a + b\frac{1+\sqrt{5}}{2}) - (m + n\frac{1+\sqrt{5}}{2})) = (a - m + \frac{b-n}{2})^2 - \frac{5}{4}(b-n)^2.$$

Since $0 \leq |a - m| \leq \frac{1}{2}$ and $0 \leq |b - n| \leq \frac{1}{2}$, we have

$$0 \leq |a - m + \frac{b-n}{2}| \leq |a - m| + |\frac{b-n}{2}| \leq \frac{3}{4}, \quad 0 \leq (a - m + \frac{b-n}{2})^2 \leq \frac{9}{16},$$
$$0 \leq \frac{5}{4}(b-n)^2 \leq \frac{5}{16},$$
$$-\frac{5}{16} \leq N_{K/\mathbb{Q}}(\frac{\alpha}{\beta} - \gamma) \leq \frac{9}{16}.$$

Hence we have

$$|N_{K/\mathbb{Q}}(\alpha - \beta\gamma)| = |N_{K/\mathbb{Q}}(\beta)||N_{K/\mathbb{Q}}(\frac{\alpha}{\beta} - \gamma)| \leq \frac{9}{16}|N_{K/\mathbb{Q}}(\beta)| < |N_{K/\mathbb{Q}}(\beta)|.$$

q.e.d.

Corollary 4.1. *We have a system of generators for $PSL(2, \mathcal{O})$ as follows:*

(i) When $d = 8$

$$PSL(2, \mathcal{O}) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & \sqrt{2}-1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle,$$

(ii) when $d = 5$

$$PSL(2, \mathcal{O}) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle.$$

When $d = 5, 8$, $h(K) = 1$ and $N_{K/\mathbb{Q}}(\varepsilon) = -1$. So from Theorem 3.3, $\rho' : PSL(2, \mathcal{O}) \to PO(A, \mathbb{Z})_0$ is an isomorphism. Thus we obtain a system of generators for $PGL(A, \mathbb{Z})_0$, sending by ρ' the generators for $PSL(2, \mathcal{O})$.

Proposition 4.5. *We have a system of generators for $PO(A, \mathbb{Z})_0$ as follows:*

(i) When $d = 8$,

$$PO(A, \mathbb{Z})_0 = \left\langle \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle;$$

(ii) When $d = 5$,

$$PO(A, \mathbb{Z})_0 = \left\langle \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle.$$

5 Polarized Abelian Surfaces with Real Multiplication

Conventionally the Hilbert modular group operates on $\mathbb{H}^2$. The quotient space $\mathbb{H}^2/PSL(2,\mathcal{O})$,after compactification and minimal resolutions of all singularities, is called the Hilbert modular surface. In this section, we show that $\mathbb{H}\times\mathbb{H}_-/PSL(2,\mathcal{O})$ is more natural than $\mathbb{H}^2/PSL(2,\mathcal{O})$ as a moduli space of polarized abelian surfaces with real multiplication.
We put

$$D=\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad \text{with } \ d_1|d_2.$$

Let $\mathcal{O}^*$ be the dual ideal of $\mathcal{O}$ w.r.t. the trace: $Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*)\subset\mathbb{Z}$.
Let $\mathfrak{a}$ be an invertible $\mathcal{O}$-module of rank 1 with $\mathfrak{a}\subset\mathcal{O}^*$. We use the following notation:

$$\begin{aligned} Sp(4,\mathbb{R}) &= \left\{ g\in GL(4,\mathbb{R}) \mid g\begin{bmatrix} 0 & I_2 \\ -I_2 & 0\end{bmatrix} {}^t g = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0\end{bmatrix}\right\}, \\ Sp(4,\mathbb{Z},D) &= GL(4,\mathbb{Z})\cap \begin{bmatrix} I_2 & 0 \\ 0 & D\end{bmatrix} Sp(4,\mathbb{R}) \begin{bmatrix} I_2 & 0 \\ 0 & D^{-1}\end{bmatrix}, \\ PSL_2(\mathcal{O},\mathfrak{a}) &= \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta\end{bmatrix} \in PSL(2,K) \mid \alpha,\delta\in\mathcal{O}, \quad \beta\in\mathfrak{a}, \quad \gamma\in\mathfrak{a}^{-1}\right\}. \end{aligned} \tag{5.1}$$

In particular $PSL_2(\mathcal{O},\mathcal{O})=PSL(2,\mathcal{O})$. When $D=I_2$, $Sp(4,\mathbb{Z},D)=Sp(4,\mathbb{Z})$.
Fix $\tau=(\tau_1,\tau_2)\in\mathbb{H}^2$. Let $V=\mathbb{C}^2$, $L=\{(z_1,z_2)\in V \mid z_1=\alpha\tau_1+\beta, z_2=\overline{\alpha}\tau_2+\overline{\beta}$ where $\alpha\in\mathcal{O}, \quad \beta\in\mathfrak{a}\}$.
So we have $L\cong\mathcal{O}\oplus\mathfrak{a}$.
For any two points $z=(z_1,z_2), z'=(z_1',z_2')\in V$, there exist $\alpha_1,\alpha_2,\beta_1,\beta_2\in K_{\mathbb{R}}$ such that

$$z_1=\alpha_1^{(1)}\tau_1+\beta_1^{(1)}, \quad z_2=\alpha_1^{(2)}\tau_2+\beta_1^{(2)}, z_1'=\alpha_2^{(1)}\tau_1+\beta_2^{(1)}, \quad z_2'=\alpha_2^{(2)}\tau_2+\beta_2^{(2)}.$$

We define a bilinear form $E(,)$ on V over $\mathbb{C}$ by

$$E(z,z')=\sum_{i=1}^{2}(\alpha_1^{(i)}\beta_2^{(i)}-\alpha_2^{(i)}\beta_1^{(i)}).$$

If $\quad \alpha_i\in K$ and $\beta_i\in K \quad (i=1,2),\quad$ then $E(z,z')=Tr_{K/\mathbb{Q}}(\alpha_1\beta_2-\alpha_2\beta_1)$.

