A variation of the Ramanujan-Nagell type Diophatine equation

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Introduction

In this article we study a Ramanujan-Nagell type Diophantine equation:

$$x^2 + D = 25 \cdot 2^n, \tag{0.1}$$

where D is a given integer. We look for the solution (x, n) in positive integers. This is an extension of the equation

$$x^2 + D = 2^n, \tag{0.2}$$

that is studied by F. Beukers [1], [2]. Originally the study is initiated by Ramanujan for the specific equation

$$x^2 + 7 = 2^n$$

which allows five solutions $n = 3, 4, 5, 7, 15$. We use the method discovered by Beukers with a help of a special solution of (0.1) (see (4.5)). (0.1) is a rare case that enables us to apply Beukers' method.

1 General tactics

The solution (x, n) with even $n = 2\ell$ is obatained by considering the factorization

$$D = (5 \cdot 2^\ell - x)(5 \cdot 2^\ell + x).$$

And we have the bound

$$\ell < \frac{\log D - \log 5}{\log 2}. \tag{1.1}$$

This is an easy case. So we restrict our attention only for the odd case $n = 2\ell + 1$.

$$\sqrt{2} - \frac{x}{5 \cdot 2^\ell} = \frac{D}{2^{2\ell}} \frac{1}{\sqrt{2} + x/2^\ell}.$$

The upper bound for ℓ is obtained by the evaluation

$$\left| \sqrt{2} - \frac{x}{5 \cdot 2^\ell} \right| > \frac{c}{2\sigma}, \quad \text{with } \sigma < 2. \tag{1.2}$$

In fact, from (1.2) and

$$\left| \sqrt{2} - \frac{x}{2^\ell} \right| = \frac{|D|}{2^{2\ell}} \left(\sqrt{2} + \frac{x}{2^\ell} \right)^{-1} < \frac{|D|}{2^{2\ell}\sqrt{2}},$$

we obtain

$$2^{(2-\sigma)\ell} < c \ \cdot |D|. \tag{1.3}$$

2 Results

We show the following main results.

Theorem 1. For any integers x and ℓ we have

$$\left|\sqrt{2}-\frac{x}{5\cdot 2^{\ell}}\right| > 2^{-103.07}\cdot\frac{1}{2^{1.94\ell}}$$

Theorem 2. For any solution (x,n) with given D, it holds

$$n < 3420+\frac{100}{3\log 2}(\log|D|-\log 25).$$

Theorem 3. For any solution (x,n) with given $|D|<250000$, it holds

$$n \leq 57.$$

3 Padé approximation by the hypergeometric polynomials

To prove Theorem 1, we use the Padé approximation given by Beukers ([1]). Let us recall the properties of this approximation. By using the notation of Gauss hypergeometric series

$$F(a,b,c,z)=1+\sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}z^n \ \left(\,(x,n)=x(x+1)\cdots(x+n-1)\right)$$

, for positive integers n_1, n_2 we define hypergeometric polynoomials

$$\begin{aligned} G(z) &= F(-\frac{1}{2}-n_2,-n_1,-n,z),\\ H(z) &= F(\frac{1}{2}-n_1,n_2,-n,z),\\ E(z) &= \frac{F(n_2+1,n_1+\frac{1}{2},n+2,z)}{F(n_2+1,n_1+\frac{1}{2},n+2,1)} \quad \text{with } n=n_1+n_2. \end{aligned}$$

The degree of $G(z)$ is n_1 and the degree of $H(z)$ is n_2.

Lemma 3.1 *We have*

$$G(z)-\sqrt{1-z}H(z)=z^{n+1}G(1)E(z). \tag{3.1}$$

We see this equality by observing the Riemann schemes of the Gauss hypergeometric diffrential equations corresponding to the above hypergeometric polynomials. According to Beukers ([1]) we have the following Lemma. The main property (a) is deduced from (3.1).

