# Fuchsian Initial Value Problem on $\boldsymbol{P}^2(\boldsymbol{C})$ with Hypergeometric Functions as Data along $\boldsymbol{P}^1(\boldsymbol{C})$

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## 1 Introduction

Let $[\tau : \xi : \eta]$ be a system of homogeneous coordinates for 2-dimensional complex projective space $\boldsymbol{P}^2(\boldsymbol{C})$, and let $(t, x) = \left(\frac{\tau}{\eta}, \frac{\xi}{\eta}\right)$ be its affine coordinates in $\boldsymbol{P}^2(\boldsymbol{C}) \smallsetminus \{\eta = 0\} \simeq \boldsymbol{C}^2$. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - t^{2m}\frac{\partial^2 u}{\partial x^2} + \frac{1-\lambda-\mu}{t}\frac{\partial u}{\partial t} + at^{m-1}\frac{\partial u}{\partial x} + \frac{\lambda\mu}{t^2}u = 0 \tag{1.1}$$

in $\boldsymbol{P}^2(\boldsymbol{C})$, where $m$ is a nonnagative integer and $\lambda$, $\mu$ and $a$ are complex nunbers. This equation has regular singularity along the line $\{t = 0\} \simeq \boldsymbol{P}^1(\boldsymbol{C})$ in the sense of Kashiwara-Oshima [13]. We seek a solution of the equation (1.1) under the condition that, as $t \to 0$, $u(t, x)$ behaves like $t^\lambda\{x^p(1-x)^q F(\alpha, \beta, \gamma, x)\}$, where $F(\alpha, \beta.\gamma, x)$ denotes the Gauss hypergeometric function and $p, q, \alpha, \beta, \gamma$ are complex parameters. We will call this type of problem *Fuchsian initial value problem*, the precise definition of which will be given in section 2.

The reason why we consider the Fuchsian initial value problem instead of the usual Cauchy problem is that we want to obtain a class of solutions in $\boldsymbol{P}^2(\boldsymbol{C})$ having sufficiently many parameters. In fact, we can obtain by solving our problem a class of solutions which involves Appell's 2-dimensional hypaegeometric functions $\mathrm{F}_1$ and $\mathrm{F}_2$ as subclasses, which is impossible by the usual Cauchy paoblem for lack of some parameter.

We must mention that it is already known that, in the case $m \geq 1$, even if the initial data have poles, the solution of the Cauchy problem for the above equation, without the terms having singularities along $t = 0$, can be expressed not by elementary functions but by the use of the Gauss hypergeometric function

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(cf. Nakane [17], Urabe [20]). For example, the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - t^{2m}\dfrac{\partial^2 u}{\partial x^2} = 0, \\ u(0,x) = \dfrac{1}{x^k}, \\ \dfrac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

is

$$u(t,x) = \left(x + \frac{t^{m+1}}{m+1}\right)^{-k} F\left(\frac{m}{2(m+1)},\ k,\ \frac{m}{m+1},\ \frac{2t^{m+1}}{(m+1)t+t^{m+1}}\right).$$

The natural question arises. What function would we obtain then, if we give hypergeometric functions as data? This question was the motive for our study.

# 2 Fuchsian initial value problem

It is necessary for our study to explain what is Fuchsian initial value problem. For our purpose, we refer to Friom [8, 9, 10]. Let $(t,x) = (t, x_1, \dots, x_n) \in \boldsymbol{C}^{n+1}$ be independent variables, and consider the equation

$$\sum_{|\nu|\le m} \frac{a_\nu(t,x)}{t^{n-\nu_0-1+\delta_{\nu_0|\nu|}}} \frac{\partial^{|\nu|}u(t,x)}{\partial t^{\nu_0}\partial x_1^{\nu_1}\dots\partial x_n^{\nu_n}} = 0, \tag{2.1}$$

where

$$\begin{aligned} &\nu = (\nu_0, \nu_1, \dots, \nu_n), \qquad |\nu| = \nu_0 + \nu_1 + \dots + \nu_m, \\ &a_{n0\dots0}(t,x) = 1, \\ &a_{i0\dots0}(0,x) = a^0_{i0\dots0} \qquad (i = 0, 1, \dots n), \end{aligned}$$

$a^0_{i0\dots0}$ being complex constants, and $\delta_{ik}$ are the Kronecker delta, and $a_\nu(t,x)$ are elements of ${}_{n+1}\mathcal{O}_0$, where ${}_{n+1}\mathcal{O}_0$ denotes the set of germs of holomorphic functions at $(t,x) = (0,0) \in \boldsymbol{C}^{n+1}$. The *indicial equation* associated to the partial differential equation (2.1) is defined by

$$f(\rho) = \sum_{i=0}^{n} a^0_{i0\dots0}\,\rho(\rho-1)\dots(\rho-i+1) = 0, \tag{2.2}$$

and its roots $\rho_1, \rho_2, \dots, \rho_n$ are called *the characteristic exponents along the hypersurface* $\{t = 0\}$. We assume that $\rho_i - \rho_j \notin \boldsymbol{Z}$ if $i \neq j$. Let ${}_{n+1}\tilde{\mathcal{O}}_0$ be the set of germs of functions holomorphic in the universal covering space of $U \setminus \{t = 0\}$, $U$ being small neighborhood of the origin $(t,x) = (0,0)$. We denote by $\tilde{\mathcal{S}}_0$ be the set of all solutions of the equation (2.1) which belong to ${}_{n+1}\tilde{\mathcal{O}}_0$.

The following theorem is due to Froim [10].

**Theorem 2.1.** $\tilde{\mathcal{S}}_0 \simeq {}_n^{'}\mathcal{O}_0^m$.

Here ${}_n'\mathcal{O}_0^m$ denotes the direct product of $m$ copies of the set of germs of holomorphic functions in $n$ variables. Precisely speaking, any solution $u(t,x) \in \tilde{\mathcal{S}}_0$ can be expressed as

$$u(t,x) = \sum_{j=1}^{m} t^{\rho_j} u_{\rho_j}(t,x)$$

where $u_{\rho_j} \in {}_n'\mathcal{O}_0$ $(j = 1,\dots,m)$ and the above isomorphism is given by the map

$$A_t \colon \tilde{\mathcal{S}}_0 \longrightarrow {}_n'\mathcal{O}_0^m$$

defined by

$$A_t u = (u_{\rho_1}(0,x),\dots,u_{\rho_m}(0,x)).$$

Let $A_t^{\rho_j} : \tilde{\mathcal{S}}_0 \to {}_n'\mathcal{O}_0$ be the map defined by

$$A_t^{\rho_j} u = u_{\rho_j}(0,x).$$

The set of functions $(A_t^{\rho_1} u,\dots,A_t^{\rho_m} u) = (u_{\rho_1}(0,x),\dots,u_{\rho_m}(0,x))$ is called the *Fuchsian initial data*.