Proposition 5.1. (van der Geer [3],p.148) *Under the notation above, we have*

(i) $E(z,z')=\sum_{i=1}^2(\Im\tau_i)^{-1}\Im(z_i\overline{z_i'})$,

(ii) E *is alternating, non-degenerate, bi-linear and integral on* L.

(iii) $E(iz,iz')=E(z,z')$,

(iv) $E(iz,z)>0$

[proof].

(i) We have

$$\alpha_1^{(i)}=\frac{z_i-\overline{z_i}}{\tau_i-\overline{\tau_i}}=(\Im\tau_i)^{-1}\Im z_i, \quad \text{likewise } \ \alpha_2^{(i)}=(\Im\tau_i)^{-1}\Im z_i'.$$

Hence:

$$\begin{aligned} \alpha_1^{(i)}\beta_2^{(i)}-\alpha_2^{(i)}\beta_1^{(i)} &= \alpha_1^{(i)}(z_i'-\alpha_2^{(i)}\tau_i)-\alpha_2^{(i)}(z_i-\alpha_1^{(i)}\tau_i) \\ &= \Re\left(\alpha_1^{(i)}(z_i'-\alpha_2^{(i)}\tau_i)-\alpha_2^{(i)}(z_i-\alpha_1^{(i)}\tau_i)\right) \\ &= \alpha_1^{(i)}\Re z_i'-\alpha_2^{(i)}\Re z_i \\ &= (\Im\tau_i)^{-1}(\Im z_i\Re z_i'-\Im z_i'\Re z_i) \\ &= (\Im\tau_i)^{-1}\Im(z_i\overline{z_i'}). \end{aligned}$$

(ii) E is alternating because

$$E(z',z)=\sum_{i=1}^{2}(\Im\tau_i)^{-1}\Im(z'_i\overline{z_i})=-\sum_{i=1}^{2}(\Im\tau_i)^{-1}\Im(z_i\overline{z'_i})=-E(z,z').$$

E is bilinear because we have for $c_1,c_2\in\mathbb{R},\quad z,z',w\in V$,

$$E(c_1z+c_2z',w)=\sum_{i=1}^{2}(\Im\tau_i)^{-1}\Im((c_1z_1+c_2z')\overline{w})=c_1E(z,w)+c_2E(z',w).$$

E is itegral on L because we have for $(\alpha_i,\beta_i)\in\mathcal{O}\oplus\mathfrak{a}\quad(i=1,2)$,

$$E(\alpha_1\tau+\beta_1,\alpha_2\tau+\beta_2)=Tr_{K/\mathbb{Q}}(\alpha_1\beta_2-\alpha_2\beta_1)\in Tr_{K/\mathbb{Q}}(\mathcal{O}\mathfrak{a})\subset Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*)\subset\mathbb{Z}.$$

(iii) $E(iz,iz')=\sum_{i=1}^{2}(\Im\tau_i)^{-1}\Im(iz\overline{(iz')})=E(z,z')$.

(iv) $E(iz,z)=\sum_{i=1}^{2}(\Im\tau_i)^{-1}\Im(iz_i\overline{z_i})\geq 0$. $E(iz,z)=0$ if and only if $z=0$. So E is also non-degenerate.

q.e.d.

In this way V/L becomes a polarized abelian surface $\mathcal{A}(\tau,\mathcal{O}\oplus\mathfrak{a})$ with E, a Riemann form on V w.r.t. lattice L. We call $\mathcal{A}(\tau,\mathcal{O}\oplus\mathfrak{a})$ a polarized abelian surface with multiplication by $\mathcal{O}$.

We use the notation

$$\Phi(x)=\begin{bmatrix}x & 0\\ 0 & \overline{x}\end{bmatrix}\qquad for\quad x\in K.$$

Proposition 5.2. (Shimizu [9],p.231 Theorem7.3) *$\mathcal{A}(\tau,\mathcal{O}\oplus\mathfrak{a})$ is isomorphic to $\mathcal{A}(\tau',\mathcal{O}\oplus\mathfrak{a})$ if and only if there exists $g\in PSL_2(\mathcal{O},\mathfrak{a})$ such that $\tau=g[\tau']$.*

[Proof]. In this proof, let an element in $V=\mathbb{C}^2$ be represented by a column vector $z=\begin{pmatrix}z_1\\ z_2\end{pmatrix}$. We can write

$$L=\Phi(\mathcal{O})\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}+\Phi(\mathfrak{a})\begin{pmatrix}1\\ 1\end{pmatrix},$$

and let

$$L'=\Phi(\mathcal{O})\begin{pmatrix}\tau'_1\\ \tau'_2\end{pmatrix}+\Phi(\mathfrak{a})\begin{pmatrix}1\\ 1\end{pmatrix}.$$

Let

$$\theta:V/L\to V/L'$$

be an isomorphism, and let

$$\Theta:V\to V$$

be a linear transformation which represents θ. Since θ is an isomorphism, Θ is commutable with $\Phi(\alpha)(\alpha\in K)$. So $\Theta=\left[\begin{smallmatrix}\Theta_1 & 0\\ 0 & \Theta_2\end{smallmatrix}\right]$ $(\Theta_1,\Theta_2\in\mathbb{C}^\times)$. There exist $\kappa_{ij}\in K\quad(i,j=1,2)$ such that

$$\begin{aligned}\Theta\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}&=\Phi(\kappa_{11})\begin{pmatrix}\tau'\\ \tau'\end{pmatrix}+\Phi(\kappa_{12})\begin{pmatrix}1\\ 1\end{pmatrix},\\ \Theta\begin{pmatrix}1\\ 1\end{pmatrix}&=\Phi(\kappa_{21})\begin{pmatrix}\tau'\\ \tau'\end{pmatrix}+\Phi(\kappa_{22})\begin{pmatrix}1\\ 1\end{pmatrix}.\end{aligned}\qquad(5.2)$$