Lemma 3.2 (a) *For* $|z|<1$ *it holds*

$$\left|G(z)-\sqrt{1-z}H(z)\right| < G(1)\,|z|^{n+1}$$

(b) *For* $0<z<1$ *we have*

$$G(1)<G(z)<G(0)=1.$$

(c)

$$G(1)=\binom{n}{n_1}^{-1}\prod_{m=1}^{n_1}\left(1-\frac{1}{2m}\right).$$

Lemma 3.3 $\binom{n}{n_1}G(4z)$ *and* $\binom{n}{n_1}H(4z)$ *are polynomials of* z *with integer coefficients.*

[proof] The coefficient of $z^k \quad (k \leq n_1)$ in $\binom{n}{n_1} G(4z)$ is

$$\binom{n}{n_1} \frac{(-\frac{1}{2}-n_2,k)(-n_1,k)}{(-n,k)\,k!}4^k = \binom{n}{n_1} \frac{(-\frac{1}{2}-n_2)\cdots(-\frac{1}{2}-n_2+k-1)}{k!} \cdot \frac{n_1(n_1-1)\cdots(n_1-k+1)}{n(n-1)\cdots(n-k+1)}4^k$$
$$= (-4)^k \frac{n!}{n_1!n_2!} \cdot \frac{(n_2+\frac{1}{2})(n_2-\frac{1}{2})\cdots(n_2+\frac{1}{2}-k+1)}{k!} \cdot \frac{n_1!/(n_1-k)!)}{n!/(n-k)!}$$
$$= (-4)^k \binom{n_2+\frac{1}{2}}{k} \frac{(n-k)!}{n_2!(n_1-k)!} = (-4)^k \binom{n_2+\frac{1}{2}}{k} \binom{n-k}{n_2}.$$

Because $\binom{n-k}{n_2}$ is an integer, just we may show

$$4^k \binom{m+\frac{1}{2}}{k} \in \boldsymbol{Z} \quad \text{for} \quad m,k \in \boldsymbol{N}.$$

We have

$$\begin{aligned} 4^k\,(\,m+\tfrac{1}{2}\,)\,k &= 4^k \frac{\left(m+\frac{1}{2}\right)\left(m+\frac{1}{2}-1\right)\cdots\left(m+\frac{1}{2}-k+1\right)}{k!} \\ &= \frac{2^k(2m+1)(2m-1)\cdots(2m-2k+3)}{k!}. \end{aligned}$$

So we may show $\mathrm{Ord}_p((2m+1)(2m-1)\cdots(2m-2k+3)2^k) \geq \mathrm{Ord}_p(k!)$ for every prime p, where $\mathrm{Ord}_p(x)$ means the p-adic order of x. This is an easy observation.

q.e.d.

Lemma 3.4 *It holds* $\binom{n}{k} < \frac{1}{2}\left(\frac{3}{\sqrt[3]{4}}\right)^n$ *for any positive integers with* $n \geq 3k$.

[proof] Put $n = 3m+\delta$ $(m \in \boldsymbol{N}, \delta \in \{0,1,2\})$.
From $n \geq 3k$ we have $\binom{n}{k} = \binom{3m+\delta}{k} \leq \binom{3m+\delta}{m}$.
Note that

$$\frac{\binom{3m+\delta}{m}}{\binom{3(m-1)+\delta}{m-1}} = \frac{(3m+\delta)(3m+\delta-1)(3m+\delta-2)}{m(2m+\delta)(2m+\delta-1)}$$

is an increasing sequence for $m \geq 2$. And it holds

$$\lim_{m\to\infty} \frac{(3m+\delta)(3m+\delta-1)(3m+\delta-2)}{m(2m+\delta)(2m+\delta-1)} = \frac{27}{4}$$

we have

$$\frac{\binom{3m_0+\delta}{m_0}}{\binom{3(m_0-1)+\delta}{m_0-1}} < \frac{27}{4}$$

for $m_0 \geq 2$. Let us make a product of (3.2) for $m_0 = 2,3,\cdots,m$. Then

$$\binom{3m+\delta}{m} < \binom{3+\delta}{1}\left(\frac{27}{4}\right)^{m-1} = (3+\delta)\left(\frac{27}{4}\right)^{\frac{n}{3}}\left(\frac{4}{27}\right)^{1+\frac{\delta}{3}}$$

By using inequality $(3+\delta)\left(\frac{4}{27}\right)^{1+\frac{\delta}{3}} < \frac{1}{2} \quad (\delta = 0,1,2)$ we obtain the assertion.

