

The Variants of Kappa Function

Yuzen TANAKA

Technical Reports of Mathematical Sciences
Chiba University, Vol. 21 (2005), No. 6

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Abstract: M. KANEKO and M.YOSHIDA introduced the kappa function with the property $J(\kappa(z)) = \lambda(z)$ in [KY], where J is the elliptic modular function and λ is the Jacobi's elliptic λ -function. In this paper the author constructs the variants of $\kappa(z)$ using other pairs of simple modular functions instead of $(J(z), \lambda(z))$, and gives explicit Fourier expansions for them.

1. Variants of the λ function

We rewrite some well-known factor about the modular function.

Put

$$\begin{aligned}\Gamma &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid ad - bc = \pm 1 \right\}, \\ \Gamma(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \begin{array}{l} a \equiv d \equiv 1 \pmod{2} \\ b \equiv c \equiv 0 \pmod{2} \end{array} \right\}, \\ \Gamma_{1,2} &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid ab \equiv cd \equiv 0 \pmod{2} \right\}.\end{aligned}$$

Setting

$$\gamma_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{\rho_2} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

we have

$$\Gamma = \langle \gamma_i, \gamma_{\rho_2} \rangle, \quad \Gamma(2) = \langle \gamma_\infty, \gamma_0 \rangle, \quad \Gamma_{1,2} = \langle \gamma_i, \gamma_0 \rangle.$$

And we define

$$\Gamma^*(2) = \left\langle \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \right\rangle.$$

Because of

$$\begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix},$$

we have

$$\Gamma^*(2) = \langle \gamma_{1+i}, \gamma_{\sqrt{2}i} \rangle,$$

with

$$\gamma_{1+i} = \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \gamma_{\sqrt{2}i} = \begin{pmatrix} 0 & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}.$$

Any of these groups acts on the upper half complex plane \mathbb{H} .

(1) The modular function λ^\sharp :

Let us make a biholomorphic map from the fundamental domain $F(2) = \{z \in \mathbb{H}; 0 < \operatorname{Re} z < 1, |z - \frac{1}{2}| < \frac{1}{2}\}$ of $\Gamma(2)$, that is indicated as a shaded part in Figure 1.1, onto the lower half complex plane \mathbb{H}^- . We assume the boundary points 0, 1 and ∞ of \mathbb{H}^- correspond to 0, $i\infty$ and 1, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda^\sharp(z)$ defined on the whole \mathbb{H} .

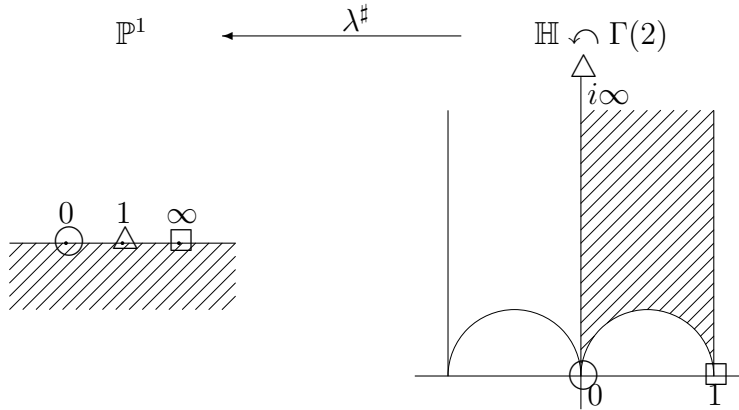


FIGURE 1.1

Here λ^\sharp is the function λ in [KY] and we have

$$\lambda^\sharp(z) = \frac{\theta_0(z)^4}{\theta_3(z)^4}, \quad \left(\theta_0(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}, \theta_3(z) = \sum_{n \in \mathbb{Z}} q^{n^2} \right),$$

where $q = e^{\pi iz}$.

(2) The fundamental domain of λ^b :

Let us make a biholomorphic map from the fundamental domain $F(2) = \{z \in \mathbb{H}; 0 < \operatorname{Re} z < 1, |z - \frac{1}{2}| < \frac{1}{2}\}$ of $\Gamma(2)$, that is indicated as a shaded part in Figure 1.2, onto the lower half complex plane \mathbb{H} . We assume the boundary points 0, 1 and ∞ of \mathbb{H} correspond to 1, $i\infty$ and 0, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda^b(z)$ defined on the whole \mathbb{H} .

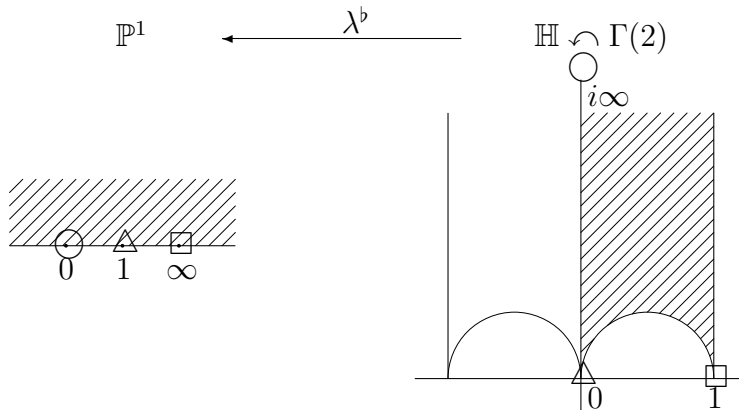


FIGURE 1.2

Here we have

$$\lambda^b(z) = \frac{\theta_2(z)^4}{\theta_3(z)^4}, \quad \left(\theta_2(z) = \sum_{n \in \mathbb{Z}} q^{\frac{(2n+1)^2}{4}}, \theta_3(z) = \sum_{n \in \mathbb{Z}} q^{n^2} \right),$$

where $q = e^{\pi iz}$.