Let

$$g=\begin{bmatrix}\kappa_{11} & \kappa_{12}\\ \kappa_{21} & \kappa_{22}\end{bmatrix}.$$

Since $\Theta L \subset L'$, we have for $(\alpha, \beta) \in \mathcal{O} \oplus \mathfrak{a}$

$$\Theta L \ni \Theta \left[\Phi(\alpha) \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \Phi(\beta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = \Theta[\Phi(\alpha), \Phi(\beta)] \begin{pmatrix} \tau_1 \\ \tau_2 \\ 1 \\ 1 \end{pmatrix} = [\Phi(\alpha), \Phi(\beta)] \begin{pmatrix} \Theta \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \\ \Theta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= [\Phi(\alpha), \Phi(\beta)] \begin{bmatrix} \Phi(\kappa_{11}) & \Phi(\kappa_{12}) \\ \Phi(\kappa_{21}) & \Phi(\kappa_{22}) \end{bmatrix} \begin{pmatrix} \tau_1' \\ \tau_2' \\ 1 \\ 1 \end{pmatrix} \subset L'.$$

So $(\alpha, \beta)g \in \mathcal{O} \oplus \mathfrak{a}$. i.e., $(\mathcal{O} \oplus \mathfrak{a})g \subset \mathcal{O} \oplus \mathfrak{a}$. On the other hand, since $\Theta^{-1}L' \subset L$, $(\mathcal{O} \oplus \mathfrak{a})g^{-1} \subset \mathcal{O} \oplus \mathfrak{a}$. Hence

$$(\mathcal{O} \oplus \mathfrak{a})g = \mathcal{O} \oplus \mathfrak{a}. \tag{5.3}$$

Let $E(E')$ be the Riemann form of $V/L(V/L' resp.)$. Since the isomorphism θ preserves the value of the Riemann form, we have for any $(a_1, a_2), (b_1, b_2) \in \mathcal{O} \oplus \mathfrak{a}$

$$\begin{aligned}
& Tr_{K/\mathbb{Q}} \left((a_1, a_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) \\
&= Tr_{K/\mathbb{Q}}(a_1 b_2 - a_2 b_1) \\
&= E \left(\Phi(a_1) \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} + \Phi(a_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Phi(b_1) \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} + \Phi(b_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\
&= E' \left(\Theta \left(\Phi(a_1) \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} + \Phi(a_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \Theta \left(\Phi(b_1) \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} + \Phi(b_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \\
&= E' \left(\Phi(a_1) \Theta \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} + \Phi(a_2) \Theta \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Phi(b_1) \Theta \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} + \Phi(b_2) \Theta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\
&= E' \left(\Phi(a_1) \left(\Phi(\kappa_{11}) \begin{pmatrix} \tau_1' \\ \tau_2' \end{pmatrix} + \Phi(\kappa_{12}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \Phi(a_2) \left(\Phi(\kappa_{21}) \begin{pmatrix} \tau_1' \\ \tau_2' \end{pmatrix} + \Phi(\kappa_{22}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \right. \\
&\qquad \left. \Phi(b_1) \left(\Phi(\kappa_{11}) \begin{pmatrix} \tau_1' \\ \tau_2' \end{pmatrix} + \Phi(\kappa_{12}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \Phi(b_2) \left(\Phi(\kappa_{21}) \begin{pmatrix} \tau_1' \\ \tau_2' \end{pmatrix} + \Phi(\kappa_{22}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \\
&= Tr_{K/\mathbb{Q}}((a_1\kappa_{11} + a_2\kappa_{21})(b_1\kappa_{12} + b_2\kappa_{22}) - (a_1\kappa_{12} + a_2\kappa_{22})(b_1\kappa_{11} + b_2\kappa_{21})) \\
&= Tr_{K/\mathbb{Q}} \left((a_1\kappa_{11} + a_2\kappa_{21}, a_1\kappa_{12} + a_2\kappa_{22}) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} b_1\kappa_{11} + b_2\kappa_{21} \\ b_1\kappa_{12} + b_2\kappa_{22} \end{pmatrix} \right) \\
&= Tr_{K/\mathbb{Q}} \left((a_1, a_2) \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \kappa_{11} & \kappa_{21} \\ \kappa_{12} & \kappa_{22} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right).
\end{aligned}$$

Hence

$$g \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} {}^t g = \begin{bmatrix} 0 & \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} \\ -(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{5.4}$$

From (5.3),(5.4), we have

$$g \in PSL_2(\mathcal{O}, \mathfrak{a}).$$

From (5.2), we have

$$\Theta_1 \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} = g \begin{pmatrix} \tau_1' \\ 1 \end{pmatrix}, \quad \Theta_2 \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix} = \overline{g} \begin{pmatrix} \tau_2' \\ 1 \end{pmatrix}, \quad \text{where } \overline{g} = [\overline{\kappa_{ij}}]_{ij}.$$

Therefore

$$(\tau_1, \tau_2) = g[(\tau_1', \tau_2')].$$

q.e.d.

Hence the isomorphism classes of $\{\mathcal{A}(\tau, \mathcal{O} \oplus \mathfrak{a}) \mid \tau \in \mathbb{H}^2\}$ are parameterized by $\mathbb{H}^2/PSL_2(\mathcal{O}, \mathfrak{a})$.