q.e.d

4 Proofs of the theorems

Let (x_0, δ, ν) be a fixed special solution of (0.1):

$$x_0^2 + \delta = 25 \cdot 2^\nu,$$

with positive δ and odd ν. Put $w = 2^\nu$. Suppose a general solution

$$x^2 + D = 25 \cdot 2^{2\ell+1}, \quad 2\ell + 1 \geq \nu.$$

Put $p = 2^{2\ell+1}$ and choose n_1, n_2 and $\lambda \in \boldsymbol{N}$ so that we have

$$(4w)^\lambda \geq \frac{\sqrt{p}}{\sqrt{w}} > (4w)^{\lambda-1}, \quad \frac{2}{3}\lambda - \frac{2}{3} \leq n_1 \leq \frac{2}{3}\lambda + 1, \quad n_2 = n_1 + \lambda. \tag{4.1}$$

We note that λ is determined in a unique way as a positive integer, but we have two possibilities of n_1. We don't determine here alternative choice. According to Lemma 3.3 we may put

$$\binom{n}{n_1} G\left(\frac{\delta}{25w}\right) = \frac{A}{(4 \cdot 25w)^{n_1}}, \quad \binom{n}{n_1} H\left(\frac{\delta}{25w}\right) = \frac{B}{(4 \cdot 25w)^{n_2}}$$

with some positive integers A and B. By Lemma 3.2 and the fact that (x_0, δ, ν) is a solution, we obtain

$$\left| \frac{A}{(4 \cdot 25w)^{n_1}} - \frac{x_0}{5\sqrt{w}} \frac{B}{(4 \cdot 25w)^{n_2}} \right| < G(1) \left(\frac{\delta}{25w}\right)^{n+1} \binom{n}{n_1}.$$

Namely

$$\left| 1 - \frac{x_0 B}{5\sqrt{w}(4 \cdot 25w)^{n_2-n_1} A} \right| < \frac{(4 \cdot 25w)^{n_1}}{A} G(1) \left(\frac{\delta}{25w}\right)^{n+1} \binom{n}{n_1}.$$

Set

$$\varepsilon = \left| \frac{x}{5\sqrt{p}} - 1 \right|, \ K = \left| \frac{x}{5\sqrt{p}} - \frac{x_0 B}{5\sqrt{w}(4 \cdot 25w)^{n_2-n_1} A} \right|.$$

It holds

$$K = \left| \frac{x\sqrt{w}(4 \cdot 25w)^{n_2-n_1} A - x_0 B\sqrt{p}}{5\sqrt{p}\sqrt{w}(4 \cdot 25w)^{n_2-n_1} A} \right| = \frac{1}{5\sqrt{w}(4 \cdot 25w)^{n_2-n_1} A} \left(x\frac{\sqrt{w}}{\sqrt{p}}(25 \cdot 4w)^\lambda A - x_0 B \right).$$

According to the choice (4.1), $\frac{\sqrt{w}}{\sqrt{p}}(25 \cdot 4w)^\lambda$ becomes to be a positive integer. And now we determine the alternative choice of n_1 so that we have $K \neq 0$. Then we have

$$\frac{1}{5\sqrt{w}(4 \cdot 25w)^{n_2-n_1} A} \leq K. \tag{4.2}$$

By use of the triangle inequality, we obtain

$$K \leq \left| \frac{x}{5\sqrt{p}} - 1 \right| + \left| 1 - \frac{x_0 B}{5\sqrt{w}(4 \cdot 25w)^{n_2-n_1} A} \right| < \varepsilon + \frac{(4 \cdot 25w)^{n_1}}{A} G(1) \left(\frac{\delta}{25w}\right)^{n+1} \binom{n}{n_1}. \tag{4.3}$$

Connecting (4.2) and (4.3) we obtain

$$1 < \varepsilon I + J \tag{4.4}$$

with

$$I = 5\sqrt{w}(4 \cdot 25w)^{n_2-n_1}|A|, \ J = 5\sqrt{w}(4 \cdot 25w)^{n_2} G(1) \left(\frac{\delta}{25w}\right)^{n+1} \binom{n}{n_1}.$$

We are looking for an evaluation of the shape (1.2). We use the special solution

$$926819^2 + 439 = 25 \cdot 2^{35} \tag{4.5}$$

i.e.

