Supplement to Extended Gauss AGM, Quartic Relations among MatsumotoTheta functions

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This is a supplementary part of the joint work "Extended Gauss AGM and corresponding Picard modular forms" by Kenji Koike and myself. We give here detailed proofs of the linear relations among quartic power of Matsumoto theta functions induced from the family of Matsumoto hyperelliptic curves $C(x,y) : w^4 = z^2(z-1)^2(z-x)(z-y)$ as Theorem 1.1, and the simplified expression of the main theorem in the original paper of Matsumoto as Theorem 1.2.

1 Preambles

We recall the necessary notations.

$$\boldsymbol{B}^2 = \{(u,v) \in \boldsymbol{C}^2 : 2\,\mathrm{Im}\,v - |u|^2 > 0\} = \{\xi = [\xi_0, \xi_1, \xi_2] \in \boldsymbol{P}^2 : \xi H {}^t\overline{\xi} < 0\},$$

$$\text{where} H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, u = \frac{\xi_2}{\xi_0}, v = \frac{\xi_1}{\xi_0},$$

$$\Omega(u,v) = \begin{pmatrix} \frac{iu^2}{2} + v & -\frac{u^2}{2} & -iu \\ -\frac{u^2}{2} & -\frac{iu^2}{2} + v & u \\ -iu & u & i \end{pmatrix}, \quad (u,v) \in \boldsymbol{B}^2$$

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1+i & 1 & 1-i \\ -1-i & 0 & i \end{pmatrix}, g_2 = \begin{pmatrix} 2+i & -1-i & -1-i \\ 1+i & -i & -1-i \\ 1-i & -1+i & i \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} i & 1-i & 1-i \\ 0 & i & 0 \\ 0 & -1-i & -1 \end{pmatrix}, g_5 = \begin{pmatrix} 2+i & -1-i & 1-i \\ 1+i & -i & 1-i \\ -1-i & 1+i & i \end{pmatrix},$$

$$g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, g_7 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g_8 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$G = \langle g_1, g_2, g_3, g_4, g_5 \rangle, G_1 = \langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle,$$

$$G_2 = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_8 \rangle, G_0 = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \rangle,$$

where G, G_1 and G_2 act on ${}^t(\xi_1, \xi_2, \xi_0)$ from left. As projective groups we have $\pi(G_0) = PU(2,1;\boldsymbol{Z}[i]), \pi(G_1) = \pi(G(1+i))$, where π indicates projectification and $SU(2,1,\boldsymbol{Z}[\sqrt{-1}]) = \{g \in SL(3,\boldsymbol{Z}[\sqrt{-1}]) :\ gH{}^t\overline{g} = H\}$, $G(1+i) = \{g \in U(2,1;\boldsymbol{Z}[i] :\ g \equiv E \pmod{(1+i)}\}$. We need the rational representation $N(\delta_6), N(\delta_8)$ of g_6, g_8 , respectively:

$$N(g_6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \ N(g_8) = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We use

$$\vartheta \begin{bmatrix} p \\ q \end{bmatrix} (v) = \sum_{n \in \boldsymbol{Z}} \exp[\pi i (n + \frac{p}{2})^2 v + 2\pi i (n + \frac{p}{2}) \frac{q}{2}] \quad (p, q \in \boldsymbol{Z},\ \mathrm{Im}\ (\mathrm{v}) > 0),$$

$$\vartheta \begin{bmatrix} p \\ q \end{bmatrix} (z, \Omega) = \vartheta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (z, \Omega)$$

$$= \sum_{n=(n_1,n_2,n_3) \in \boldsymbol{Z}^3} \exp[\pi i (n + \frac{p}{2}) \Omega^t (n + \frac{p}{2}) + 2\pi i (n + \frac{p}{2})^t (z + \frac{q}{2})],$$

$(z \in \boldsymbol{C}^3, \Omega \in \boldsymbol{H}_3$ (the Siegel upper half space of degree 3)$, p, q \in \boldsymbol{Z}^3)$