On the other hand, an abelian surface with polarization $\left[\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right]$ is by definition the quotient space V/L' with Riemann form E where L is a lattice generated by $u_1, u_2, u_3, u_4 \in V$ satisfying

$$[E(u_i, u_j)]_{ij} = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}, \qquad (i, j = 1, \cdots, 4).$$

The Siegel upper half-space of degree 2 is by definition $\mathfrak{S}_2 = \{Z \in M(2, \mathbb{C}) \mid {}^tZ = Z, \Im Z > 0\}$. The action of $Sp(4, \mathbb{R})$ on $\mathfrak{S}_2$ is defined as follows:
For $G = \left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right] \in Sp(4, \mathbb{R})$ with $A_i (i = 1, \cdots, 4)$ being a 2×2 matrix and $Z \in \mathfrak{S}_2$,

$$G[Z] = (A_1 Z + A_2)(A_3 Z + A_4)^{-1}.$$

As is well known, the isomorphism classes of abelian surfaces $\mathcal{P}(Z, D)$ with polarization $\left[\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right]$ are parameterized by $\mathfrak{S}_2/Sp(4, \mathbb{Z}, D)$. We now construct an embedding from $\mathbb{H}^2/PSL_2(\mathcal{O}, \mathfrak{a})$ into $\mathfrak{S}_2/Sp(4, \mathbb{Z}, D)$.

Lemma 5.1. *Suppose* $\mathfrak{a} = \mathbb{Z}d_1 w_1' + \mathbb{Z}d_2 w_2'$ *in terms of a suitable basis* $\{w_1', w_2'\}$ *of* $\mathcal{O}^*$ *which is dual to a basis* $\{w_1, w_2\}$ *of* $\mathcal{O}$ *and* $d_1|d_2$. *Let* $\mathfrak{a}^*$ *be the dual of* $\mathfrak{a}$ *w.r.t. the trace.*
Then $\mathfrak{a}^* = \mathbb{Z}d_1^{-1}w_1 + \mathbb{Z}d_2^{-1}w_2$.

[proof]. Let $\alpha \in \mathcal{O}^*$ and $\alpha = c_1 w_1' + c_2 w_2'$. Then we have

$$\begin{aligned} \alpha \in \mathfrak{a}^* &\iff \text{for any} \quad \beta \in \mathfrak{a} \quad Tr_{K/\mathbb{Q}}(\alpha\beta) \in \mathbb{Z} \\ &\iff Tr_{K/\mathbb{Q}}(\alpha d_j w_j') \in \mathbb{Z} \ \ (j = 1, 2) \\ &\iff d_i Tr_{K/\mathbb{Q}}(\alpha w_j') \in \mathbb{Z} \ \ (j = 1, 2). \end{aligned}$$

Here

$$Tr_{K/\mathbb{Q}}(\alpha w_j') = \sum_{i=1}^{2} c_i Tr_{K/\mathbb{Q}}(w_i w_j') = c_j.$$

Hence $c_j \in d_j^{-1}\mathbb{Z}$.

q.e.d.

Lemma 5.2. *Set here* $\{w_1, w_2\} = \{1, \frac{d+\sqrt{d}}{2}\}$, $\{w_1', w_2'\} = \{\frac{1-\sqrt{d}}{2}, \frac{\sqrt{d}}{d}\}$. *We can write* $\mathfrak{a} = \mathcal{O}^* t$ *with* $t = n_1 w_1 + n_2 w_2 \in \mathcal{O}$. *Then* $\mathcal{O}^*/\mathfrak{a} \cong \mathbb{Z}/\mathbb{Z}d_1 \times \mathbb{Z}/\mathbb{Z}d_2$ *where* $d_1 = g.c.d.(n_1, n_2)$, $d_1 d_2 = |N_{K/\mathbb{Q}}(t)|$.

[proof]. Let $\{u_1, u_2\}$ be a basis of $\mathfrak{a}$ such that $u_1 = tw_1', u_2 = tw_2'$. Then we have

$$(u_1, u_2) = (tw_1', tw_2') = (w_1', w_2')F, \quad \text{where} \quad F = \begin{bmatrix} n_1 & n_2 \\ -\frac{d(d-1)}{4}n_2 & n_1 + dn_2 \end{bmatrix}.$$

The determinant of F equals $N_{K/\mathbb{Q}}(t)$:

$$|F| = n_1^2 + dn_1 n_2 + \frac{d(d-1)}{4}n_2^2 = N_{K/\mathbb{Q}}(\mathfrak{a})/N_{K/\mathbb{Q}}(\mathcal{O}^*) = N_{K/\mathbb{Q}}(t).$$

By multiplying from right a suitable unimodular matrix T, F changes to the form

$$FT = \begin{bmatrix} g.c.d.(n_1, n_2) & 0 \\ * & * \end{bmatrix}.$$

$T = T_1 \cdots T_t$ where $T_i (1 \le i \le t) \in GL(2, \mathbb{Z})$ and each T_i performs a step of the Euclidian algorithm on n_1, n_2 or an exchange of the columns. Since the lower elements of FT represented by $*$ is a multiple of $g.c.d.(n_1, n_2)$, by multiplying from left a suitable unimodular matrix $S \in GL(2, \mathbb{Z})$ we finally get

$$SFT = \begin{bmatrix} g.c.d.(n_1, n_2) & 0 \\ 0 & |N_{K/\mathbb{Q}}(t)|/g.c.d.(n_1, n_2) \end{bmatrix}.$$

Let $\{u_1', u_2'\}$ be a basis of $\mathfrak{a}$ such that $(u_1', u_2') = (u_1, u_2)T$. Let $\{w_1'', w_2''\}$ be a basis of $\mathcal{O}^*$ such that $(w_1'', w_2'') = (w_1', w_2')S^{-1}$. Then we have

$$(u_1', u_2') = (w_1'', w_2'')SFT.$$

q.e.d.

Proposition 5.3. (van der Geer [3],p.148) *Let f be a map*

$$\begin{aligned} f: \ \mathbb{H}^2 &\longrightarrow \mathfrak{S}^2 \\ z=(z_1,z_2) &\mapsto W\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} {}^tW. \end{aligned}$$

Let φ be a homomorphism

$$\begin{aligned} \varphi: \ PSL(\mathcal{O},\mathfrak{a}) &\longrightarrow Sp(4,\mathbb{R}) \\ g=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} &\mapsto \begin{bmatrix} W & 0 \\ 0 & {}^tW^{-1} \end{bmatrix} \begin{bmatrix} \Phi(\alpha) & \Phi(\beta) \\ \Phi(\gamma) & \Phi(\delta) \end{bmatrix} \begin{bmatrix} W^{-1} & 0 \\ 0 & {}^tW \end{bmatrix}, \end{aligned}$$

Then f and φ are well defined and $f(g[z]) = \varphi(g)[f(z)]$.