(3) The fundamental domain of $\lambda_{1,2}^\sharp$:

Let us make a biholomorphic map from the fundamental domain $F_{1,2}^\sharp = \{z \in \mathbb{H}; -1 < \operatorname{Re} z < 0, |z| > 1\}$ of $\Gamma_{1,2}$, that is indicated as a shaded part in Figure 1.3, onto the lower half complex plane \mathbb{H} . We assume the boundary points $0, 1$ and ∞ of \mathbb{H} correspond to $i, i\infty$ and -1 , respectively. By the Schwarz reflection principle we obtain a modular function $\lambda_{1,2}^\sharp(z)$ defined on the whole \mathbb{H} .

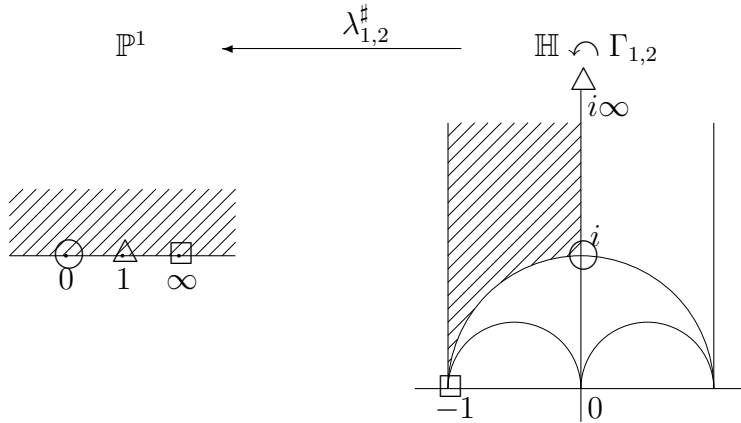


FIGURE 1.3

(4) The fundamental domain of $\lambda_{1,2}^b$:

Let us make a biholomorphic map from the fundamental domain $F_{1,2}^b = \{z \in \mathbb{H}; 0 < \operatorname{Re} z < 1, |z| > 1\}$ of $\Gamma_{1,2}$, that is indicated as a shaded part in Figure 1.4, onto the lower half complex plane \mathbb{H} . We assume the boundary points $0, 1$ and ∞ of \mathbb{H} correspond to $i\infty, i$ and 1 , respectively. By the Schwarz reflection principle we obtain a modular function $\lambda_{1,2}^b(z)$ defined on the whole \mathbb{H} .

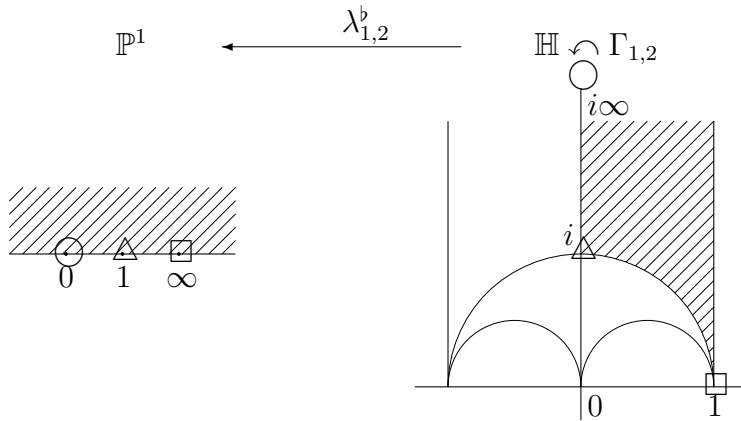


FIGURE 1.4

(5) The fundamental domain of J :

Let us make a biholomorphic map from the fundamental domain $F = \{z \in \mathbb{H}; -\frac{1}{2} < \operatorname{Re} z < 0, |z| > 1\}$ of Γ , that is indicated as a shaded part in Figure 1.5, onto the lower half complex plane \mathbb{H} . We assume the boundary points $0, 1$ and ∞ of \mathbb{H} correspond to ρ_2, i and $i\infty$, respectively. By the Schwarz reflection principle we obtain a modular function J defined on the whole \mathbb{H} .

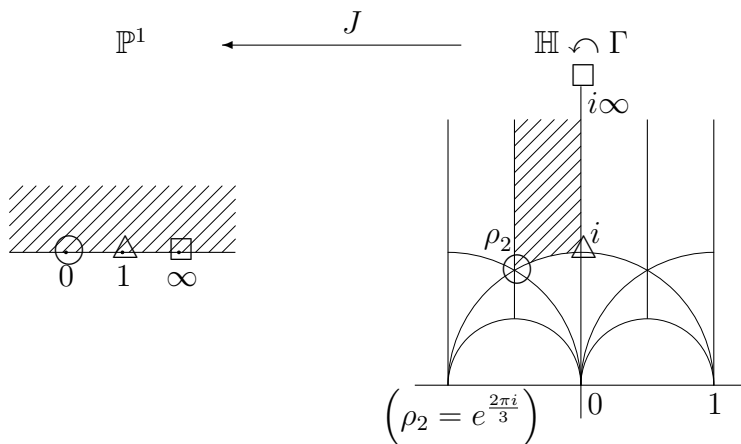


FIGURE 1.5

(6) The fundamental domain of $\lambda_{(2)}^*$:

Let us make a biholomorphic map from the fundamental domain $F^*(2) = \{z \in \mathbb{H}; 0 < \operatorname{Re} z < 1, |z| > 1\}$ of $\Gamma^*(2)$, that is indicated as a shaded part in Figure 1.6, onto the lower half complex plane \mathbb{H} . We assume the boundary points $0, 1$ and ∞ of \mathbb{H} correspond to $\sqrt{2}i, 1+i$ and $i\infty$, respectively. By the Schwarz reflection principle we obtain a modular function $\lambda(2)^*$ defined on the whole \mathbb{H} .