The above theorem is extended by Tahara [19] to the case where the indicial equation involves functions of $x$. For our study, it is enough to refer to the above elementary result.

Briefly speaking, *Fuchsian initial value problem is a problem to solve an equation of the form* (2.1) *with given Fuchsian initial data along* $\{t = 0\}$.

# 3 Setting of the problem

It is easy to verify that the characteristic exponents of the equation (1.1) along $\{t = 0\}$ are $\lambda$ and $\mu$.

We assume that $\lambda - \mu \notin \boldsymbol{Z}$ for the time being, but it is sufficient to assume that $\lambda \neq \mu$, as can be seen from later argument.

Now our problem is stated as follows.

*Consider the partial differential equation* (1.1) *in* $\boldsymbol{P}^2(\boldsymbol{C})$ *with Fuchsian initial data*

$$A_t^{\lambda} u = x^p (1-x)^q F(\alpha,\beta,\gamma,\,x), \tag{3.1}$$

$$A_t^{\mu} u = 0, \tag{3.2}$$

*where* $F(\alpha,\beta,\gamma,\,x)$ *denotes the Gauss hypergeometric function and* $p, q, \alpha, \beta$ *and* $\gamma$ *are complex parameters.*

Analytic solution of this problem is uniquely determined by Theorem 2.1. What kind of function would we obtain by this problem?

*Remark* 3.1. As the equation (1.1) is symmmetric with respect to the parameters $\lambda$ and $\mu$, it is sufficient to consider the above case where one of the data is zero because of the principle of superposition.

*Remark* 3.2. If $(\lambda, \mu) = (0,1)$ or $(1,0)$, the Fuchsian initial value problem is reduced to the usual Cauchy problem.

*Remark* 3.3. The solution space of a linear ordinary differential equation of order two with three regular singular points $0, 1$ and $\infty$ is spanned by two functions of the form of the right hand of (3.1) by the theory of Riemann's P-function. In this sense, the above data might be natural.

# 4 Characteristic algebraic curves in $\boldsymbol{P}^2(\boldsymbol{C})$

Local property of singularities of solutions of partial differential equation was studied by several authors. The pioneering work of Leray [15], Hamada [11] and Hamada-Leray-Wagschal [12] shows that, roughly speaking, the singularity of solutions of partial differential equations appear on characteristic varieties.

The *characteristic variety* is a complex analytic variety of codimension 1 with the property that any contact element on this variety annihilates the principal symbol of the operator.

In our case, by Chow's theorem (cf. Mumford [16]), such variety is projective algebraic curve in $\boldsymbol{P}^2(\boldsymbol{C})$. In the affine plane $\boldsymbol{C}^2 = \{[\tau : \xi : \eta] \in \boldsymbol{P}^2(\boldsymbol{C});\, \eta \neq 0\}$, the affine algebraic curve

$$V = \{(t,x) \in \boldsymbol{C}^2;\, f(t,x) = 0\} \subset \boldsymbol{C}^2,$$

where $f$ is a polynomial, is characteristic with respect to the principal symbol of the equation (1.1) divided by $t^2$, namely $T^2 - t^{2m}X^2$, where $(T, X)$ are the dual variables of $(t, x)$, if and only if either

$$\frac{\partial f}{\partial t}(t,x) - t^m \frac{\partial f}{\partial x}(t,x) = 0 \tag{4.1$_+$}$$

or

$$\frac{\partial f}{\partial t}(t,x) + t^m \frac{\partial f}{\partial x}(t,x) = 0 \tag{4.1$_-$}$$

is satisfied whenever $(t,x) \in V$. It is easy to see that general solution of (4.1$_+$) and (4.1$_-$) are

$$f(t,x) = G(x \pm \frac{t^{m+1}}{m+1}), \tag{4.2$_\pm$}$$

where the function $G$ is arbitrary. Therefore,

$$V_\pm^c = \{(t,x) \in \boldsymbol{C}^2;\, x \pm \frac{t^{m+1}}{m+1} - c = 0\}$$

($c \in \boldsymbol{C}$) are 1-parameter families of characteristic algebraic varieties. In homogeneous coordinates,

$$V_\pm^c = \{[\tau, \xi, \eta] \in \boldsymbol{P}^2(\boldsymbol{C});\, \xi\eta^m - c\eta^{m+1} \pm \frac{\tau^{m+1}}{m+1} = 0\}. \tag{4.3}$$

If we let $c \to \infty$ in (4.3) (see Remark 4.1 below), then we get the line at infinity

$$V^\infty = \{[\tau, \xi, \eta] \in \boldsymbol{P}^2(\boldsymbol{C});\ \eta = 0\}. \tag{4.4}$$

The line $V^\infty$ is also characteristic, for, by the coordinates $(t', x') = (\frac{\tau}{\xi}, \frac{\eta}{\xi})$, the equation (1.1) is written as

$$\begin{aligned}(t'^{2m+2} - x'^{2m})\frac{\partial^2 u}{\partial t'^2} + 2t'^{2m+1}x'\frac{\partial^2 u}{\partial t' \partial x'} + t'^{2m}x'^m\frac{\partial^2 u}{\partial x'^2} \\ + \{t'^{2m+1} - (1-\lambda-\mu)t'x'^{2m-2} + at'^m x'^m\}\frac{\partial u}{\partial t'} \\ + \{2t'^{2m}x' + at'^{m-1}x'^{m+1}\}\frac{\partial u}{\partial x'} - \lambda\mu t'^2 x'^{2m} u = 0,\end{aligned}$$

and all contact elements of $V^\infty$ annihilate the principal symbol of the above equation. In fact, we have

$$(t'^{2m+2} - x'^{2m})\left(\frac{\partial x'}{\partial t'}\right)^2 + 2t'^{2m+1}x'\left(\frac{\partial x'}{\partial t'}\right)\left(\frac{\partial x'}{\partial x'}\right) + t'^{2m}x'^2\left(\frac{\partial x'}{\partial x'}\right)^2\Bigg|_{x'=0} = 0.$$

*Remark* 4.1. The meaning of "$c \to \infty$" is as follows. Fix the variables $(\tau, \xi)$ and consider the algebraic equation of order $m+1$ in $\eta$:

$$\xi\eta^m - c\eta^{m+1} \pm \frac{\tau^{m+1}}{m+1} = 0.$$

Let $\eta_\pm^1(c), \eta_\pm^2(c), \ldots, \eta_\pm^{m+1}(c)$ be the roots of the above equation. Then, by Rouche's theorem, we have

$$\lim_{c\to\infty} \eta_\pm^j(c) = 0 \qquad (j = 1, \ldots, m+1).$$

Thus, in this sense, we have $\lim_{c\to\infty} V^c = V^\infty$.