Matsumoto theta functions are defined by

$$\Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (u, v) = \vartheta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (\Omega(u, v))$$

$$= \sum_{n=(n_1,n_2,n_3) \in \boldsymbol{Z}^3} \exp[\pi i (n + \frac{p}{2}) \Omega(u, v)^t (n + \frac{p}{2}) + \pi i (n + \frac{p}{2})^t q].$$

We have 20 different Matsumoto theta functions $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (u, v)$:

number	name	characteristic $\{a, b\}$
n.1	Θ_{h1}	$\{\{0,0,0\},\{0,0,0\}\}$
n.2		$\{\{1,0,0\},\{0,0,0\}\}$
n.3		$\{\{0,0,0\},\{1,0,0\}\}$
n.4		$\{\{0,0,1\},\{0,0,0\}\}$
n.5	Θ_{h2}	$\{\{1,1,0\},\{0,0,0\}\}$
n.6	Θ_{yN}	$\{\{1,0,1\},\{0,0,0\}\}$
n.7		$\{\{1,0,0\},\{0,1,0\}\}$
n.8	Θ_{xN}	$\{\{1,0,0\},\{0,0,1\}\}$
n.9	Θ_{h3}	$\{\{0,0,0\},\{1,1,0\}\}$
n.10	Θ_{xZ}	$\{\{0,0,1\},\{1,0,0\}\}$
n.11	Θ_{yZ}	$\{\{0,0,1\},\{0,1,0\}\}$
n.12		$\{\{1,1,1\},\{0,0,0\}\}$
n.13	Θ_{yD}	$\{\{1,0,1\},\{0,1,0\}\}$
n.14	Θ_{xD}	$\{\{1,0,0\},\{0,1,1\}\}$
n.15		$\{\{0,0,1\},\{1,1,0\}\}$
n.16		$\{\{1,0,1\},\{1,0,1\}\}$
n.17		$\{\{1,1,1\},\{1,1,0\}\}$
n.18		$\{\{1,1,1\},\{1,0,1\}\}$
n.19		$\{\{1,0,1\},\{1,1,1\}\}$
n.20		$\{\{1,1,0\},\{1,1,0\}\}$

Table 3.1

Here we note $\Theta \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} (u, v) = 0$, it reflects the fact that our curve $C(x, y)$ is hyperelliptic.

A holomorphic function $f(u, v)$ on $\boldsymbol{B}^2$ is said to be a modular form of weight d with respect to G_0 provided

$$f(g(u, v)) = ((a_1 + a_2 v + a_3 u))^d f(u, v), \quad g = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G_0. \tag{1.1}$$

Let $M_d(G_0)$ denote the vector space of such modular forms. We define the modular form for other subgroups of G_0 by the same way. We use the same notation for the vector spaces of such modular forms. Already we have

[Fact 1] $\dim M_4(G) = 4,\ \dim M_8(G) = 9$.

[Fact 2] Fourth powers of Matsumoto theta functions $\Theta_{h1}^4, \Theta_{h2}^4, \Theta_{h3}^4, \Theta_{xN}^4, \Theta_{yN}^4, \Theta_{xD}^4, \Theta_{yD}^4, \Theta_{xZ}^4, \Theta_{yZ}^4$ belong to $M_4(G)$, and they are a system of generators of $M_4(G)$.

We claim here

Theorem 1.1 (Fourth powers relations)

$$\begin{aligned}
&\Theta_{xN}^4 - \Theta_{xD}^4 = \Theta_{yN}^4 - \Theta_{yD}^4,\\
&\Theta_{h2}^4 = 2(\Theta_{xN}^4 - \Theta_{xD}^4),\\
&\Theta_{h1}^4 - \Theta_{h3}^4 = 2(\Theta_{xD}^4 + \Theta_{yN}^4),\\
&\Theta_{xZ}^4 - \Theta_{yZ}^4 = \Theta_{xN}^4 - \Theta_{yN}^4,\\
&\Theta_{h1}^4 - \Theta_{h2}^4 + \Theta_{h3}^4 = 2(\Theta_{xZ}^4 + \Theta_{yZ}^4),\\
\\
&\Theta_{xD}^4\Theta_{yN}^4 = \Theta_{xZ}^4(\Theta_{xN}^4 - (\Theta_{xD}^4),\\
&\Theta_{xN}^4\Theta_{yD}^4 = \Theta_{yZ}^4(\Theta_{yN}^4 - \Theta_{yD}^4).
\end{aligned}$$