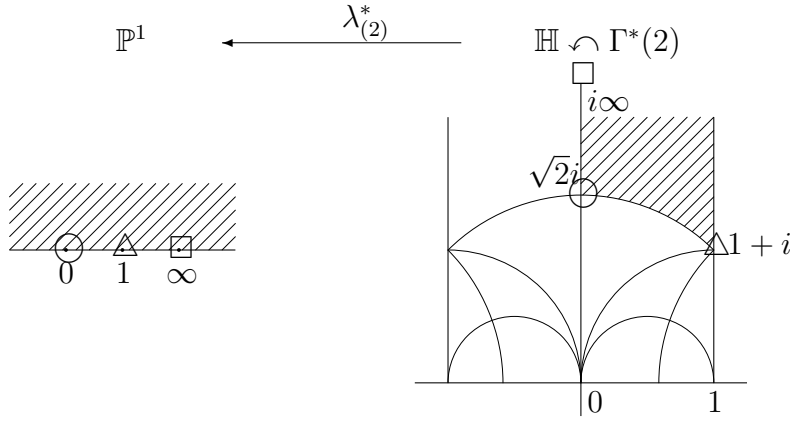


FIGURE 1.6

Proposition

We have the relations among the above modular functions as follows:

$$\begin{aligned}
 J &= P(\lambda^\sharp) = P(\lambda^b), & \text{where } P(l) &= \frac{4}{27} \frac{(1-l+l^2)^3}{l^2(1-l)^2}, \\
 \lambda_{1,2}^\sharp &= P^\sharp(\lambda^\sharp) = P^\sharp(\lambda^b), & \text{where } P^\sharp(l) &= (1-2l)^2, \\
 \lambda_{1,2}^b &= P^b(\lambda^\sharp) = P^b(\lambda^b), & \text{where } P^b(l) &= 4l(1-l), \\
 \lambda_{(2)}^* &= P^*(\lambda^\sharp) = P^*(\lambda^b), & \text{where } P^*(l) &= -\frac{1}{16} \frac{(1-6l+l^2)^2}{l(1-l)^2},
 \end{aligned}$$

and

$$\begin{aligned}
 \lambda^\sharp &= 1 - \lambda^b, \\
 \lambda_{1,2}^\sharp &= 1 - \lambda_{1,2}^b, \\
 J &= P_{1,2}^\sharp(\lambda_{1,2}^\sharp) = P_{1,2}^b(\lambda_{1,2}^b), \text{ where } P_{1,2}^\sharp(l) = \frac{1}{27} \frac{(3+l)^3}{(1-l)^2} \text{ and } P_{1,2}^b = \frac{1}{27} \frac{(4-l)^3}{l^2}.
 \end{aligned}$$

We have some values of the above modular functions as following:

$z \in \mathbb{H}$	$\lambda^\sharp(z)$	$\lambda^\flat(z)$	$\lambda_{1,2}^\sharp(z)$	$\lambda_{1,2}^\flat(z)$	$J(z)$	$\lambda_{(2)}^*(z)$
$i\infty$	1	0	1	0	∞	∞
$\sqrt{2}i$	$-2 - 2\sqrt{2}$	$3 - 2\sqrt{2}$	$57 - 40\sqrt{2}$	$-8(7 - 5\sqrt{2})$	$\frac{125}{27}$	0
i	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	$-\frac{49}{32}$
0	0	1	1	0	∞	∞
$\frac{1+i}{2}$	-1	2	9	-8	1	$-\frac{49}{32}$
$\frac{2+\sqrt{2}i}{3}$	$-2 - 2\sqrt{2}$	$3 + 2\sqrt{2}$	$57 + 40\sqrt{2}$	$-8(7 + 5\sqrt{2})$	$\frac{125}{27}$	0
1	∞	∞	∞	∞	∞	∞
$1+i$	2	-1	9	-8	1	1
$\frac{1+\sqrt{3}i}{2}$	$\frac{1-\sqrt{3}i}{2}$	$\frac{1+\sqrt{3}i}{2}$	-3	4	0	$\frac{25}{16}$

2. Variants of the κ function

2.1 κ function

In this subsection we recall the κ function studied in [KY]. Because of $J(z) = P(\lambda^\sharp(z))$ we have the following diagram:

$$\begin{array}{ccc}
 \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowright \Gamma \\
 P \uparrow & & \parallel \\
 \mathbb{P}^1 & \xleftarrow{\lambda^\sharp} & \mathbb{H} \curvearrowright \Gamma(2)
 \end{array}$$

Let us make a biholomorphic map from the fundamental domain $F(2)$ of $\Gamma(2)$ to that of Γ that sends 0 ($1, i\infty$) to $\rho_1 = e^{\frac{\pi i}{3}}$ ($i\infty, i$), respectively. By the Schwarz reflection principle we obtain a mapping defined on the whole \mathbb{H} . We denote it by κ^\sharp .

$$\begin{array}{ccc}
 \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowright \Gamma \\
 \parallel & & \uparrow \kappa^\sharp \\
 \mathbb{P}^1 & \xleftarrow{\lambda^\sharp} & \mathbb{H} \curvearrowright \Gamma(2)
 \end{array}$$

We note that κ^\sharp induces a 1:1 correspondence between two shaded parts in Figure 2.1.