Let us denote the initial line by $L$, namely

$$L = \{[\tau, \xi, \eta] \in \boldsymbol{P}^2(\boldsymbol{C});\ \tau = 0\}.$$

Then we have the following proposition.

**Proposition 4.1.** *$V_\pm^c$ $(c \in \boldsymbol{C})$, $V^\infty$ and $L$ are the only characteristic algebraic curves for the equation* (1.1).

*Proof.* Let $V$ be an arbitrary irreducible characteristic variety different from $V^\infty$ and $L$. We will show that $V$ must coincide one of the characteristic curves $V_\pm^c$ $(c \in \boldsymbol{C})$.

As the variety $V$ is different from $V^\infty$, we may take affine coordinates $(t, x) = (\frac{\tau}{\eta}, \frac{\xi}{\eta})$. Let

$$f(t, x) = \sum_{i+j\le l} a_{ij}t^i x^j \qquad (a_{ij} \in \boldsymbol{C}) \tag{4.5}$$

be the irreducible polynomial with zero locus $V$. If the curve $V$ does not intersect $L$ in this affine space, then we must have

$$f(t,x) = t - b$$

for some $b \in \boldsymbol{C}$, but it is obvious that contact elements of $\{(t,x);\, t-b=0\}$ never annihilate our principal symbol. So we may assume that $V$ intersect $L$ in the affine space. Let $(0,c)$ be one of the points of $V \cap L$. Since $V \neq L$, the equation $f(t,x)=0$ can be solved locally in a neighborhood of $(0,c)$ as $x=\phi(t)$, where $\phi(t)$ is a convergent Puiseux series with $\phi(0)=c$. We have of course the identity

$$f(t,\phi(t)) = 0, \tag{4.6}$$

in a neighborhood of $t=0$. Differentiating the equality (4.6) with respect to $t$, we have

$$\frac{\partial f}{\partial t}(t,\phi(t)) + \phi'(t)\frac{\partial f}{\partial x}(t,\phi(t)) = 0. \tag{4.7}$$

On the other hand, as $V$ is characteristic, we have

$$\frac{\partial f}{\partial t}(t,\phi(t)) - t^m\frac{\partial f}{\partial x}(t,\phi(t)) = 0 \tag{4.8$_+$}$$

or

$$\frac{\partial f}{\partial t}(t,\phi(t)) + t^m\frac{\partial f}{\partial x}(t,\phi(t)) = 0 \tag{4.8$_-$}$$

Comparing $(4.8_\pm)$ with (4.7), we have

$$(\phi'(t) \pm t^m)\frac{\partial f}{\partial x}(t,\phi(t)) = 0$$

in a neighborhood of $t=0$. As the function $\frac{\partial f}{\partial x}(t,\phi(t))$ cannnot be identically zero, for the polynomial must involve $x$, it follows that

$$\phi'(t) - t^m = 0 \qquad \text{or} \qquad \phi'(t) + t^m = 0.$$

Solving the differential equation, since $\phi(0)=c$, we have

$$\phi(t) = \frac{t^{m+1}}{m+1} + c \qquad \text{or} \qquad \phi(t) = -\frac{t^{m+1}}{m+1} + c.$$

Accordingly we have $V = V_+^c$ or $V = V_-^c$ in a neighborhood of $(0,c)$. By the theorem of identity, $V$ must globally coincide $V_+^c$ or $V_-^c$, concluding the proof of the proposition. □

Since the Fuchsian initial data (3.1) has singular points at $[0:0:1]$, $[0:1:1]$ and $[0:1:0]$, we may expext that the solution of the Fuchsian initial value problem has singularity along $V_\pm^0$, $V_\pm^1$, $V^\infty$ and $L$. We will see later that this is true.

The following proposition is easily verified (see Figere 1 when m=0).