Theorem 1.2 (Simplified Matsumoto hyperelliptic theta map theorem)

$$(x, y) = \left(\frac{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u, v)}{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} (u, v)}, \ \frac{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u, v)}{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} (u, v)} \right).$$

Remark 1.1 *The coincidence of the parameters $x = y$ corresponds to the divisor $u = 0$ on $\boldsymbol{B}^2$. In this case this formula is reduced to the classical Jacobi identity*

$$\lambda = \frac{\vartheta^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau)}{\vartheta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau)}.$$

2 Necessary tools

We use the following established tools.

Lemma 2.1 (Permutation Lemma after Matsumoto [Mat1])

$$\Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (u, v) = \exp[\pi i p_1 q_1] \Theta \begin{bmatrix} p_2 & p_1 & p_3 \\ q_2 & q_1 & q_3 \end{bmatrix} (iu, v) \tag{2.1}$$

Lemma 2.2 (Exchange formula after Matsumoto（The multiplication factor is misprinted in the original paper [Mat1])) *We have*

$$\Theta \begin{bmatrix} p_2 & p_1 & q_3 \\ q_2 & q_1 & p_3 \end{bmatrix} (u, v) = \exp[\pi i p_2 q_2 + \frac{\pi i}{2} p_3 q_3] \cdot \Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (u, v).$$

Proposition 2.1 (Transformation formula (see for example [I])) *Let $g \in G_0$, and let $N_g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$ be its symplectic representation. Then we have*

$$\Theta \left[N_g \circ \begin{bmatrix} a \\ b \end{bmatrix} \right] (g \circ (u, v)) = \varepsilon(g, a, b) \left(\det\ (C_g \Omega(u, v) + D_g)\right)^{1/2} \Theta \begin{bmatrix} a \\ b \end{bmatrix} (u, v).$$

Where

$$N_g \circ \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} D_g a - C_g b + \mathrm{dv}(C_g {}^t D_g) \\ -B_g a + A_g b + \mathrm{dv}(A_g {}^t B_g) \end{bmatrix},$$

("dv" *indicates the diagonal vector*) *and $\varepsilon(g, a, b)$ is a certain 8-th root of unity.*

Theorem 2.1 (Riemann Quartic Relation, see [Mum])

$$\vartheta\begin{bmatrix} a+b+c+d \\ e+f+g+h \end{bmatrix}\cdot\vartheta\begin{bmatrix} a+b-c-d \\ e+f-g-h \end{bmatrix}\cdot\vartheta\begin{bmatrix} a-b+c-d \\ e-f+g-h \end{bmatrix}\cdot\vartheta\begin{bmatrix} a-b-c+d \\ e-f-g+h \end{bmatrix}$$
$$= \frac{1}{2^3}\sum_{\alpha,\beta\in \boldsymbol{Z}^3/2\boldsymbol{Z}^3} \exp[-\frac{\pi i}{2}{}^t\beta(a+b+c+d)]\cdot\vartheta\begin{bmatrix} a+\alpha \\ e+\beta \end{bmatrix}\cdot\vartheta\begin{bmatrix} b+\alpha \\ f+\beta \end{bmatrix}\cdot\vartheta\begin{bmatrix} c+\alpha \\ g+\beta \end{bmatrix}\cdot\vartheta\begin{bmatrix} d+\alpha \\ h+\beta \end{bmatrix}$$

for $a, b, \ldots, g, h \in \boldsymbol{Z}^3/2\boldsymbol{Z}^3$.

3 Quartic Theta relations

We define the mapping P of theta characteristics:

$$P : \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} \to \begin{bmatrix} p_2 & p_1 & p_3 \\ q_2 & q_1 & q_3 \end{bmatrix}.$$

We extend the action of P on the polynomial of $\Theta\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$'s in a natural way.