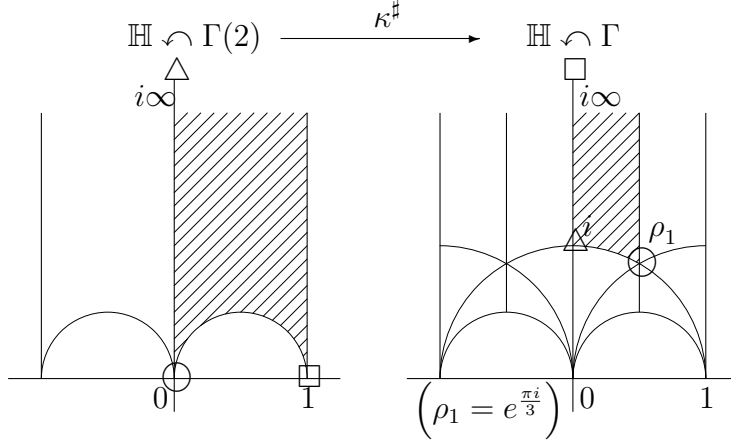


FIGURE 2.1

Now we use the symbol $\ll \cdot \gg$ to denote the smallest normal subgroup containing \cdot of a given group. Then the isomorphism $\Gamma(2)/\ll \gamma_\infty^3, \gamma_0^2 \gg \cong \Gamma$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2)/\ll \gamma_\infty^3, \gamma_0^2 \gg & \longrightarrow & \Gamma \\ \gamma_\infty & \longmapsto & \gamma_{\rho_2} \\ \gamma_0 & \longmapsto & \gamma_i \end{array}$$

Thus the function κ^\sharp satisfies

$$\kappa^\sharp(z+2) = -\frac{1}{\kappa^\sharp(z)} \quad \text{and} \quad \kappa^\sharp\left(\frac{z}{2z+1}\right) = 1 - \frac{1}{\kappa^\sharp(z)}.$$

2.2. Construction of new κ functions

By the same method, we obtain new κ functions using variants of λ functions. Starting from the correspondence of the fundamental domains of both modular groups, we obtain our variants of κ function by the Schwarz reflection principle.

(1) The function $\tilde{\kappa}_{(2)} : \mathbb{H} \curvearrowright \Gamma(2) \longrightarrow \mathbb{H} \curvearrowright \Gamma(2)$.

That is defined by the equation $\lambda^\sharp(z) = \lambda^b(\tilde{\kappa}_{(2)}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{\lambda^\sharp} & \mathbb{H} \curvearrowright \Gamma(2) \\ \parallel & & \uparrow \tilde{\kappa}_{(2)} \\ \mathbb{P}^1 & \xleftarrow{\lambda^b} & \mathbb{H} \curvearrowright \Gamma(2) \end{array}$$

The automorphism $\tilde{\kappa}_{(2)} \in \text{Aut}(\Gamma(2))$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2) & \longrightarrow & \Gamma(2) \\ \gamma_\infty & \longmapsto & \gamma_0 \\ \gamma_0 & \longmapsto & \gamma_\infty \end{array}$$

Thus the function $\tilde{\kappa}_{(2)}$ satisfies

$$\tilde{\kappa}_{(2)}(z+2) = \frac{\tilde{\kappa}_{(2)}(z)}{-2\tilde{\kappa}_{(2)}(z)+1} \quad \text{and} \quad \tilde{\kappa}_{(2)}\left(\frac{z}{2z+1}\right) = \tilde{\kappa}_{(2)}(z) - 2.$$

(2) The function $\kappa^b : \mathbb{H} \curvearrowright \Gamma(2) \longrightarrow \mathbb{H} \curvearrowright \Gamma$.

That is defined by the equation $\lambda^b(z) = J(\kappa^b(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowright \Gamma \\ \parallel & & \uparrow \kappa^b \\ \mathbb{P}^1 & \xleftarrow{\lambda^b} & \mathbb{H} \curvearrowright \Gamma(2) \end{array}$$

The isomorphism $\Gamma(2)/\langle\langle \gamma_\infty^3, \gamma_0^2 \rangle\rangle \cong \Gamma$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2)/\langle\langle \gamma_\infty^3, \gamma_0^2 \rangle\rangle & \longrightarrow & \Gamma \\ \gamma_\infty & \longmapsto & \gamma_{\rho_2} \\ \gamma_0 & \longmapsto & \gamma_i \end{array}$$

Thus the function κ^b satisfies

$$\kappa^b(z+2) = -\frac{1}{\kappa^b(z)} \quad \text{and} \quad \kappa^b\left(\frac{z}{2z+1}\right) = 1 - \frac{1}{\kappa^b(z)}.$$

(3) The function $\kappa_{1,2} : \mathbb{H} \curvearrowright \Gamma(2) \longrightarrow \mathbb{H} \curvearrowright \Gamma_{1,2}$.

That is defined by the equation $\lambda^{\sharp}(z) = \lambda_{1,2}^{\sharp}(\kappa_{1,2}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{\lambda_{1,2}^{\sharp}} & \mathbb{H} \curvearrowright \Gamma_{1,2} \\ \parallel & & \uparrow \kappa_{1,2} \\ \mathbb{P}^1 & \xleftarrow{\lambda^b} & \mathbb{H} \curvearrowright \Gamma(2) \end{array}$$

The isomorphism $\Gamma(2)/\langle\langle \gamma_\infty^2 \rangle\rangle \cong \Gamma_{1,2}$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma(2)/\langle\langle \gamma_\infty^2 \rangle\rangle & \longrightarrow & \Gamma_{1,2} \\ \gamma_\infty & \longmapsto & \gamma_i \\ \gamma_0 & \longmapsto & \gamma_0 \end{array}$$

Thus the function $\kappa_{1,2}$ satisfies

$$\kappa_{1,2}(z+2) = -\frac{1}{\kappa_{1,2}(z)} \quad \text{and} \quad \kappa_{1,2}\left(\frac{z}{2z+1}\right) = \frac{\kappa_{1,2}(z)}{2\kappa_{1,2}(z)+1}.$$

(4) The function $\tilde{\kappa}_{1,2} : \mathbb{H} \curvearrowright \Gamma_{1,2} \longrightarrow \mathbb{H} \curvearrowright \Gamma_{1,2}$.
That is defined by the equation $\lambda_{1,2}^\sharp(z) = \lambda_{1,2}^\flat(\tilde{\kappa}_{1,2}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{\lambda_{1,2}^\flat} & \mathbb{H} \curvearrowright \Gamma_{1,2} \\ \parallel & & \uparrow \tilde{\kappa}_{1,2} \\ \mathbb{P}^1 & \xleftarrow{\lambda_{1,2}^\sharp} & \mathbb{H} \curvearrowright \Gamma_{1,2} \end{array}$$