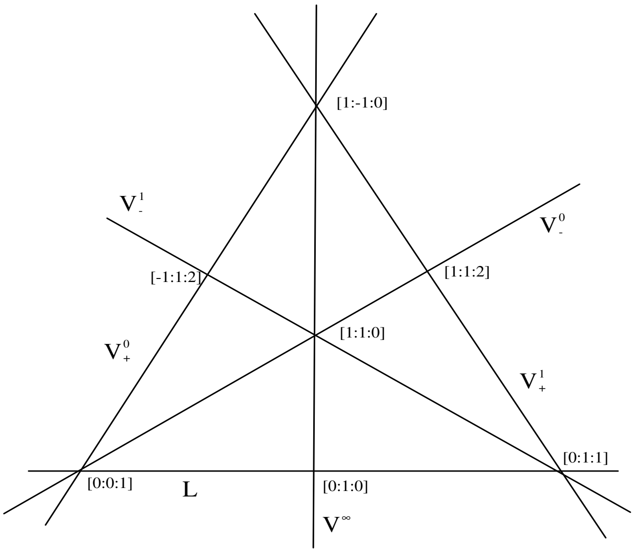


Figure 1: the case $m = 0$

**Proposition 4.2.**

$$\begin{aligned}
L \cap V^0_{\pm} &= \{[0:0:1]\} \qquad \textit{if } m = 0,\\
&= \{[0:0:1],[0:1:0]\} \qquad \textit{if } m \geq 1.\\
L \cap V^1_{\pm} &= \{[0:1:1]\} \qquad \textit{if } m = 0,\\
&= \{[0:1:1],[0:1:0]\} \qquad \textit{if } m \geq 1.\\
V^0_{\pm} \cap V^1_{\pm} &= \{[1:\mp 1:0]\} \qquad \textit{if } m = 0,\\
&= \{[0:1:0]\} \qquad \textit{if } m \geq 1.\\
V^0_{+} \cap V^1_{-} &= \{[-1:1:2]\} \qquad \textit{if } m = 0,\\
&= \{[0:1:0],[2^{\frac{1}{m+1}}(m+1)^{\frac{1}{m+1}}e^{\frac{2k+1}{m+1}\pi i}:1:2]\ (k=0,\dots,m)\} \quad \textit{if } m \geq 1.\\
V^0_{-} \cap V^1_{+} &= \{[1:1:2]\} \qquad \textit{if } m = 0,\\
&= \{[0:1:0],[2^{\frac{m}{m+1}}(m+1)^{\frac{1}{m+1}}e^{\frac{2k}{m+1}\pi i}:1:2]\ (k=0,\dots,m)\} \quad \textit{if } m \geq 1.\\
V^{\infty} \cap V^0_{\pm} &= \{[1:\mp 1:0]\} \qquad \textit{if } m = 0,\\
&= \{[0:1:0]\} \qquad \textit{if } m \geq 0.\\
V^{\infty} \cap V^1_{\pm} &= \{[1:\mp 1:0]\} \qquad \textit{if } m = 0,\\
&= \{[0:1:0]\} \qquad \textit{if } m \geq 0.
\end{aligned}$$

$$
\begin{aligned}
V^\infty \cap L &= \{[0:1:0]\}.\\
V_+^0 \cap V_-^0 &= \{[0:0:1]\} \qquad \textit{if } m = 0,\\
&= \{[0:0:1],[0:1:0]\} \qquad \textit{if } m \geq 1.\\
V_+^1 \cap V_-^1 &= \{[0:1:1]\} \qquad \textit{if } m = 0,\\
&= \{[0:1:1],[0:1:0]\}.
\end{aligned}
$$

# 5 Reduction to the Euler-Poisson-Darboux equation

In order to solve the equation (1.1) with the Fuchsian initial condition (3.1) and (3.2), we first change the unknown $u(t,x)$ to a new unknown $v(t,x)$ by

$$u(t,x) = t^\lambda v(t,x). \tag{5.1}$$

After an easy calculation, we obtain the equation

$$\frac{\partial^2 v}{\partial t^2} - t^{2m}\frac{\partial^2 v}{\partial x^2} + \frac{1+\lambda-\mu}{t}\frac{\partial v}{\partial t} + at^{m-1}\frac{\partial v}{\partial x} = 0 \tag{5.2}$$

and the condition

$$A_t^0 v = x^p(1-x)^q F(\alpha,\beta,\gamma,\, x), \tag{5.3}$$

$$A_t^{\mu-\lambda} v = 0. \tag{5.4}$$

If we take the characteristic coordinates

$$z_1 = x + \frac{t^{m+1}}{m+1}, \qquad z_2 = x - \frac{t^{m+1}}{m+1}, \tag{5.5}$$

the equation (5.2) is reduced to the Euler-Poisson-Darboux equation

$$\frac{\partial^2 v}{\partial z_1 \partial z_2} - \frac{m+1+\lambda-\mu+a}{2(m+1)(z_1-z_2)}\frac{\partial v}{\partial z_1} + \frac{m+1+\lambda-\mu-a}{2(m+1)(z_1-z_2)}\frac{\partial v}{\partial z_2} = 0. \tag{5.6}$$

For the sake of simplicity, we introduce new parameters $\varepsilon$ and $\delta$ by

$$\varepsilon = \frac{m+1+\lambda-\mu+a}{2(m+1)}, \qquad \delta = \frac{m+1+\lambda-\mu-a}{2(m+1)}. \tag{5.7}$$

Then the equation (5.6) takes the form

$$\frac{\partial^2 v}{\partial z_1 \partial z_2} - \frac{\varepsilon}{z_1-z_2}\frac{\partial v}{\partial z_1} + \frac{\delta}{z_1-z_2}\frac{\partial v}{\partial z_2} = 0. \tag{5.8}$$

## 6 Integral representation of the solution

The following theorem is due to Darboux [4].

**Theorem 6.1.** *Under the conditions*

$$0 < \operatorname{Re}\delta, \operatorname{Re}\varepsilon < 1, \tag{6.1}$$

*and*

$$\delta + \varepsilon \neq 1, \tag{6.2}$$

*general solution of* (5.8) *is represented by*

$$\begin{aligned} v = (z_1 - z_2)^{1-\delta-\varepsilon} \int_0^1 \Phi(z_1 + (z_2 - z_1)\zeta)\zeta^{-\delta}(1-\zeta)^{-\varepsilon}\, d\zeta \\ + \int_0^1 \Psi(z_1 + (z_2 - z_1)\zeta)\zeta^{\varepsilon-1}(1-\zeta)^{\delta-1}\, d\zeta, \end{aligned} \tag{6.3}$$

*where the functions* $\Phi$ *and* $\Psi$ *are arbitrary. There is no other solution than this representation.*

*Proof.* See Darboux [4]. □

The condition (6.2) implies, in our case, $\lambda \neq \mu$, which is obviously satisfied (see section 3).

In order to remove the restriction (6.1), we take the integration along the Pochhammmer's double loop. As is easily seen, analytic continuation as a function of $\delta$ and $\varepsilon$ of (6.3) is given by the double loop integral

$$\begin{aligned} v = \frac{(z_2 - z_1)^{1-\delta-\varepsilon}}{(1-e^{-2\pi i\delta})(1-e^{-2\pi i\varepsilon})} \oint_{[1\pm,0\pm]} \Phi(z_1 + (z_2 - z_1)\zeta)\zeta^{-\delta}(1-\zeta)^{-\varepsilon}\, d\zeta \\ + \frac{1}{(1-e^{2\pi i\delta})(1-e^{2\pi i\varepsilon})} \oint_{[1\pm,0\pm]} \Psi(z_1 + (z_2 - z_1)\zeta)\zeta^{\varepsilon-1}(1-\zeta)^{\delta-1}\, d\zeta, \end{aligned} \tag{6.4}$$

where $[1\pm, 0\pm]$ is a closed curve which first rounds $\zeta = 1$ counterclockwise, next rounds $\zeta = 0$ also counterclockwise, then rounds $\zeta = 1$ clockwise, and finally rounds $\zeta = 0$ clockwise.