Lemma 3.1 *We have*

$$xN^4 - xD^4 = yN^4 - yD^4,\ xZ^4 - xN^4 = yZ^4 - yN^4.$$

proof] It holds

$$xN(u,v)^4 - xD(u,v)^4 = \Theta^4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}(u,v) - \Theta^4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}(u,v)$$
$$= \Theta^4 P\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)(u,v) - \Theta^4 P\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right)(u,v) = P(yN(u,v)^4 - yD(u,v)^4)$$

We apply Theorem 2.1 for

$$\begin{bmatrix} a \\ e \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} b \\ f \end{bmatrix} = \begin{bmatrix} c \\ g \end{bmatrix} = \begin{bmatrix} d \\ h \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$$8\Theta^4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \Theta^4\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Theta^4\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \Theta^4\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \Theta^4\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
$$+\Theta^4\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \Theta^4\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \Theta^4\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \Theta^4\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Here we used the fact that the theta constants with odd theta characteristics are identically zero, and that $\Theta\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = 0$. By the same manner we have

$$8\Theta^4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \Theta^4\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Theta^4\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \Theta^4\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} - \Theta^4\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
$$+\Theta^4\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} + \Theta^4\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \Theta^4\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \Theta^4\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We have the similar equalities for yN^4 and yD^4 by the action of P. By taking their differences we have

$$
\begin{aligned}
&9((xN^4 - xD^4) - (yN^4 - yD^4)) \\
&= (\Theta^4 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \Theta^4 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}) - P(\Theta^4 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \Theta^4 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}) = 0.
\end{aligned}
$$

Here we made the reduction of the theta's by using Matsumoto exchange formula .

We can show the second equality by the same method.

q.e.d.

Proposition 3.1 *We have* $\dim M_4(G_1) = 3$*, and* $M_4(G_1) = \langle \Theta_{h1}, \Theta_{h2}, \Theta_{h3} \rangle$.

proof].

We have the diagram

$$
\begin{array}{ccc}
\boldsymbol{P}^1 \times \boldsymbol{P}^1 & \xrightarrow{\quad\Phi\quad} & \overline{\boldsymbol{B}^2/G} \cong \{X_0X_3 = X_1X_2\} \subset \boldsymbol{P}^3 \\
(x, y) & & (u, v) \\
\downarrow \pi & & \downarrow \pi \\
\boldsymbol{P}^2 = (\boldsymbol{P}^1 \times \boldsymbol{P}^1)/\iota & \xrightarrow{\quad\Psi\quad} & \overline{\boldsymbol{B}^2/G_1} \cong \boldsymbol{P}^2
\end{array}
$$

Diagram 1: Period diagram

,where ι means the involution $(x, y) \mapsto (y, x)$. Then G_1 determines an orbifold with the arrangement

name	divisor	weight
D_{0x}	$\{(x, y) :\ x = 0\}$	4
D_{1x}	$\{(x, y) :\ x = 1\}$	4
$D_{\infty x}$	$\{(x, y) :\ x = \infty\}$	4
D_{xy}	$\{(x, y) :\ x = y\}$	4

.

By observing the pull back of the 2-form $du \wedge dv$ on $\boldsymbol{B}^2$ by the isomorphism

$$\Psi : \boldsymbol{P}^2 = (\boldsymbol{P}^1 \times \boldsymbol{P}^1)/\iota \to \overline{\boldsymbol{B}^2/G_1}$$

, we have

$$(du \wedge dv) = -\frac{3}{4}D_{0x} - \frac{3}{4}D_{1x} - \frac{3}{4}D_{\infty x} - \frac{3}{4}D_{xy}.$$

For $f(u, v) \in M_4(G_1)$, we have the relation

$$\Big(f(u, v)(du \wedge dv)^{4/3}\Big) \geq -D_{0x} - D_{1x} - D_{\infty x} - D_{xy}.$$

Let H be a hyperplane on $\boldsymbol{P}^2 = (\boldsymbol{P}^1 \times \boldsymbol{P}^1)/\iota$. The divisor D_{xy} is linearly equivalent to $2H$. So we have

$$M_4(G_1) \cong H^0(\boldsymbol{P}^2, \mathscr{O}(5H + \frac{4}{3}K)).$$

And

$$H^0(\boldsymbol{P}^2, \mathscr{O}(5H + \frac{4}{3}K)) = H^0(\boldsymbol{P}^2, \mathscr{O}(H)) = \langle 1, s, t \rangle,$$

where s, t are afine coordinates of $\boldsymbol{P}^2$.Hence we obtain the required conclusion.

q.e.d.