The isomorphism $\langle \gamma_\infty \rangle / \langle \gamma_\infty^2 \rangle \cong \langle \gamma_i \rangle$ is given by the following correspondence:

$$\begin{array}{ccc} \langle \gamma_\infty \rangle / \langle \gamma_\infty^2 \rangle & \longrightarrow & \langle \gamma_i \rangle \\ \gamma_\infty & \longmapsto & \gamma_i \end{array}$$

Thus the function $\tilde{\kappa}_{1,2}$ satisfies

$$\tilde{\kappa}_{1,2}(z+2) = -\frac{1}{\tilde{\kappa}_{1,2}(z)}.$$

(5) The function $\check{\kappa}_{1,2} : \mathbb{H} \curvearrowright \Gamma_{1,2} \longrightarrow \mathbb{H} \curvearrowright \Gamma$.
That is defined by the equation $\lambda_{1,2}^\flat(z) = J(\check{\kappa}_{1,2}(z))$.

$$\begin{array}{ccc} \mathbb{P}^1 & \xleftarrow{J} & \mathbb{H} \curvearrowright \Gamma \\ \parallel & & \uparrow \check{\kappa}_{1,2} \\ \mathbb{P}^1 & \xleftarrow{\lambda_{1,2}^\flat} & \mathbb{H} \curvearrowright \Gamma_{1,2} \end{array}$$

The isomorphism $\Gamma_{1,2} / \ll \gamma_\infty^3 \gg \cong \Gamma$ is given by the following correspondence:

$$\begin{array}{ccc} \Gamma_{1,2} / \ll \gamma_\infty^3 \gg & \longrightarrow & \Gamma \\ \gamma_\infty & \longmapsto & \gamma_{\rho_2} \\ \gamma_i & \longmapsto & \gamma_i \end{array}$$

Thus the function $\check{\kappa}_{1,2}$ satisfies

$$\check{\kappa}_{1,2}(z+2) = -1 - \frac{1}{\check{\kappa}_{1,2}(z)} \quad \text{and} \quad \check{\kappa}_{1,2}\left(-\frac{1}{z}\right) = -\frac{1}{\check{\kappa}_{1,2}(z)}.$$

(6) The function $\kappa_{(2)}^* : \mathbb{H} \curvearrowright \Gamma(2) \longrightarrow \mathbb{H} \curvearrowright \Gamma^*(2)$.
That is defined by the equation $\lambda_{(2)}^\flat(z) = \lambda_{(2)}^*(\kappa_{(2)}^*(z))$.

$$\begin{array}{ccc}
\mathbb{P}^1 & \xleftarrow{\lambda_{(2)}^*} & \mathbb{H} \curvearrowright \Gamma^*(2) \\
\parallel & & \uparrow \kappa_{(2)}^* \\
\mathbb{P}^1 & \xleftarrow{\lambda^b} & \mathbb{H} \curvearrowright \Gamma(2)
\end{array}$$

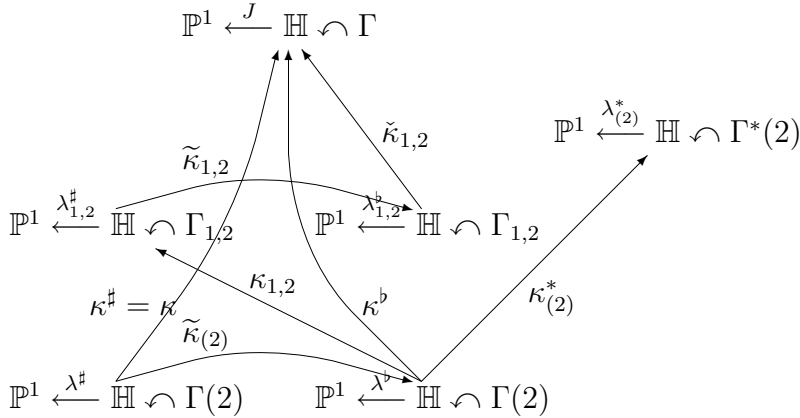
The isomorphism $\Gamma(2)/\langle\langle \gamma_\infty^2, \gamma_0^4 \rangle\rangle \cong \Gamma^*(2)$ is given by the following correspondence:

$$\begin{array}{ccc}
\Gamma(2)/\langle\langle \gamma_\infty^2, \gamma_0^4 \rangle\rangle & \longrightarrow & \Gamma^*(2) \\
\gamma_\infty & \longmapsto & \gamma_{\sqrt{2}i} \\
\gamma_0 & \longmapsto & \gamma_{1+i}
\end{array}$$

Thus the function $\kappa_{(2)}^*$ satisfies

$$\kappa_{(2)}^*(z+2) = -\frac{2}{\kappa_{(2)}^*(z)} \quad \text{and} \quad \kappa_{(2)}^*\left(\frac{z}{2z+1}\right) = -\frac{2}{\kappa_{(2)}^*(z)-2}.$$

By the section 2.1 and (1), (2), \dots , (6) we have the figure of the relations between the modular functions as following:



3. Fourier expansion

Let $q_n = e^{\frac{\pi iz}{n}}$ and

$$\sigma_k^-(n) = \sum_{d|n} (-1)^k d^k.$$

Each κ function, except $\kappa_{(2)}$, in the section 2 has the Fourier series expansion at $z = i\infty$ and $\kappa_{(2)}$ is a linear transformation as following:

Theorem 1

(0) The function $\kappa^\#$ ([KY]):

$$\kappa^\#(z) = i(1 + a_1 q_2 + a_2 q_2^2 + a_3 q_2^3 + \dots),$$

where

$$a_1 = -i \frac{32}{\sqrt{3}} \frac{\pi^2}{\Gamma(\frac{1}{4})^4} = -1.0552729262852 \dots \times i,$$

and the coefficients a_n satisfies the reccurent relation

$$\begin{aligned}
a_2 &= \frac{a_1^2}{2} \quad \text{and} \\
2n(n-1)(n-2)a_1a_n \\
&= - \sum_{m=2}^{n-1} i(n+1-m) (2(n+1)^2 - 7m(n+1) + 5m^2 + 1) a_m a_{n+1-m} \\
&\quad - \sum_{j=1}^{n-1} b_j \sum_{m=1}^{n-j} i(n+1-j-m) a_m a_{n+1-j-m}, \quad (n \geq 3),
\end{aligned}$$

and the series b_n are

$$b_n = \begin{cases} 1, & n = 0 \\ 0, & \text{for } n:\text{odd}, \\ (-1)^{\frac{n}{2}} \frac{64}{9} \sigma_3^- \left(\frac{n}{2} \right), & \text{for } n \equiv 2 \pmod{4}, \\ (-1)^{\frac{n}{2}} \frac{64}{9} \sigma_3^- \left(\frac{n}{2} \right) + \frac{80}{9} \sigma_3^- \left(\frac{n}{4} \right), & \text{for } n \equiv 0 \pmod{4}, \end{cases}$$

i.e.

$$\begin{aligned}
a_2 &= \frac{1}{2} a_1^2, \\
a_3 &= \frac{1}{4} a_1^3 - \frac{16}{27} a_1, \\
a_4 &= \frac{1}{8} a_1^4 - \frac{16}{27} a_1^2, \\
a_5 &= \frac{1}{16} a_1^5 - \frac{4}{9} a_1^3 + \frac{98}{1215} a_1, \\
a_6 &= \frac{1}{32} a_1^6 - \frac{8}{27} a_1^4 + \frac{934}{3645} a_1^2, \\
a_7 &= \frac{1}{64} a_1^7 - \frac{5}{27} a_1^5 + \frac{787}{2430} a_1^3 - \frac{1504}{6561} a_1, \\
a_8 &= \frac{1}{128} a_1^8 - \frac{1}{9} a_1^6 + \frac{41}{135} a_1^4 - \frac{9088}{32805} a_1^2, \\
&\dots
\end{aligned}$$

(1) The function $\tilde{\kappa}_{(2)}$:

$$\tilde{\kappa}_{(2)}(z) = -\frac{1}{z}$$

(2) The function κ^b :

$$\kappa^b(z) = \rho_2(1 + a_1q_3 + a_2q_3^2 + a_3q_3^3 + \cdots),$$

where

$$a_1 = -i \frac{3^2 \Gamma\left(\frac{5}{6}\right)^2 \Gamma\left(\frac{2}{3}\right)^2}{2^{\frac{4}{3}} \pi^2} = -0.8454860396348 \cdots \times i,$$

and the coefficients a_n satisfies the reccurent relation

$$\begin{aligned} a_2 &= \frac{1}{3^{\frac{1}{2}}} e^{\frac{\pi i}{6}} a_1^2 \quad \text{and} \\ 2n(n-1)(n+1)a_1a_{n+1} \\ &= -a_1^2b_n + \sum_{m=1}^{n-1} \left((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \right. \\ &\quad \left. - b_{n-m} \sum_{j=0}^m (j+1)(m-j+1)a_{j+1}a_{m-j+1} \right), \quad (n \geq 2), \end{aligned}$$

and the series b_n are

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 4, 5 \pmod{6}, \\ 36(-1)^n \sigma_3^- \left(\frac{n}{3} \right), & \text{for } n \equiv 3 \pmod{6}, \\ 36(-1)^n \sigma_3^- \left(\frac{n}{3} \right) - 20\sigma_3^- \left(\frac{n}{6} \right), & \text{for } n \equiv 0 \pmod{6}, \end{cases}$$

i.e.

$$\begin{aligned} a_2 &= \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}} a_1^2, \\ a_3 &= \frac{1}{3} e^{\frac{\pi i}{3}} a_1^3, \\ a_4 &= \frac{1}{3\sqrt{3}} e^{\frac{\pi i}{2}} a_1^4 - \frac{3}{4} a_1, \\ a_5 &= \frac{1}{9} e^{\frac{2\pi i}{3}} a_1^5 - \frac{\sqrt{3}}{2} e^{\frac{\pi i}{6}} a_1^2, \\ a_6 &= \frac{1}{9\sqrt{3}} e^{5\pi i} 6a_1^6 - \frac{3}{4} e^{\frac{\pi i}{3}} a_1^3, \\ a_7 &= \frac{1}{27} e^{\pi i} a_1^7 - \frac{1}{\sqrt{3}} e^{\frac{\pi i}{2}} a_1^4 + \frac{79}{420} a_1, \\ a_8 &= \frac{1}{27\sqrt{3}} e^{\frac{7\pi i}{6}} a_1^8 - \frac{5}{12} e^{\frac{2\pi i}{3}} a_1^5 + \frac{1577}{1680\sqrt{3}} e^{\frac{\pi i}{6}} a_1^2, \end{aligned}$$

...

(3) The function $\kappa_{1,2}$:

$$\kappa_{1,2}(z) = i(1 + a_1q_2 + a_2q_2^2 + a_3q_2^3 + \cdots),$$

where

$$a_1 = -\frac{32\pi^2}{\Gamma\left(\frac{1}{4}\right)^4} = -1.8277863241779 \cdots,$$

and the coefficients a_n satisfies the reccurent relation

$$\begin{aligned} a_2 &= \frac{a_1^2}{2} \quad \text{and} \\ 2n(n-1)(n+1)a_1a_{n+1} \\ &= -a_1^2b_n + \sum_{m=1}^{n-1} \left((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \right. \\ &\quad \left. - b_{n-m} \sum_{j=0}^m (j+1)(m-j+1)a_{j+1}a_{m-j+1} \right), \quad (n \geq 2), \end{aligned}$$

and the series b_n are

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 3 \pmod{4}, \\ 16\sigma_3^-\left(\frac{n}{4}\right), & \text{for } n \equiv 0 \pmod{4}, \end{cases}$$

i.e.

$$\begin{aligned} a_2 &= \frac{a_1^2}{2}, \\ a_3 &= \frac{a_1^3}{4}, \\ a_4 &= \frac{a_1^4}{8}, \\ a_5 &= \frac{a_1^5}{16} + \frac{2}{15}a_1, \\ a_6 &= \frac{a_1^6}{32} + \frac{2}{15}a_1^2, \\ a_7 &= \frac{a_1^7}{64} + \frac{a_1^3}{10}, \\ a_8 &= \frac{a_1^8}{128} + \frac{a_1^4}{15}, \end{aligned}$$

...