Next we see the Fuchsian initial condition (3.1) and (3.2). As

$$\begin{aligned} \frac{1}{(1-e^{-2\pi i\delta})(1-e^{-2\pi i\varepsilon})} \oint_{[1\pm,0\pm]} \Phi(z_1 + (z_2 - z_1)\zeta)\zeta^{-\delta}(1-\zeta)^{-\varepsilon}\, d\zeta \Bigg|_{t=0} \\ = \oint_{[1\pm,0\pm]} \Phi(x)\zeta^{-\delta}(1-\zeta)^{-\varepsilon}\, d\zeta \\ = B(-\delta+1, -\varepsilon+1)\Phi(x), \end{aligned}$$

and

$$\frac{1}{(1-e^{2\pi i\delta})(1-e^{2\pi i\varepsilon})}\oint_{[1\pm,0\pm]}\Psi(z_1+(z_2-z_1)\zeta)\zeta^{\varepsilon-1}(1-\zeta)^{\delta-1}\,d\zeta\Big|_{t=0} = B(\varepsilon,\delta)\Psi(x),$$

where $B(\cdot,\cdot)$ is the beta function, and since

$$(z_2-z_1)^{1-\delta-\varepsilon}=\left(\frac{-2}{m+1}\right)^{\frac{\mu-\lambda}{m+1}}t^{\mu-\lambda},$$

we have, from (3.2)

$$\Phi=0, \tag{6.5}$$

and from (3.1)

$$\Psi(x)=\frac{\Gamma(\varepsilon+\delta)}{\Gamma(\varepsilon)\Gamma(\delta)}x^p(1-x)^qF(\alpha,\beta,\gamma,\,x). \tag{6.6}$$

Therefore we get the equality

$$\begin{aligned} v=\frac{1}{(1-e^{2\pi i\delta})(1-e^{2\pi i\varepsilon})}\frac{\Gamma(\varepsilon+\delta)}{\Gamma(\varepsilon)\Gamma(\delta)}\oint_{[1\pm,0\pm]}(z_1+(z_2-z_1)\zeta)^p \\ (1-z_1-(z_2-z_1)\zeta)^qF(\alpha,\beta,\gamma,z_1+(z_2-z_1)\zeta)\zeta^{\varepsilon-1}(1-\zeta)^{\delta-1}\,d\zeta, \end{aligned} \tag{6.7}$$

and consequently we obtain the equality

(6.8)

$$\begin{aligned} v(t,x)=\frac{1}{(1-e^{2\pi i\delta})(1-e^{2\pi i\varepsilon})}\frac{\Gamma(\varepsilon+\delta)}{\Gamma(\varepsilon)\Gamma(\delta)}\oint_{[1\pm,0\pm]}\left(x+\frac{t^{m+1}}{m+1}-\frac{2t^{m+1}}{m+1}\zeta\right)^p \\ \left(1-x+\frac{t^{m+1}}{m+1}-\frac{2t^{m+1}}{m+1}\zeta\right)^qF\Big(\alpha,\beta,\gamma,\,x+\frac{t^{m+1}}{m+1}-\frac{2t^{m+1}}{m+1}\zeta\Big)\zeta^{\varepsilon-1}(1-\zeta)^{\delta-1}\,d\zeta. \end{aligned}$$

Substituting into the above equality the Euler integral representation of the hypergeometric function

$$\begin{aligned} F(\alpha,\beta,\gamma,\,z)=\frac{1}{(1-e^{2\pi i\alpha})(1-e^{2\pi i(\gamma-\alpha)})}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \\ \oint_{[1\pm,0\pm]}\sigma^{\alpha-1}(1-\sigma)^{\gamma-\alpha-1}(1-\sigma z)^{-\beta}\,d\sigma, \end{aligned}$$

we obtain the integral representation of the solution $u(t,x)$.

**Theorem 6.2** (Main Theorem)**.** *The unique solution of the Fuchsian initial value problem* (1.1) *with conditions* (3.1) *and* (3.2) *has the integral representation*

(6.9)

$$
\begin{aligned}
u(t,x) = {} & \frac{t^{\lambda}}{(1-e^{2\pi i\delta})(1-e^{2\pi i\varepsilon})(1-e^{2\pi i\alpha})(1-e^{2\pi i(\gamma-\alpha)})} \\
& \frac{\Gamma(\varepsilon+\delta)\Gamma(\gamma)}{\Gamma(\varepsilon)\Gamma(\delta)\Gamma(\alpha)\Gamma(\gamma-\alpha)} \oint_{[1\pm,0\pm]} d\zeta \\
& \oint_{[1\pm,0\pm]} \left(x+\frac{t^{m+1}}{m+1}-\frac{2t^{m+1}}{m+1}\zeta\right)^{p} \left(1-x+\frac{t^{m+1}}{m+1}-\frac{2t^{m+1}}{m+1}\zeta\right)^{q} \\
& \zeta^{\varepsilon-1}(1-\zeta)^{\delta-1}\sigma^{\alpha-1}(1-\sigma)^{\gamma-\alpha-1}\left(1-x\sigma-\frac{t^{m+1}}{m+1}\sigma+\frac{2t^{m+1}}{m+1}\sigma\zeta\right)^{-\beta} d\sigma,
\end{aligned}
$$

*where*

$$
\varepsilon = \frac{m+1+\lambda-\mu+a}{2(m+1)}, \qquad \delta = \frac{m+1+\lambda-\mu-a}{2(m+1)},
$$

*and two double loops with respect to $d\zeta$ and $d\sigma$ must satisfy conditions A and B which will be stated in the following proof (they depends on the path of analytic continuation). Furthermore, the solution $u(t,x)$ can be analytically continued to a function holomorphic in the universal covering space of*

$$
\boldsymbol{P}^2(\boldsymbol{C}) \setminus (L \cup V_+^0 \cup V_-^0 \cup V_+^1 \cup V_-^1 \cup V^\infty).
$$

*Proof.* We will explain how to choose double loops.