We obtain the following by an analogous argument.

Proposition 3.2 *We have* $\dim M_4(G_2) = 1$*, and* $M_4(G_2) = \langle \Theta_{h2}^4 \rangle$.

proof] We obatain $\dim M_4(G_2) = 1$ by a similar Riemann-Roch argument as above. It is sufficient just to show $\Theta_{h2}^4 \in M_4(G_2)$. g_8 has the rational representation

$$N(g_8) = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

According to the Transformation formula we know

$$\Theta_{h2}(u,v) = \vartheta N(g_8) \circ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (N(g_8) \circ \Omega(u,v))$$
$$= \vartheta \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (N(g_8) \circ \Omega(u,v)) = \rho \exp(2\pi i) \vartheta \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\Omega(u,v)),$$

where ρ is an 8-th root of unity. So it holds

$$\Theta_{h2}^4(g_8 \circ (u,v)) = \pm \Theta_{h2}^4(u,v).$$

We can determine the signature by an approximate calculation, and we find that it is positive. Namely $\Theta_{h2}^4(u,v) \in M_4(G_2)$.

q.e.d.

Lemma 3.2 *We have* $\Theta_{h2}^4 = 2(\Theta_{xN}^4 - \Theta_{xD}^4)$.

proof] According to Lemma 3.1 $f(u,v) = \Theta_{xN}^4 - \Theta_{xD}^4$ belongs to $M_4(G_1)$. So we show it belongs to $M_4(G_2)$. For it, we should see the automorphic property just for g_8:

$$f(g_8(u,v)) = f(u,v). \tag{3.1}$$

According to the transformation formula we have

$$\Theta N(g_8) \circ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (g_8(u,v)) = \Theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} (u, v+1) = \varepsilon \sqrt{-1} \Theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u,v),$$

where ε is a 8-th root of unity that is independent of the characteristic. So we have

$$f(g_8(u,v)) = \pm f(u,v).$$

We can determine the signature by an approximate calculation, and it is positive. So we obtain the equality (3.1) as desired.

As both Θ_{h2}^4 and $2(\Theta_{xN}^4 - \Theta_{xD}^4)$ belong to $M_4(G_2)$, they differ just a constant factor. By observing the shape of $\Omega(u,v)$ we have

$$\Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (0,v) = \vartheta \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} (v) \cdot \vartheta \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} (v) \cdot \vartheta \begin{bmatrix} p_3 \\ q_3 \end{bmatrix} (i).$$

By using the Jacobi identity $\vartheta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vartheta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \vartheta^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have

$$\Theta_{h2}^4(0,v) - 2(\Theta_{xN}^4(0,v) - \Theta_{xD}^4(0,v)) = \vartheta^8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (v) \{ \vartheta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (i) - 2\vartheta^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (i) \} = 0.$$

q.e.d.

Lemma 3.3 *We have* $2(\Theta_{xZ}^4 + \Theta_{yZ}^4) = \Theta_{h1}^4 - \Theta_{h2}^4 + \Theta_{h3}^4$.

proof]. Set $f_1 = 2(\Theta^4_{xZ} + \Theta^4_{yZ})$ and set $f_2 = \Theta^4_{h1} - \Theta^4_{h2} + \Theta^4_{h3}$. Both of then are symmetric under the action of P. So they belong to the one dimensional vector space $M_4(G_1)$. Then it is sufficient to show that the ratio $f_1/f_2 \equiv 1$. For it, we consider the values on the divisor $\{u = 0\}$.