$$(x_0, \delta, \nu) = (926819, 439, 35).$$

With this special solution we show

Proposition 4.1

$$J < 0.831753.$$

and
Proposition 4.2

$$I < 2^{100.011} p^{0.97}.$$

[proof of Proposition 4.1].
According to Lemma 3.2

$$G(1)\binom{n}{n_1} = \prod_{m=1}^{n_1}\left(1 - \frac{1}{2m}\right) < 1.$$

So we have

$$\begin{aligned} J &< 5\sqrt{w}(4\cdot 25w)^{n_2}\left(\frac{\delta}{25w}\right)^{n+1} = 5\sqrt{w}(4\cdot 25w)^{n_1+\lambda}\left(\frac{\delta}{25w}\right)^{2n_1+\lambda+1} \\ &= \frac{\delta}{5\sqrt{w}}\left(\frac{4\delta^2}{25w}\right)^{n_1}(4\delta)^{\lambda}. \end{aligned}$$

Easy evaluation $4\delta^2/(25w) < 1$ and the condition $n_1 \geq \dfrac{2}{3}(\lambda - 1)$ in (4.1) lead to

$$J < \frac{\delta}{5\sqrt{w}}\left(\frac{4\delta^2}{25w}\right)^{\frac{2}{3}\lambda - \frac{2}{3}}(4\delta)^{\lambda} = \frac{(25w)^{1/6}}{2^{4/3}\delta^{1/3}}\left(\frac{2^{10/3}\delta^{7/3}}{(25w)^{2/3}}\right)^{\lambda}.$$

Putting

$$\varphi(\lambda) = \frac{(25w)^{1/6}}{2^{4/3}\delta^{1/3}}\left(\frac{2^{10/3}\delta^{7/3}}{(25w)^{2/3}}\right)^{\lambda},$$

we have a computer aded calculation $\varphi(1) = 0.831752\cdots$, $\varphi(2) = 0.135889\cdots$. The above argument does not cover the case when we have $\ell \leq 17$. But in this case we can examine by a direct approximation of $\sqrt{2}$. So we have the required inequality.

q.e.d.

5 Solutions

To list up the exact solutions for bounded values of D, we use the following:

Proposition 5.1 *Suppose $|D| < 2^{82}$. For any solution of (0.1), it holds*

$$n < 26 + \frac{2}{\log 2}(\log|D| - \log 5).$$

[proof]
Let us suppose a solution $x^2 + D = 2^{2\ell+1}$. If $|D| < 2^{82}$, then the upper bound in tells us

$$\ell \leq 2998.$$

Among the binomial expansiom of $5\sqrt{2}$ it appears at most 12 consecutive zeros up to 2^{-3000}. So it holds

$$\left|x - 5\sqrt{2}\cdot 2^{\ell}\right| \geq 2^{-13}$$

for any $x \in \boldsymbol{Z}$, $\ell \leq 3000$. Again by the factorization $|D| = |x^2 - 25\cdot 2^{2\ell+1}| = |x - 5\sqrt{2}\cdot 2^{\ell}||x + 5\sqrt{2}\cdot 2^{\ell}|$ we have

$$\frac{|D|}{5\sqrt{2}\cdot 2^{\ell}} > \frac{|D|}{x + 5\sqrt{2}\cdot 2^{\ell}} = |x - 5\sqrt{2}\cdot 2^{\ell}| \geq 2^{-13}.$$

Namely it holds $\ell < 12.5 + \frac{1}{\log 2}(\log|D| - \log 5)$. By considering the easy bound for even n case (1.1), we get the assertion.

q.e.d.

We put the list of solutions for odd D without the factor 25 up to $|D| < 1000$.

solutions of $x^2 + D = 25 \cdot 2^n$ $(0 < D < 1000)$							
D	(x,n)	D	(x,n)	D	(x,n)	D	(x,n)
1	$(7,1)$	19	$(9,2)$	31	$(13,3),(113,9)$	39	$(19,4)$
41	$(3,1)$	49	$(1,1)$	51	$(7,2)$	71	$(27,5)$
79	$(11,3),(39,6)$	91	$(3,2)$	99	$(1,2)$	111	$(17,4)$
119	$(9,3)$	151	$(7,3)$	159	$(79,8)$	191	$(3,3)$
199	$(1,3)$	231	$(13,4),(37,6)$	271	$(23,5)$	279	$(11,4)$
319	$(9,4),(159,10)$	351	$(7,4)$	359	$(21,5)$	391	$(3,4),(53,7)$
399	$(1,4)$	439	$(19,5),(926819,35)$	471	$(77,8)$	479	$(111,9)$
511	$(17,5),(33,6)$	599	$(51,7)$	631	$(13,5)$	639	$(31,6),(319,12)$
679	$(11,5)$	719	$(9,5)$	751	$(7,5)$	759	$(29,6)$
791	$(3,5)$	799	$(1,5),(49,7)$	871	$(27,6)$	919	$(109,9)$
951	$(157,10)$	991	$(47,7)$				