Let $V(z_1,z_2)$ be the right hand of the equality (6.7). Then we have

$$
\begin{aligned}
V(z_1,z_2) = {} & \frac{1}{(1-e^{2\pi i\delta})(1-e^{2\pi i\varepsilon})(1-e^{2\pi i\alpha})(1-e^{2\pi i(\gamma-\alpha)})} \\
& \frac{\Gamma(\varepsilon+\delta)\Gamma(\gamma)}{\Gamma(\varepsilon)\Gamma(\delta)\Gamma(\alpha)\Gamma(\gamma-\alpha)} \oint_{[1\pm,0\pm]} d\zeta \\
& \oint_{[1\pm,0\pm]} (z_1+(z_2-z_1)\zeta)^p(1-z_1-(z_2-z_1)\zeta)^q\zeta^{\varepsilon-1}(1-\zeta)^{\delta-1} \\
& \quad \sigma^{\alpha-1}(1-\sigma)^{\gamma-\alpha-1}(1-z_1\sigma-(z_2-z_1)\sigma\zeta)^{-\beta}\, d\sigma.
\end{aligned} \tag{6.10}
$$

We note that the integrand (as a function of four variables $(z_1,z_2,\sigma,\zeta)$) has singularities along the set defined by the following equations:

$$
\begin{aligned}
& z_1+(z_2-z_1)\zeta = 0, \\
& 1-z_1-(z_2-z_1)\zeta = 0, \\
& \zeta = 0,1, \\
& \sigma = 0,1, \\
& 1-z_1\sigma-(z_2-z_1)\sigma\zeta = 0.
\end{aligned}
$$

It suffices to prove that $V(z_1, z_2)$ can be analytically continued to a function holomorphic in the universal covering space of

$$D = \boldsymbol{C}^2 \setminus [\{z_1 = 0\} \cup \{z_1 = 1\} \cup \{z_2 = 0\} \cup \{z_2 = 1\} \cup \{z_1 = z_2\}] .$$

Let

$$C = \{(z_1(s), z_2(s));\ 0 \le s \le 1\}$$

be an arbitrary path starting from a point on the line $z_1 = z_2$ satisfying

$$\begin{cases} z_1(0) = z_2(0), & \\ z_1(s) \neq 0, 1 & \text{for all } s \in [0, 1], \\ z_2(s) \neq 0, 1 & \text{for all } s \in [0, 1], \\ z_1(s) \neq z_2(s) & \text{for all } s \in [0, 1]. \end{cases} \tag{6.11}$$

We will show that $V(z_1, z_2)$ can be analytically continued along $C$.

We define two double loops $\Gamma_{\sigma,s}$, $\Gamma_{\zeta,s}$ with respect to $d\sigma$ and $d\zeta$ respectively which are continuously deformed as the parameter $s$ goes from 0 to 1.

(i) When $s = 0$, *the initial double loop* $\Gamma_{\sigma,0}$ in the $\sigma$-plane is to be so taken that the point $\frac{1}{z_1(0)} = \frac{1}{z_2(0)}$ is in the exterior of $\Gamma_{\sigma,0}$, which is possible, for $z_1(0) \neq 0, 1$. We may take $\Gamma_{\zeta,0}$ in the $\zeta$-plane arbitrary.

(ii) The double loop $\Gamma_{\sigma,s}$ $(0 < s \le 1)$, which is continuously deformed as $s$ goes from 0 to 1, must be so taken as to satisfy the following condition.

*Condition A*. Let

$$\Delta_{\sigma,s} = \left\{ \frac{1 - z_1(s)\sigma}{(z_2(s) - z_1(s))\sigma};\ \sigma \in \Gamma_{\sigma,s} \right\} \subset \boldsymbol{P}^1(\boldsymbol{C})$$

be the double loop obtained from $\Gamma_{\sigma,s}$ thus by the linear fractional transformation. $\Delta_{\sigma,s}$ must satisfy the following:

$$\operatorname{dist}(0, \bigcup_{0 \le s \le 1} \Delta_{\sigma,s}) > 0,$$
$$\operatorname{dist}(1, \bigcup_{0 \le s \le 1} \Delta_{\sigma,s}) > 0,$$

and moreover, the points 0 and 1 are never "surrounded" by $\Delta_{\sigma,s}$ for all $\sigma \in [0, 1]$, namely, the rotation nunbers around 0 and 1 are zero for any closed subarc of $\Delta_{\sigma,s}$.

This is possible. For, if $s = 0$, we have obviously $\Delta_{\sigma,0} = \{\infty\}$, and, if $0 < s \le 1$, it is only when

$$\sigma = \frac{1}{z_1(s)}, \frac{1}{z_2(s)}$$

that

$$\frac{1 - z_1(s)\sigma}{(z_2(s) - z_1(s))\sigma} = 0, 1,$$

and therefore, we can obviously deform the initial loop and obtain $\Gamma_{\sigma,s}$ $(0 < s \le 1)$ such that the points $1/z_1(s), 1/z_2(s)$ are always in the exterior of $\Gamma_{\sigma,s}$ for all $s \in (0, 1]$.

(iii) Lastly $\Gamma_{\zeta,s}$ $(0 < s \leq 1)$, which is obtained by continuous deformation of $\Gamma_{\zeta,0}$, must satisfy the following condition.
*Condition B.*
(a) $\dfrac{1 - z_1(s)\sigma}{(z_2(s) - z_1(s))\sigma} \notin \Gamma_{\zeta,s}$ for all $\sigma \in \Gamma_{\sigma,s}$ and all $s \in [0, 1]$,
(b) the point $\dfrac{z_1(s)}{z_1(s) - z_2(s)}$ is in the exterior of $\Gamma_{\zeta,s}$ for all $s \in [0, 1]$,
(c) the point $\dfrac{1 - z_1(s)}{z_1(s) - z_2(s)}$ is also in the exterior of $\Gamma_{\zeta,s}$ for all $s \in [0, 1]$.

This is also possible. For, from condition A, 0 and 1 are never "surrounded" by $\Delta_{\sigma,s}$ for all $\sigma \in [0, 1]$.

Using the above introdeced double loops, it is now obvious that the function $V(z_1, z_2)$ admits holomorphic continuation along the path $C = \{(z_1(s), z_2(s)); 0 \leq s \leq 1\}$.

Since $C$ was arbitrary as far as (6.11) is satisfied, it follows that the function $V(z_1, z_2)$ can be analytically continued to a function holomorphic on the universal covering space of $D$.