We have

$$f_2(0,v) = \Theta^4_{h1}(0,v) - \Theta^4_{h2}(0,v) + \Theta^4_{h3}(0,v) = \vartheta^4\begin{bmatrix}0\\0\end{bmatrix}(i)\left\{\vartheta^8\begin{bmatrix}0\\0\end{bmatrix}(v) - \vartheta^8\begin{bmatrix}1\\0\end{bmatrix}(v) + \vartheta^8\begin{bmatrix}0\\1\end{bmatrix}(v)\right\}$$
$$= 2\vartheta^4\begin{bmatrix}1\\0\end{bmatrix}(i)\left\{\left(\vartheta^4\begin{bmatrix}1\\0\end{bmatrix}(v) + \vartheta^4\begin{bmatrix}0\\1\end{bmatrix}(v)\right)^2 - \vartheta^8\begin{bmatrix}1\\0\end{bmatrix}(v) + \vartheta^8\begin{bmatrix}0\\1\end{bmatrix}(v)\right\}$$
$$= 4\vartheta^4\begin{bmatrix}1\\0\end{bmatrix}(i)\vartheta^4\begin{bmatrix}0\\1\end{bmatrix}(v)\left(\vartheta^4\begin{bmatrix}0\\1\end{bmatrix}(v) + \vartheta^4\begin{bmatrix}1\\0\end{bmatrix}(v)\right) = 4\vartheta^4\begin{bmatrix}1\\0\end{bmatrix}(i)\vartheta^4\begin{bmatrix}0\\1\end{bmatrix}(v)\vartheta^4\begin{bmatrix}0\\0\end{bmatrix}(v).$$

And we have

$$f_1(0,v) = 2(\Theta^4_{xZ}(0,v) + \Theta^4_{yZ})(0,v) = 4\Theta^4_{xZ}(0,v)$$
$$= 4\vartheta^4\begin{bmatrix}0\\1\end{bmatrix}(v)\vartheta^4\begin{bmatrix}0\\0\end{bmatrix}(v)\vartheta^4\begin{bmatrix}1\\0\end{bmatrix}(i).$$

So we have $f_1(0,v) = f_2(u,v)$. It means the required equality on the whole domain $\boldsymbol{B}^2$.

q.e.d.

The rest of the equalities in Proposition 1.1 can be showed by the same manner. So we obtain all the required equalities.

4 Reduction of the Matsumoto theta formula

The original Matsumoto theta formula ([Mat1])is given by

$$x = \frac{\Theta^2\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}\Theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\Theta^2\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\Theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}}{\Theta^2\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}\Theta^2\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\Theta^2\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}\Theta^2\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}},$$
$$y = \frac{\Theta^2\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}\Theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\Theta^2\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\Theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}}{\Theta^2\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}\Theta^2\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\Theta^2\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}\Theta^2\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}}.$$

According to Exchange formula we have

$$\begin{cases}\Theta^2\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\Theta^2\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix} = \Theta^4\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\\ \Theta^2\begin{bmatrix}0&1&1\\1&0&0\end{bmatrix}\Theta^2\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix} = \Theta^4\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}\\ \Theta^2\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\Theta^2\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix} = \Theta^4\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\\ \Theta^2\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}\Theta^2\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix} = \Theta^4\begin{bmatrix}0&1&0\\1&0&1\end{bmatrix}.\end{cases}$$

For the proof of Theorem 1.2, it is sufficient to show

$$\Theta^2\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}(u,v)\Theta^2\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}(u,v) = \Theta^2\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}(u,v)\Theta^2\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}(u,v). \quad (4.1)$$

As we discussed in the proof of Dimension formula ([Ko-Shi2]), a meromorphic modular form $f(u,v)$ with respect to G belongs to $M_4(G)$ if and only if

$$f(u,v)(du \wedge dv)^{4/3} \geq -3D_{0x} - 3D_{0y} - \frac{2}{3}D_{xy}.$$

So $f(u,v) \in M_4(g)$ is zero on $\{u=0\}$ if and only if

$$f(u,v)(du \wedge dv)^{4/3} \geq -3D_{0x} - 3D_{0y} + \frac{1}{3}D_{xy}.$$

We have

$$H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}(\frac{8}{3}D_{0x} + \frac{8}{3}D_{0y} + \frac{4}{3}K)) \cong H^0(\boldsymbol{P}^1 \times \boldsymbol{P}^1, \mathscr{O}) \cong \boldsymbol{C}.$$

Then

$$\dim_{\boldsymbol{C}}\{f(u,v) \in M_4(G) : (f) \geq \{u=0\}\} = 1.$$

Because $\vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (v) = 0$, both sides of (4.1) are zero on $\{u=0\}$. So they differ only a constant factor. Let us determine this constant.