Solutions of $x^2 + D = 25 \cdot 2^n$ $(-1000 < D < 0)$							
D	(x,n)	D	(x,n)	D	(x,n)	D	(x,n)
−21	$(11,2)$	−31	$(9,1)$	−41	$(21,4),(29,5)$	−49	$(57,7)$
−69	$(13,2)$	−71	$(11,1)$	−81	$(41,6)$	−89	$(17,3)$
−119	$(13,1)$	−129	$(23,4)$	−161	$(19,3),(31,5),(81,8)$	−189	$(17,2)$
−239	$(17,1)$	−241	$(21,3)$	−249	$(43,6)$	−261	$(19,2)$
−281	$(59,7)$	−289	$(33,5)$	−311	$(19,1)$	−321	$(161,10)$
−329	$(23,3),(27,4),(227,11)$	−341	$(21,2)$	−391	$(21,1)$	−409	$(453,13)$
−429	$(23,2)$	−441	$(29,4)$	−479	$(23,1)$	−489	$(83,8)$
−521	$(61,7)$	−529	$(27,3)$	−561	$(31,4)$	−569	$(37,5)$
−609	$(47,6)$	−629	$(27,2)$	−641	$(29,3),(321,12)$	−679	$(27,1)$
−689	$(33,4)$	−721	$(39,5)$	−741	$(29,2)$	−761	$(31,3)$
−769	$(63,7)$	−791	$(29,1)$	−861	$(31,2)$	−881	$(41,5)$
−889	$(33,3),(117,9)$	−911	$(31,1)$	−969	$(37,4),(163,10)$	−989	$(33,2)$

Also we put the list of plural solutions for odd D without the factor 25 up to $|D| < 250000$. In this case we have an upper bound $n \leq 74$.

Double solutions of $x^2 + D = 25 \cdot 2^n (0 < D < 250000)$							
D		D		D		D	
31	$n = 3,9$	79	$n = 3,6$	231	$n = 4,6$	319	$n = 4,10$
391	$n = 4,7$	439	$n = 5,35$	511	$n = 5,6$	639	$n = 6,12$
799	$n = 5,7$	1071	$n = 6,8$	1351	$n = 7,9$	1519	$n = 6,7$
1911	$n = 8,12$	2191	$n = 9,10$	2359	$n = 7,11$	3151	$n = 7,8$
3199	$n = 7,13$	4879	$n = 8,9$	6279	$n = 8,10$	9991	$n = 9,11$
10591	$n = 9,12$	11439	$n = 10,14$	19039	$n = 10,13$	24511	$n = 10,19$
32431	$n = 11,15$	40399	$n = 12,17$	121279	$n = 13,25$		

Remark 5.1 *We did not find any plural solution among* $121279 < D < 10^8$*. So the equation (0.1) may admits pluri solutions only finite values of D except the solutions coming from Beukers' case for positive numbers D. And for a negative number D it may have at most 3 solutions. But we don't have proofs for these two conjectures.*

multiple solutions of $x^2 + D = 25 \cdot 2^n$ $(0 > D > -250000)$							
D		D		D		D	
−41	$n = 4,5$	−161	$n = 3,5,8$	−329	$n = 3,4,11$	−641	$n = 3,12$
−889	$n = 3,9$	−969	$n = 4,10$	−1169	$n = 3,8$	−1281	$n = 4,14$
−1649	$n = 3,6$	−1841	$n = 7,9$	−1881	$n = 6,8$	−2009	$n = 3,5$
−2201	$n = 3,4$	−2921	$n = 5,17$	−3281	$n = 3,21$	−3689	$n = 5,7$
−4329	$n = 6,10$	−4361	$n = 4,9$	−4529	$n = 5,12$	−4641	$n = 4,6$