By the obvious correspondence

$$\begin{array}{ll} z_1 = 0 \longleftrightarrow V_+^0, & z_1 = 1 \longleftrightarrow V_+^1, \\ z_2 = 0 \longleftrightarrow V_-^0, & z_2 = 1 \longleftrightarrow V_-^1, \\ z_1 = z_2 \longleftrightarrow L, & \text{line at infinity} \longleftrightarrow V^\infty, \end{array}$$

the solution $u(t, x)$ is analytically continued to a function holomorphic in the universal covering space of $\boldsymbol{P}^2(\boldsymbol{C}) \setminus (L \cup V_+^0 \cup V_-^0 \cup V_+^1 \cup V_-^1 \cup V^\infty)$. Thus the theorem is established. □

# 7 Power series expression of the solution

**Lemma 7.1.** *The unique solution of the Fuchsian initial value problem*

$$\left\{ \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - t^{2m} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{1 - \lambda - \mu}{t} \dfrac{\partial u}{\partial t} + a t^{m-1} \dfrac{\partial u}{\partial x} + \dfrac{\lambda \mu}{t^2} u = 0, \\ A_t^\lambda u = x, \\ A_t^\mu u = 0 \end{array} \right. \tag{7.1}$$

*is*

(7.2)
$$u(t, x) = t^\lambda \left( x + \frac{t^{m+1}}{m+1} \right)^\alpha \frac{\Gamma(\varepsilon + \delta)}{\Gamma(\varepsilon)\Gamma(\delta)} F \left( -\alpha, \varepsilon, \varepsilon + \delta, \frac{2t^{m+1}}{(m+1)x + t^{m+1}} \right).$$

*Proof.* By the same argument as that given in section 6, it follows that

$$u(t, x) = \frac{t^\lambda}{(1 - e^{2\pi i \delta})(1 - e^{2\pi i \varepsilon})} \frac{\Gamma(\varepsilon + \delta)}{\Gamma(\varepsilon)\Gamma(\delta)} \left( x + \frac{t^{m+1}}{m+1} \right)^\alpha$$
$$\oint_{[1\pm, 0\pm]} \left( 1 - \frac{2t^{m+1}}{(m+1)x + t^{m+1}} \zeta \right)^\alpha \zeta^{\varepsilon - 1} (1 - \zeta)^{\delta - 1} \, d\zeta.$$

The equality (7.2) follows immediately by the Euler integral representation of the Gauss hypergeometric function. □

**Theorem 7.2.** *The unique solution of the equation* (1.1) *with Fuchsian initial conditions* (3.1) *and* (3.2) *is represented by the series*

$$\begin{aligned}
u(t,x) &= \frac{\Gamma(\varepsilon+\delta)t^\lambda}{\Gamma(\varepsilon)\Gamma(\delta)} \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha,l)(\beta,l)(-q,n)}{(\gamma,l)(1,l)(1,n)} \\
&\quad \cdot F\left(-p-l-n, \varepsilon, \varepsilon+\delta,\ \frac{2t^{m+1}}{(m+1)x+t^{m+1}}\right)\left(x+\frac{t^{m+1}}{m+1}\right)^{p+l+n} \\
&= \frac{\Gamma(\varepsilon+\delta)t^\lambda}{\Gamma(\varepsilon)\Gamma(\delta)} \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\alpha,l)(\beta,l)(-q,n)(-p-l-n,k)(\varepsilon,k)}{(\gamma,l)(1,l)(1,n)(\varepsilon+\delta,k)(1,k)} \\
&\quad \cdot \left(\frac{2t^{m+1}}{(m+1)x+t^{m+1}}\right)^k \left(x+\frac{t^{m+1}}{m+1}\right)^{p+l+n}
\end{aligned} \tag{7.3}$$

*Proof.* Since

$$(1-x)^q = \sum_{n=0}^{\infty}\binom{q}{n}(-x)^n = \sum_{n=0}^{\infty}\frac{(-q,n)}{(1,n)}x^n,$$

we have

$$x^p(1-x)^q F(\alpha,\beta,\gamma,\ x) = \sum_{l=o}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,l)(\beta,l)(-q,n)}{(\gamma,l)(1,l)(1,n)}x^{p+l+n}.$$

The expression (7.3) follows immediately by the use of Lemma 7.1. □

# 8 The obtained class of solutions involves Appell's $F_1$

Appell's 2-dimensional hypergeometric function $F_1$, which is defined by the convergent power series in a neighborhood of the origin:

$$F_1(\alpha,\beta,\beta',\gamma,\ z_1,z_2) = \sum_{m,n=0}^{\infty}\frac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}z_1^m z_2^n,$$

has the Euler integral representation (see Kimura [14])

$$\begin{aligned}
F_1(\alpha,\beta,\beta',\gamma,\ z_1,z_2) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \\
&\quad \cdot \iint_{k,l,1-k-l\geq 0} k^{\beta-1}l^{\beta'-1}(1-k-l)^{\gamma-\beta-\beta'-1}(1-z_1k-z_2l)^{-\alpha}\,dkdl
\end{aligned} \tag{8.1}$$

if $\operatorname{Re}\beta>0$, $\operatorname{Re}\beta'>0$, $\operatorname{Re}(\gamma-\beta-\beta')>0$.

**Theorem 8.1.** *If $p = q = \lambda = 0$ and $\beta = \varepsilon + \delta$, where $\varepsilon$ and $\delta$ are as* (5.7), *then the solution $u(t, x)$ of the equation* (1.1) *with conditions* (3.1) *and* (3.2) *is written by Appell's $F_1$:*

$$u(t, x) = F_1\left(\alpha, \delta, \varepsilon, \gamma,\, x + \frac{t^{m+1}}{m+1},\, x - \frac{t^{m+1}}{m+1}\right).$$

*Remark* 8.1. In the above theorem, four parameters $\alpha$, $\delta$, $\varepsilon$ and $\gamma$ can take any prescribed values wherever $F_1$ is defined. In fact, we have only to take $\mu$ and $a$ by

$$\begin{aligned} \mu &= (m+1)(1 - \varepsilon - \delta), \\ a &= (m+1)(\varepsilon - \delta). \end{aligned}$$

This is why we considered Fuchsian initial value problem instead of the usual Cauchy problem.