[Proof].

(i) $f(z) = {}^tf(z)$ is obvious. $\Im f(z) = W\begin{bmatrix} \Im z & 0 \\ 0 & \Im z \end{bmatrix} {}^tW > 0$. Therefore $f(z) \in \mathfrak{S}_2$.

(ii) We show that $\varphi(g)\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} {}^t\varphi(g) = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}$:

$$\begin{bmatrix} W^{-1} & 0 \\ 0 & {}^tW \end{bmatrix}\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}\begin{bmatrix} {}^tW^{-1} & 0 \\ 0 & W \end{bmatrix} = \begin{bmatrix} 0 & W^{-1} \\ -{}^tW & 0 \end{bmatrix}\begin{bmatrix} {}^tW^{-1} & 0 \\ 0 & W \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix},$$

$$\begin{aligned} \begin{bmatrix} \Phi(\alpha) & \Phi(\beta) \\ \Phi(\gamma) & \Phi(\delta) \end{bmatrix}\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}\begin{bmatrix} \Phi(\alpha) & \Phi(\beta) \\ \Phi(\gamma) & \Phi(\delta) \end{bmatrix} &= \begin{bmatrix} -\Phi(\beta) & \Phi(\alpha) \\ -\Phi(\delta) & \Phi(\gamma) \end{bmatrix}\begin{bmatrix} \Phi(\alpha) & \Phi(\beta) \\ \Phi(\gamma) & \Phi(\delta) \end{bmatrix} \\ &= \begin{bmatrix} -\Phi(\beta)\Phi(\alpha)+\Phi(\alpha)\Phi(\beta) & -\Phi(\beta)\Phi(\gamma)+\Phi(\alpha)\Phi(\delta) \\ -\Phi(\delta)\Phi(\alpha)+\Phi(\gamma)\Phi(\alpha) & -\Phi(\delta)\Phi(\gamma)+\Phi(\gamma)\Phi(\delta) \end{bmatrix} \\ &= \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} W & 0 \\ 0 & {}^tW^{-1} \end{bmatrix}\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}\begin{bmatrix} {}^tW & 0 \\ 0 & W^{-1} \end{bmatrix} = \begin{bmatrix} 0 & W \\ -{}^tW^{-1} & 0 \end{bmatrix}\begin{bmatrix} {}^tW & 0 \\ 0 & W^{-1} \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}.$$

(iii) Let $z' = g[z]$. We show that $\varphi(g)(f(z)) = f(z')$:

$$\begin{aligned} \begin{bmatrix} W^{-1} & 0 \\ 0 & {}^tW \end{bmatrix}\left[W\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}{}^tW\right] &= W^{-1}\left(W\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}{}^tW\right){}^tW^{-1} = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}, \\ \begin{bmatrix} \Phi(\alpha) & \Phi(\beta) \\ \Phi(\gamma) & \Phi(\delta) \end{bmatrix}\left[\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}\right] &= \left(\Phi(\alpha)\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}+\Phi(\beta)\right)\left(\Phi(\gamma)\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}+\Phi(\delta)\right)^{-1} \\ &= \begin{bmatrix} \alpha z_1+\beta & 0 \\ 0 & \overline{\alpha}z_2+\overline{\beta} \end{bmatrix}\begin{bmatrix} \gamma z_1+\delta & 0 \\ 0 & \overline{\gamma}z_2+\overline{\delta} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} z_1' & 0 \\ 0 & z_2' \end{bmatrix}, \\ \begin{bmatrix} W & 0 \\ 0 & {}^tW^{-1} \end{bmatrix}\left[\begin{bmatrix} z_1' & 0 \\ 0 & z_2' \end{bmatrix}\right] &= W\begin{bmatrix} z_1' & 0 \\ 0 & z_2' \end{bmatrix}{}^tW = f(z'). \end{aligned}$$

q.e.d.

Lemma 5.3. (van der Geer [3],p.148) *There exists an embedding*

$$\mathbb{H}^2/PSL_2(\mathcal{O},\mathfrak{a}) \hookrightarrow \mathfrak{S}_2/Sp(4,\mathbb{Z},D).$$

[Proof]. We use the notation of Lemma 5.1 and Proposition 5.3. Let ν be an automorphism

$$\begin{aligned} \nu: \ \mathfrak{S}^2 &\longrightarrow \mathfrak{S}^2 \\ Z &\longmapsto ZD^{-1}. \end{aligned}$$

Let ψ be a homomorphism

$$\begin{array}{rccc}\varphi: & Sp(4,\mathbb{R}) & \longrightarrow & Sp(4,\mathbb{R}) \\ & G & \mapsto & \begin{bmatrix} I_2 & 0 \\ 0 & D \end{bmatrix} G \begin{bmatrix} I_2 & 0 \\ 0 & D^{-1} \end{bmatrix}. \end{array}$$