multiple solutions of $x^2 + D = 25 \cdot 2^n \quad (0 > D > -250000)$							
D		D		D		D	
-5129	$n = 3, 5$	-5729	$n = 3, 10$	-5841	$n = 4, 12$	-5921	$n = 8, 11$
-5969	$n = 6, 9$	-7081	$n = 9, 15$	-7161	$n = 10, 12$	-7721	$n = 3, 13$
-7849	$n = 5, 11$	-8249	$n = 4, 7$	-9009	$n = 4, 6, 14$	-9401	$n = 4, 5, 9$
-9569	$n = 7, 13$	-9809	$n = 5, 11$	-10001	$n = 3, 9$	-10241	$n = 8, 20$
-10721	$n = 6, 15$	-11169	$n = 6, 12$	-12369	$n = 4, 8$	-13289	$n = 4, 13$
-13961	$n = 3, 7$	-16441	$n = 3, 9$	-17249	$n = 7, 10$	-17969	$n = 5, 11, 16$
-18921	$n = 4, 10$	-18929	$n = 13, 19$	-21489	$n = 8, 10$	-22001	$n = 3, 5$
-23681	$n = 6, 9$	-24449	$n = 3, 9$	-24689	$n = 7, 17$	-25361	$n = 7, 15$
-27089	$n = 5, 8$	-27761	$n = 5, 10, 11$	-28841	$n = 4, 7$	-29041	$n = 3, 15$
-29729	$n = 3, 6$	-31961	$n = 5, 17$	-33449	$n = 10, 16$	-36281	$n = 3, 13$
-36401	$n = 7, 10$	-36449	$n = 5, 8$	-38009	$n = 5, 7$	-38409	$n = 4, 10$
-38801	$n = 5, 6$	-39401	$n = 3, 11$	-43329	$n = 8, 14$	-44321	$n = 3, 9$
-46529	$n = 7, 17$	-47561	$n = 4, 15$	-48209	$n = 9, 13$	-48929	$n = 5, 10$
-51129	$n = 4, 10$	-52241	$n = 3, 10$	-54281	$n = 9, 13$	-56321	$n = 5, 13$
-59409	$n = 6, 16$	-60809	$n = 3, 7$	-62369	$n = 11, 14$	-66281	$n = 5, 15$
-66521	$n = 6, 12$	-72929	$n = 6, 13$	-75761	$n = 7, 17$	-76529	$n = 3, 12$
-77441	$n = 4, 10$	-79289	$n = 5, 13$	-80321	$n = 7, 16$	-81921	$n = 6, 26$
-89001	$n = 4, 6$	-90209	$n = 6, 21$	-90321	$n = 8, 12$	-93849	$n = 4, 12$
-94129	$n = 9, 27$	-95921	$n = 5, 14$	-97769	$n = 3, 17$	-101361	$n = 4, 16$
-104489	$n = 8, 14$	-106641	$n = 6, 16$	-107961	$n = 6, 18$	-111969	$n = 6, 12$
-114449	$n = 7, 11$	-116081	$n = 3, 9, 11$	-121601	$n = 3, 6$	-127281	$n = 6, 10$
-133889	$n = 5, 9$	-134561	$n = 6, 11$	-137241	$n = 4, 8$	-138929	$n = 3, 7$
-139769	$n = 11, 13$	-144809	$n = 9, 29$	-149681	$n = 7, 19$	-152849	$n = 6, 9$
-154401	$n = 8, 14$	-156009	$n = 6, 8$	-156809	$n = 5, 17$	-157409	$n = 3, 14$
-157601	$n = 6, 7$	-165249	$n = 4, 16$	-166481	$n = 5, 14$	-168521	$n = 4, 13$
-172529	$n = 8, 11$	-180841	$n = 7, 13$	-185969	$n = 11, 12$	-193409	$n = 8, 19$
-196049	$n = 3, 9$	-201201	$n = 4, 12, 18$	-208649	$n = 3, 13$	-212321	$n = 3, 12$
-215481	$n = 10, 20$	-219161	$n = 5, 15$	-220241	$n = 6, 11$	-225401	$n = 10, 22$
-225929	$n = 6, 12$	-227329	$n = 3, 15$	-228641	$n = 5, 11$	-232889	$n = 4, 11$
-242849	$n = 3, 18$	-243761	$n = 10, 15$	-246609	$n = 4, 8$		

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References

[1] F. Beukers, On the generalized Ramanujan-Nagell equation I, Acta Arithmetica *38*,,(1981), pp389-410.

[2] F. Beukers, On the generalized Ramanujan-Nagell equation II, Acta Arithmetica *39*, (1981), pp113-123.

[3] T. Maruyama, On the Ramanujan-Nagell type equations (in Japanese), Master Thesis, Graduate school of Sci. Tech., Chiba Univ. (2007).