*Proof.* Since

$$F(\alpha, \beta, \gamma,\, x) = F(\beta, \alpha, \gamma,\, x),$$

it suffices to prove that, if

$$p = q = \lambda = 0 \qquad \text{and} \qquad \alpha = \varepsilon + \delta, \tag{8.2}$$

then

$$u(t, x) = F_1\left(\beta, \delta, \varepsilon, \gamma,\, x + \frac{t^{m+1}}{m+1},\, x - \frac{t^{m+1}}{m+1}\right) \tag{8.3}$$

In this case (8.2), the integral representation (6.10) takes the following form:

$$\begin{aligned} V(z_1, z_2) = \frac{\Gamma(\gamma)}{\Gamma(\varepsilon)\Gamma(\delta)\Gamma(\gamma - \varepsilon - \delta)} &\int_0^1 \int_0^1 \sigma^{\varepsilon+\delta-1}(1-\sigma)^{\gamma-\varepsilon-\delta-1} \\ &(1 - z_1\sigma + (z_1 - z_2)\sigma\zeta)^{-\beta}\zeta^{\varepsilon-1}(1-\zeta)^{\delta-1}\, d\sigma d\zeta, \end{aligned}$$

wherever the integral converges. We use the transformation

$$\begin{cases} k = \sigma(1 - \zeta), \\ l = \sigma\zeta. \end{cases}$$

After an easy calculation, we have

$$\begin{aligned} V(z_1, z_2) = \frac{\Gamma(\gamma)}{\Gamma(\varepsilon)\Gamma(\delta)\Gamma(\gamma - \varepsilon - \delta)} &\iint_{k,l,1-k-l\geq 0} k^{\delta-1} l^{\varepsilon-1} \\ &(1 - k - l)^{\gamma-\varepsilon-\delta-1}(1 - z_1 k - z_2 l)^{-\beta}\, dk dl, \end{aligned}$$

and therefore, after an analyric continuation with respect to parameters, we have

$$V(z_1, z_2) = F_1(\beta, \delta, \varepsilon, \gamma,\, z_1, z_2).$$

Thus the theorem follows. $\square$

# 9 The obtained class of solutions involves Appell's $F_2$

Appell's double hypergeometric function $F_2$ is defined by the convergent power series

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',z_1,z_2) = \sum_{k,l=0}^{\infty} \frac{(\alpha,k+l)(\beta,k)(\beta',l)}{(\gamma,k)(\gamma',l)(1,k)(1,l)} z_1^k z_2^l \tag{9.1}$$

in a neighborhood of the origin.

**Lemma 9.1** (transformation formulas)**.**

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',\, z_1,z_2)$$

$$= \sum_{l=0}^{\infty} \frac{(\alpha,l)(\beta,l)}{(\gamma,l)(1,l)} F(\alpha+l,\beta',\gamma',z_2)\, z_1^l \tag{9.2a}$$

$$= \sum_{l=0}^{\infty} \frac{(\alpha,l)(\beta,l)}{(\gamma,l)(1,l)} (1-z_2)^{-\alpha-l} F\big(\alpha+l,\gamma'-\beta',\gamma',\, \frac{z_2}{z_2-1}\big) z_1^l \tag{9.2b}$$

$$= \sum_{l=0}^{\infty} \frac{(\alpha,l)(\beta,l)}{(\gamma,l)(1,l)} (1-z_2)^{-\beta'} F\big(\gamma'-\alpha-l,\beta',\gamma',\, \frac{z_2}{z_2-1}\big) z_1^l \tag{9.2c}$$

$$= \sum_{l=0}^{\infty} \frac{(\alpha,l)(\beta,l)}{(\gamma,l)(1,l)} (1-z_2)^{\gamma'-\alpha-\beta'-l} F(\gamma'-\alpha-l,\gamma'-\beta',\gamma',z_2)\, z_1^l \tag{9.2d}$$

*Proof.* The equality (9.2a) is obvious. The rest are immediate consequences of Kummmer's transformation formulas:

$$\begin{aligned} F(\alpha,\beta,\gamma,z) &= (1-z)^{-\alpha} F\big(\alpha,\gamma-\alpha,\gamma,\frac{z}{z-1}\big) \\ &= (1-z)^{-\beta} F\big(\gamma-\alpha,\beta,\gamma,\frac{z}{z-1}\big) \\ &= (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma,z) \end{aligned}$$

□

**Theorem 9.2.** *If $q=0$ and $\alpha = p+\varepsilon+\delta$, then the solution of the equation* (1.1) *with conditions* (3.1) *and* (3.2) *is written by Appell's $F_2$:*

$$\begin{aligned} u(t,x) = \frac{\Gamma(\varepsilon+\delta)t^\lambda}{\Gamma(\varepsilon)\Gamma(\delta)} &\left(x + \frac{t^{m+1}}{m+1}\right)^{p+\varepsilon} \left(x - \frac{t^{m+1}}{m+1}\right)^{-\varepsilon} \\ &F_2\Big(p+\varepsilon+\delta,\beta,\gamma,\varepsilon+\delta,\, x+\frac{t^{m+1}}{m+1},\, \frac{-2t^{m+1}}{(m+1)x-t^{m+1}}\Big). \end{aligned}$$

*Remark* 9.1. Five parameters $p+\varepsilon+\delta$, $\beta$, $\varepsilon$, $\gamma$, $\varepsilon+\delta$ in the above expression in $F_2$ can take any prescribed values wherever $F_2$ is defined.

*Proof.* As $q = 0$, the representation (7.3) takes the following form:

$$u(t,x) = \frac{\Gamma(\varepsilon+\delta)t^\lambda}{\Gamma(\varepsilon)\Gamma(\delta)} \sum_{l=0}^{\infty} \frac{(\alpha,l)(\beta,l)}{(\gamma,l)(1,l)} F\Big(-p-l, \varepsilon, \varepsilon+\delta, \frac{z_1-z_2}{z_1}\Big) z_1^{p+l},$$

where $z_1$ and $z_2$ are the characteristic variables defined by (5.5). We rewrite the above equality as

$$u(t,x) = \frac{\Gamma(\varepsilon+\delta)t^\lambda}{\Gamma(\varepsilon)\Gamma(\delta)} z_1^{p+\varepsilon} z_2^{-\varepsilon}$$
$$\sum_{l=0}^{\infty} \frac{(\alpha,l)(\beta,l)}{(\gamma,l)(1,l)} \Big(\frac{z_1}{z_2}\Big)^{-\varepsilon} F\Big(-p-l, \varepsilon, \varepsilon+\delta, \frac{z_1-z_2}{z_1}\Big) z_1^{l}.$$

Since $\alpha = p + \varepsilon + \delta$, it follows from (9.2b) that

$$u(t,x) = \frac{\Gamma(\varepsilon+\delta)t^\lambda}{\Gamma(\varepsilon)\Gamma(\delta)} z_1^{p+\varepsilon} z_2^{-\varepsilon} F_2\Big(p+\varepsilon+\delta, \beta, \varepsilon, \gamma, \varepsilon+\delta, z_1, \frac{z_2-z_1}{z_2}\Big),$$

which is the conclusion of the theorem. □

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