(4) The function $\tilde{\kappa}_{1,2}$:

$$\tilde{\kappa}_{1,2}(z) = i(1 + a_1q_2 + a_2q_2^2 + a_3q_2^3 + \cdots),$$

where

$$a_1 = -\frac{64\pi^2}{\Gamma\left(\frac{1}{4}\right)^4} = -3.6555726483557 \cdots,$$

and the coefficients a_n satisfies the reccurent relation

$$\begin{aligned} a_2 &= \frac{a_1^2}{2} \quad \text{and} \\ 2n(n-1)(n+1)a_1a_{n+1} \\ &= \sum_{m=1}^{n-1} \left((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \right. \\ &\quad \left. + (4s_{n-m} - 4t_{n-m} + u_{n-m}) \left(\sum_{l=0}^m (l+1)^2(m-l+1)(3m-5l+1)a_{l+1}a_{m-l+1} \right) \right. \\ &\quad \left. - \left(\sum_{l=0}^{n-m} (s_l - 8t_l + 8u_l)s_{n-m-l} \right) \left(\sum_{l=0}^m (l+1)(m-l+1)a_{l+1}a_{m-l+1} \right) \right), \quad (n \geq 2), \end{aligned}$$

and the series s_n , t_n and u_n are

$$\begin{aligned} s_n &= \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n : \text{ odd}, \\ 16(-1)^{\frac{n}{2}}\sigma_3^-\left(\frac{n}{2}\right), & \text{for } n : \text{ even}, \end{cases} \\ t_n &= \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 3 \pmod{4}, \\ 16\sigma_3^-\left(\frac{n}{4}\right), & \text{for } n \equiv 0 \pmod{4}, \end{cases} \\ u_n &= \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n : \text{ odd}, \\ 16\sigma_3^-\left(\frac{n}{2}\right), & \text{for } n : \text{ even}, \end{cases} \end{aligned}$$

i.e.

$$\begin{aligned}
a_2 &= \frac{a_1^2}{2}, \\
a_3 &= \frac{a_1^3}{4} - 12a_1, \\
a_4 &= \frac{a_1^4}{8} - 12a_1^2, \\
a_5 &= \frac{a_1^5}{16} - 9a_1^3 + \frac{1582}{15}a_1, \\
a_6 &= \frac{a_1^6}{32} - 6a_1^4 + \frac{2662}{15}a_1^2, \\
a_7 &= \frac{a_1^7}{64} - \frac{15}{4}a_1^5 + \frac{1871}{10}a_1^3 - \frac{153128}{105}a_1, \\
a_8 &= \frac{a_1^8}{128} - \frac{9}{4}a_1^6 + \frac{2411}{15}a_1^4 - \frac{286016}{105}a_1^2, \\
&\dots
\end{aligned}$$

(5) The function $\check{\kappa}_{1,2}$:

$$\check{\kappa}_{1,2}(z) = \rho_2(1 + a_1q_3 + a_2q_3^2 + a_3q_3^3 + \dots),$$

where

$$a_1 = -i \frac{3^2 \Gamma\left(\frac{5}{6}\right)^2 \Gamma\left(\frac{2}{3}\right)^2}{2^{\frac{4}{3}} \pi^2} = -0.8454860396348 \dots \times i,$$

and the coefficients a_n satisfies the recurrent relation

$$\begin{aligned}
a_2 &= \frac{1}{3^{\frac{1}{2}}} e^{\frac{\pi i}{6}} a_1^2 \quad \text{and} \\
2n(n-1)(n+1)a_1a_{n+1} \\
&= -a_1^2b_n + \sum_{m=1}^{n-1} \left((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \right. \\
&\quad \left. - b_{n-m} \sum_{j=0}^m (j+1)(m-j+1)a_{j+1}a_{m-j+1} \right), \quad (n \geq 2),
\end{aligned}$$

and the series b_n are

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2 \pmod{3}, \\ 16(-1)^n \sigma_3^- \left(\frac{n}{3} \right), & \text{for } n \equiv 0 \pmod{3}, \end{cases}$$

i.e.

$$\begin{aligned}
a_2 &= \frac{1}{\sqrt{3}} e^{\frac{\pi i}{6}} a_1^2, \\
a_3 &= \frac{1}{3} e^{\frac{\pi i}{3}} a_1^3, \\
a_4 &= \frac{1}{3\sqrt{3}} e^{\frac{\pi i}{2}} a_1^4 - \frac{1}{3} a_1, \\
a_5 &= \frac{1}{9} e^{\frac{2\pi i}{3}} a_1^5 - \frac{2}{3\sqrt{3}} e^{\frac{\pi i}{6}} a_1^2, \\
a_6 &= \frac{1}{9\sqrt{3}} e^{5\pi i} 6a_1^6 - \frac{1}{3} e^{\frac{\pi i}{3}} a_1^3, \\
a_7 &= \frac{1}{27} e^{\pi i} a_1^7 - \frac{4}{9\sqrt{3}} e^{\frac{\pi i}{2}} a_1^4 - \frac{32}{315} a_1, \\
a_8 &= \frac{1}{27\sqrt{3}} e^{\frac{7\pi i}{6}} a_1^8 - \frac{5}{27} e^{\frac{2\pi i}{3}} a_1^5 - \frac{29}{315\sqrt{3}} e^{\frac{\pi i}{6}} a_1^2, \\
&\dots
\end{aligned}$$