It is obvious that $\nu(G[Z]) = \psi(G)[\nu(Z)]$. Hence $(\nu \circ f)(g[z]) = (\psi \circ \varphi)(g)[(\nu \circ f)(z)]$.
We show that $(\psi \circ \varphi)(PSL_2(\mathcal{O}, \mathfrak{a})) \subset Sp(4, \mathbb{Z}, D)$:

$$\psi \circ \varphi(g) = \begin{bmatrix} W\Phi(\alpha)W^{-1} & W\Phi(\beta)^t W D^{-1} \\ D^t W^{-1}\Phi(\gamma)W^{-1} & D^t W^{-1}\Phi(\delta)^t W D^{-1} \end{bmatrix}, \quad \text{where}$$

$$\begin{aligned}
W\Phi(\alpha)W^{-1} &= \begin{bmatrix} w_1\alpha w_1' + \overline{w_1\alpha w_1'} & w_1\alpha w_2' + \overline{w_1\alpha w_2'} \\ w_2\alpha w_1' + \overline{w_2\alpha w_1'} & w_2\alpha w_2' + \overline{w_2\alpha w_2'} \end{bmatrix} \\
&= [Tr_{K/\mathbb{Q}}(w_i\alpha w_j')]_{ij} \in M(2, Tr_{K/\mathbb{Q}}(\mathcal{O}\mathcal{O}^*)) = M(2,\mathbb{Z}), \\
W\Phi(\beta)^t W D^{-1} &= \begin{bmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & \overline{\beta} \end{bmatrix} \begin{bmatrix} w_1 & w_2 \\ \overline{w_1} & \overline{w_2} \end{bmatrix} \begin{bmatrix} d_1^{-1} & 0 \\ 0 & d_2^{-1} \end{bmatrix} \\
&= \begin{bmatrix} w_1\beta & \overline{w_1\beta} \\ w_2\beta & \overline{w_2\beta} \end{bmatrix} \begin{bmatrix} d_1^{-1}w_1 & d_2^{-1}w_2 \\ d_1^{-1}\overline{w_1} & d_2^{-1}\overline{w_2} \end{bmatrix} \\
&= \begin{bmatrix} w_1\beta d_1^{-1}w_1 + \overline{w_1\beta d_1^{-1}w_1} & w_1\beta d_2^{-1}w_2 + \overline{w_1\beta d_2^{-1}w_2} \\ w_2\beta d_1^{-1}w_1 + \overline{w_2\beta d_1^{-1}w_1} & w_2\beta d_2^{-1}w_2 + \overline{w_2\beta d_1^{-1}w_2} \end{bmatrix} \\
&= [Tr_{K/\mathbb{Q}}(w_i\beta d_j^{-1}w_j)]_{ij} \in M(2, Tr_{K/\mathbb{Q}}(\mathcal{O}\mathfrak{a}\mathfrak{a}^*)) = M(2,\mathbb{Z}), \\
D^t W^{-1}\Phi(\gamma)W^{-1} &= \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{bmatrix} \begin{bmatrix} w_1' & w_2' \\ \overline{w_1'} & \overline{w_2'} \end{bmatrix} \\
&= \begin{bmatrix} d_1w_1' & \overline{d_1w_1'} \\ d_2w_2' & \overline{d_2w_2'} \end{bmatrix} \begin{bmatrix} \gamma w_1' & \gamma w_2' \\ \overline{\gamma w_1'} & \overline{\gamma w_2'} \end{bmatrix} \\
&= \begin{bmatrix} d_1w_1'\gamma w_1' + \overline{d_1w_1'\gamma w_1'} & d_1w_1'\gamma w_2' + \overline{d_1w_1'\gamma w_2'} \\ d_2w_2'\gamma w_1' + \overline{d_2w_2'\gamma w_1'} & d_2w_2'\gamma w_2' + \overline{d_2w_2'\gamma w_2'} \end{bmatrix} \\
&= [Tr_{K/\mathbb{Q}}(d_iw_i\gamma w_j)]_{ij} \in M(2, Tr_{K/\mathbb{Q}}(\mathfrak{a}\mathfrak{a}^{-1}\mathcal{O}^*)) = M(2,\mathbb{Z}), \\
D^t W^{-1}\Phi(\delta)^t W D^{-1} &= \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{bmatrix} \begin{bmatrix} \delta & 0 \\ 0 & \overline{\delta} \end{bmatrix} \begin{bmatrix} w_1 & w_2 \\ \overline{w_1} & \overline{w_2} \end{bmatrix} \begin{bmatrix} d_1^{-1} & 0 \\ 0 & d_2^{-1} \end{bmatrix} \\
&= \begin{bmatrix} d_1w_1' & \overline{d_1w_1'} \\ d_2w_2' & \overline{d_2w_2'} \end{bmatrix} \begin{bmatrix} \delta & 0 \\ 0 & \overline{\delta} \end{bmatrix} \begin{bmatrix} d_1^{-1}w_1 & d_2^{-1}w_2 \\ \overline{d_1^{-1}w_1} & \overline{d_2^{-1}w_2} \end{bmatrix} \\
&= [Tr_{K/\mathbb{Q}}(d_iw_i'\delta d_j^{-1}w_j)]_{ij} \in M(2, Tr_{K/\mathbb{Q}}(\mathfrak{a}\mathcal{O}\mathfrak{a}^*)) = M(2,\mathbb{Z}).
\end{aligned}$$

Since the kernel of $\nu \circ f$ is null, $\mathbb{H}^2/PSL(\mathcal{O},\mathfrak{a}) \to \mathfrak{S}_2/Sp(4,\mathbb{Z},D)$ induced by $\nu \circ f$ is an embedding.

q.e.d.

Lemma 5.4. *(i) If there exists $t \in \mathcal{O}$ such that $\mathfrak{a} = \mathcal{O}^* t$, $N_{K/\mathbb{Q}}(t) < 0$, then*

$$\mathbb{H}^2/PSL(2,\mathcal{O}) \cong \mathbb{H}^2/PSL_2(\mathcal{O},\mathfrak{a}).$$

(ii) If there exists $t \in \mathcal{O}$ such that $\mathfrak{a} = \mathcal{O}^* t$, $N_{K/\mathbb{Q}}(t) > 0$, then

$$\mathbb{H}^2/PSL(2,\mathcal{O}^*) \cong \mathbb{H}^2/PSL_2(\mathcal{O},\mathfrak{a}).$$

(iii) If $N_{K/\mathbb{Q}}(\varepsilon) = -1$, then

$$\mathbb{H}^2/PSL(2,\mathcal{O}) \cong \mathbb{H}^2/PSL_2(\mathcal{O},\mathcal{O}^*).$$

[Proof].