Put

$$f_3(z,u,v) = \vartheta^2 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} (z, \Omega(u,v)) \cdot \vartheta^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (z, \Omega(u,v)),$$
$$f_4(z,u,v) = \vartheta^2 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} (z, \Omega(u,v)) \cdot \vartheta^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} (z, \Omega(u,v)).$$

For $p, q, p', q' \in \boldsymbol{Z}^3/2\boldsymbol{Z}^3$, we have

$$\frac{\Theta \begin{bmatrix} p \\ q \end{bmatrix} (u,v)}{\Theta \begin{bmatrix} p' \\ q' \end{bmatrix} (u,v)} = \frac{\vartheta \begin{bmatrix} p \\ q \end{bmatrix} (0, \Omega(u,v))}{\vartheta \begin{bmatrix} p' \\ q' \end{bmatrix} (0, \Omega(u,v))} = \lim_{z \to 0} \frac{\vartheta \begin{bmatrix} p \\ q \end{bmatrix} (z, \Omega(u,v))}{\vartheta \begin{bmatrix} p' \\ q' \end{bmatrix} (z, \Omega(u,v))}.$$

So it holds

$$\lim_{u \to 0} \frac{\Theta \begin{bmatrix} p \\ q \end{bmatrix} (u,v)}{\Theta \begin{bmatrix} p' \\ q' \end{bmatrix} (u,v)} = \lim_{u \to 0} \lim_{z \to 0} \frac{\vartheta \begin{bmatrix} p \\ q \end{bmatrix} (z, \Omega(u,v))}{\vartheta \begin{bmatrix} p' \\ q' \end{bmatrix} (z, \Omega(u,v))} = \lim_{z \to 0} \lim_{u \to 0} \frac{\vartheta \begin{bmatrix} p \\ q \end{bmatrix} (z, \Omega(u,v))}{\vartheta \begin{bmatrix} p' \\ q' \end{bmatrix} (z, \Omega(u,v))}$$
$$= \lim_{z \to 0} \frac{\vartheta \begin{bmatrix} p \\ q \end{bmatrix} (z, \Omega(0,v))}{\vartheta \begin{bmatrix} p' \\ q' \end{bmatrix} (z, \Omega(0,v))} = \lim_{z \to 0} \frac{\vartheta \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} (z,v) \vartheta \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} (z,v) \vartheta \begin{bmatrix} p_3 \\ q_3 \end{bmatrix} (z,i)}{\vartheta \begin{bmatrix} p'_1 \\ q'_1 \end{bmatrix} (z,v) \vartheta \begin{bmatrix} p'_2 \\ q'_2 \end{bmatrix} (z,v) \vartheta \begin{bmatrix} p'_3 \\ q'_3 \end{bmatrix} (z,i)}.$$

By applying this formula we obtain

$$\frac{f_3(z,0,v)}{f_4(z,0,v)} = 1.$$

So we have

$$\lim_{z \to 0} \frac{f_3(z,0,v)}{f_4(z,0,v)} = 1.$$

It means the required equality.

q.e.d.

References

[I] J. Igusa, Theta functions, *Springer-Verlag* , Heidelberg, New-York(1972).

[Ko-Shi1] K. Koike and H. Shiga, Isogeny formulas for the Picard modular form and a three terms arithmetic geometric mean, J. Number Theory, to appear

[Ko-Shi2] K. Koike and H. Shiga, Extended Gauss AGM and Corresponding Picard modular forms, Tech. Reports Math. Sci. Chiba Univ., 22(2006), no.1, 1-14.

[Mat1] K. Matsumoto, On Modular Functions in 2 Variables Attached to a Family of Hyperelliptic Curves of Genus 3, Ann. S.N.S. Pisa, Ser IV-XVI(1989), 557-578.

.

[Mum] D. Mumford, Tata Lectures on Theta I, *Birkhäuser*, Boston-Basel-Stuttgard(1983).