(i) Note that $\mathcal{O} = \sqrt{d}\mathcal{O}^*$. Let $s = \frac{t}{\sqrt{d}}$. We may assume $t > 0$. Then $s, \bar{s} > 0$. We have $\mathfrak{a} = \mathcal{O}s$, $\mathfrak{a}^{-1} = \mathcal{O}s^{-1}$.
Let σ be an isomorphism

$$\begin{array}{rcl} \sigma : \ \mathbb{H}^2 & \longrightarrow & \mathbb{H}^2 \\ z = (z_1, z_2) & \longmapsto & w = (sz_1, \bar{s}z_2). \end{array}$$

Let Λ be a homomorphism

$$\begin{array}{rcl} \Lambda : \ PSL(2, \mathcal{O}) & \longrightarrow & PSL_2(\mathcal{O}, \mathfrak{a}) \\ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} & \mapsto & \begin{bmatrix} \alpha & \beta s \\ \gamma s^{-1} & \delta \end{bmatrix}. \end{array}$$

Then we get

$$\sigma\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}[z]\right) = \Lambda\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right)(\sigma[z]).$$

(ii) We may assume that $t > 0$ and $\bar{t} > 0$. We have $(\mathcal{O}^*)^{-1} = \sqrt{d}\mathcal{O}$, $\mathfrak{a}^{-1} = \mathcal{O}\sqrt{d}t^{-1}$.
Let σ' be an isomorphism

$$\begin{array}{rcl} \sigma' : \ \mathbb{H}^2 & \longrightarrow & \mathbb{H}^2 \\ z = (z_1, z_2) & \longmapsto & w = (tz_1, \bar{t}z_2). \end{array}$$

Let Λ' be a homomorphism with $\alpha, \beta, \gamma, \delta \in \mathcal{O}$

$$\begin{array}{rcl} \Lambda' : \ PSL(2, \mathcal{O}) & \longrightarrow & PSL_2(\mathcal{O}, \mathfrak{a}) \\ \begin{bmatrix} \alpha & \beta\sqrt{d}^{-1} \\ \gamma\sqrt{d} & \delta \end{bmatrix} & \mapsto & \begin{bmatrix} \alpha & \beta\sqrt{d}^{-1}t \\ \gamma\sqrt{d}t^{-1} & \delta \end{bmatrix}. \end{array}$$

Then we get

$$\sigma'\left(\begin{bmatrix} \alpha & \beta\sqrt{d}^{-1} \\ \gamma\sqrt{d} & \delta \end{bmatrix}[z]\right) = \Lambda'\left(\begin{bmatrix} \alpha & \beta\sqrt{d}^{-1} \\ \gamma\sqrt{d} & \delta \end{bmatrix}\right)(\sigma'[z]).$$

(iii) Let $\mathfrak{a} = \mathcal{O}^*$ in (i).

q.e.d.

Lemma 5.5. (van der Geer [3],p.152) $\mathbb{H} \times \mathbb{H}_-/PSL(2, \mathcal{O}) \cong \mathbb{H}^2/PSL_2(\mathcal{O}, \mathcal{O}^*)$

[Proof]. Let σ'' be an isomorphism

$$\begin{array}{rcl} \sigma'' : \ \mathbb{H} \times \mathbb{H}^{-1} & \longrightarrow & \mathbb{H}^2 \\ z = (z_1, z_2) & \longmapsto & w = (\sqrt{d}^{-1}z_1, -\sqrt{d}^{-1}z_2). \end{array}$$

Let Λ'' be a homomorphism

$$\begin{array}{rcl} \Lambda'' : \ PSL(2, \mathcal{O}) & \longrightarrow & PSL_2(\mathcal{O}, \mathfrak{a}) \\ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} & \mapsto & \begin{bmatrix} \alpha & \beta\sqrt{d}^{-1} \\ \gamma\sqrt{d} & \delta \end{bmatrix}. \end{array}$$

Then we get

$$\sigma''\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}[z]\right) = \Lambda''\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right)(\sigma'[z]).$$

q.e.d

Now for a paramodular symplectic group, we obtain the following theorem:

Theorem 5.1. *Take* $\{w_1, w_2\} = \{1, \frac{d+\sqrt{d}}{2}\}$ *as a basis of* $\mathcal{O}$. *Then we have the following:*

(i) If there exists $t = n_1w_1 + n_2w_2 \in \mathcal{O}$ *such that* $N_{K/\mathbb{Q}}(t) = -d_1d_2$ *and g.c.d.*$(n_1, n_2) = d_1$, *then there exists an embedding* $\mathbb{H}^2/PSL(2,\mathcal{O}) \hookrightarrow \mathfrak{S}_2/Sp(4,\mathbb{Z},D)$.

(ii) If there exists $t = n_1w_1 + n_2w_2 \in \mathcal{O}$ *such that* $N_{K/\mathbb{Q}}(t) = d_1d_2$ *and g.c.d.*$(n_1, n_2) = d_1$, *then there exist*

(A) an embedding $\mathbb{H}^2/PSL_2(\mathcal{O},\mathcal{O}^*) \hookrightarrow \mathfrak{S}_2/Sp(4,\mathbb{Z},D)$ *and*

(B) an embedding $\mathbb{H}\times\mathbb{H}_-/PSL(2,\mathcal{O}) \hookrightarrow \mathfrak{S}_2/Sp(4,\mathbb{Z},D)$.

(iii) If $N_{K/\mathbb{Q}}(\varepsilon) = -1$, *then the conditions of (i) and (ii) are equivalent.*

In particular for the case $d_1 = d_2 = 1$ we have

Corollary 5.1. *(i) If there exists* $t = n_1w_1 + n_2w_2 \in \mathcal{O}$ *such that* $N_{K/\mathbb{Q}}(t) = -1$, *then there exists an embedding* $\mathbb{H}^2/PSL_2(\mathcal{O},\mathcal{O}) \hookrightarrow \mathfrak{S}_2/Sp(4,\mathbb{Z})$.

(ii) There exists an embedding $\mathbb{H}\times\mathbb{H}_-/PSL_2(\mathcal{O},\mathcal{O}) \hookrightarrow \mathfrak{S}_2/Sp(4,\mathbb{Z})$.

[proof]. If $N_{K/\mathbb{Q}}(t) = (n_1w_1 + n_2w_2)(n_1\overline{w_1} + n_2\overline{w_2}) = N_{K/\mathbb{Q}}(w_1)n_1^2 + tr(w_1\overline{w_2})n_1n_2 + N_{K/\mathbb{Q}}(w_2)n_2^2 = \pm1$, then g.c.d.$(n_1, n_2) = 1$.

q.e.d.

Therefore we know that $\mathbb{H}\times\mathbb{H}_-/PSL(2,\mathcal{O})$ is always embedded into $\mathfrak{S}_2/Sp(4,\mathbb{Z})$. On the other hand, it is conditional for $\mathbb{H}^2/PSL(2,\mathcal{O})$ to have an embedding in this way.

6 K3 Surfaces

It is known that $\mathcal{L} = -E_8 \oplus -E_8 \oplus U \oplus U \oplus U$ is the cohomology lattice of K3 surfaces where

$$E_8 = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

$\mathcal{L}$ is even and unimodular. $\mathcal{D}^{20}(\mathcal{L})$ is the period domain of the family of all K3 surfaces and the period map is surjective, i.e., all points of $\mathcal{D}^{20}(\mathcal{L})$ occur as period points of K3 surfaces ([1],p.339 Corollary14.2 and see p.372).

In general, we have

Proposition 6.1. (Barth [1],p.28 Theorem 2.9) *Let* L *be an even unimodular lattice containing a sublattice isometric to* $\overset{k}{\oplus} U$ *and let* M *be any even lattice.*
If rank $M \le k$, *then there exists a primitive embedding* $i : M \to L$, *i.e.,* i *is a lattice monomorphism and* $i(M)$ *is primitive.*

Since

$$(x_1, x_2)(-WU^tW)^t(x_1, x_2) = -2N_{K/\mathbb{Q}}(w_1x_1 + w_2x_2), \quad \text{for } x_1, x_2 \in \mathbb{Z},$$

$-WU^tW$ is even and identified with a primitive sublattice of $U \oplus U$. Hence $A = U \oplus (-WU^tW)$ is a primitive sublattice of $\mathcal{L}$ and $\mathcal{D}(A)$ becomes a period domain of a family $\mathcal{F}_A$ of K3 surfaces with a transcendental lattice A for a generic member. Then $\mathcal{D}(A)/PO(A,\mathbb{Z})$ becomes a a moduli space of the family $\mathcal{F}_A$. In fact, a matrix P in $PO(A,\mathbb{Z})$ represents a basis change of the cohomology class space and an orbit $PO(A,\mathbb{Z})\eta$ of $\eta \in \mathcal{D}(A)$ represents an identical member of $\mathcal{F}_A$.

From Lemma 5.4, 5.5, on the condition that there exists $t \in \mathcal{O}$ such that $N_{K/\mathbb{Q}}(t) = -1$, we have $\mathbb{H}^2/PSL(2,\mathcal{O}) \cong \mathbb{H} \times \mathbb{H}_-/PSL(2,\mathcal{O})$. As shown in Corollary 3.5, $\mathbb{H} \times \mathbb{H}_-/PSL(2,\mathcal{O})$ is again more natural than $\mathbb{H}^2/PSL(2,\mathcal{O})$ in view of the correspondence with $\mathcal{D}(A)/PO(A,\mathbb{Z})$.

In summary, the correspondence among surfaces are shown in the diagram below:

$$\begin{array}{ccccc}
\{[\Sigma_2]\} & \xrightarrow[\sim]{Jac} & \{[\mathcal{P}(Z,J)]\} & \xrightarrow{\sim} & \mathfrak{S}_2/Sp(4,\mathbb{Z}) \\
 & & \uparrow & & \uparrow \\
 & & \{[\mathcal{A}(\tau,\mathcal{O}\oplus\mathcal{O}^*)]\} & \xrightarrow{\sim} & \mathbb{H} \times \mathbb{H}_-/PSL(2,\mathcal{O}) \\
 & & & & \uparrow \\
 & & \mathcal{F}_A & \xrightarrow{\sim} & \mathbb{H} \times \mathbb{H}_-/ < \Gamma, h_2 > \cong \mathcal{D}(A)/PO(A,\mathbb{Z}),
\end{array}$$

where $Z \in \mathfrak{S}_2$, $\tau \in \mathbb{H}^2$, $J = \left[\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right]$,
$[\cdot]$: the isomorphism class of $\cdot$,
Σ_2: a closed Riemann surface of genus 2,
Jac: the Abel-Jacobi map,
$\mathcal{P}(Z,J)$: a polarized abelian surface with principal polarization J,
$\mathcal{A}(\tau,\mathcal{O}\oplus\mathcal{O}^*)$: an abelian surface with multiplication $\mathcal{O}$ and with a lattice isomorphic to $\mathcal{O}\oplus\mathcal{O}^*$,
$\mathcal{F}_A$: a family of K3 surfaces with transcendental lattice A for a generic member,

$$\Gamma = \begin{cases} PSL(2,\mathcal{O}) & \text{if } h(K)=1, N_{K/\mathbb{Q}}(\varepsilon) = -1, \\ PSL(2,\mathcal{O}) \cup PSL(2,\mathcal{O})g_\varepsilon & \text{if } h(K)=1, N_{K/\mathbb{Q}}(\varepsilon) = 1, \end{cases}$$

$$\begin{array}{rcl} h_2 : \mathbb{H} \times \mathbb{H}_- & \to & \mathbb{H} \times \mathbb{H}_- \\ (z,w) & \mapsto & (1/w, 1/z). \end{array}$$

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