(6) The function $\kappa_{(2)}^*$:

$$\kappa_{(2)}^*(z) = i(1 + a_1 q_2 + a_2 q_2^2 + a_3 q_2^3 + \dots),$$

where

$$a_1 = -\sqrt{2} \frac{\Gamma\left(\frac{7}{8}\right)^2 \Gamma\left(\frac{5}{8}\right)^2}{\pi^2} i = -0.350109339053 \dots \times i,$$

and the coefficients a_n satisfies the recurent relation

$$\begin{aligned}
a_2 &= \frac{a_1^2}{2} \quad \text{and} \\
2n(n-1)(n+1)a_1 a_{n+1} \\
&= -a_1^2 b_n + \sum_{m=1}^{n-1} \left((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1} a_{n-m+1} \right. \\
&\quad \left. - b_{n-m} \sum_{j=0}^m (j+1)(m-j+1)a_{j+1} a_{m-j+1} \right), \quad (n \geq 2),
\end{aligned}$$

and the series b_n are

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 3 \pmod{4}, \\ 12\sigma_3^-\left(\frac{n}{2}\right), & \text{for } n \equiv 2 \pmod{4}, \\ 12\sigma_3^-\left(\frac{n}{2}\right) + 4\sigma_3^-\left(\frac{n}{4}\right), & \text{for } n \equiv 0 \pmod{4}, \end{cases}$$

i.e.

$$\begin{aligned}
a_2 &= \frac{a_1^2}{2}, \\
a_3 &= \frac{a_1^3}{4} - \frac{4}{3}a_1, \\
a_4 &= \frac{a_1^4}{8} - \frac{4}{3}a_1^2, \\
a_5 &= \frac{a_1^5}{16} - a_1^3 + \frac{11}{10}a_1, \\
a_6 &= \frac{a_1^6}{32} - \frac{2}{3}a_1^4 + \frac{179}{90}a_1^2, \\
a_7 &= \frac{a_1^7}{64} - \frac{5}{12}a_1^5 + \frac{259}{120}a_1^3 - \frac{82}{105}a_1, \\
a_8 &= \frac{a_1^8}{128} - \frac{a_1^6}{4} + \frac{113}{60}a_1^4 - \frac{236}{105}a_1^2, \\
&\dots
\end{aligned}$$

Before proving Theorem 1, we show several lemmas. We use ' as

$$' = \frac{n}{\pi i} \frac{d}{dz} = q_n \frac{d}{dq_n}.$$

Lemma 1.

Let

$$f(z) = \rho \sum_{n=0}^{\infty} a_n q_2^n, \quad (a_0 = 1, \rho \in \mathbb{C}).$$

If $f(z)$ satisfies the functional equation

$$f(z+2) = \frac{\rho^2}{f(z)},$$

then the recurrent relation among the coefficients a_1, a_2, \dots, a_n (n : even, $n \geq 2$) is

$$a_n = \sum_{m=1}^{\frac{n}{2}-1} (-1)^{m-1} a_m a_{n-m} + \frac{(-1)^{\frac{n}{2}-1}}{2} a_{\frac{n}{2}}^2.$$

In particular

$$a_2 = \frac{a_1^2}{2}.$$

Proof. Since $q_2 \mapsto -q_2$ as $z \mapsto z+2$, we have

$$f(z+2) = \rho \sum_{n=0}^{\infty} (-1)^n a_n q_2^n.$$

Thus

$$\begin{aligned}
\rho^2 &= f(z)f(z+2) \\
&= \left(\rho \sum_{n=0}^{\infty} a_n q_2^n \right) \left(\rho \sum_{n=0}^{\infty} (-1)^n a_n q_2^n \right) \\
&= \rho^2 \sum_{n=0}^{\infty} \left(\sum_{m=0}^n (-1)^m a_m a_{n-m} \right) q_2^n.
\end{aligned}$$

Then we have

$$\sum_{m=0}^n (-1)^m a_m a_{n-m} = 0$$

for $n \geq 1$.

In the case n is odd, the left-hand side of this equation is always equal to 0. In the case n is even ($n \geq 2$), we have

$$\begin{aligned}
&\sum_{m=0}^n (-1)^m a_m a_{n-m} \\
&= 2a_0 a_n + \sum_{m=1}^{\frac{n}{2}-1} (-1)^m a_m a_{n-m} + (-1)^{\frac{n}{2}} a_{\frac{n}{2}}^2 + \sum_{m=\frac{n}{2}-1}^{n-1} (-1)^m a_m a_{n-m} \\
&= 2a_n + 2 \sum_{m=1}^{\frac{n}{2}-1} (-1)^m a_m a_{n-m} + (-1)^{\frac{n}{2}} a_{\frac{n}{2}}^2.
\end{aligned}$$

So we obtain Lemma 1. □

Lemma 2.

Let

$$f(z) = \rho_2 \sum_{n=0}^{\infty} a_n q_2^n, \quad (a_0 = 1, \rho_2 = e^{\frac{2\pi i}{3}}).$$

If $f(z)$ satisfies the functional equation

$$f(z+2) = -1 - \frac{1}{f(z)},$$

then the recurrent relation among the coefficients a_1, a_2, \dots, a_n ($n \equiv 0, 2 \pmod{3}$, $n \geq 2$) is

$$a_n = -\frac{\rho_2}{1 + \rho_2 + \rho_2^{n+1}} \sum_{m=1}^{n-1} \rho_2^m a_m a_{n-m},$$

i.e.

$$a_n = \begin{cases} \frac{e^{\frac{7\pi i}{6}}}{\sqrt{3}} \left(\sum_{m=1}^{\frac{n}{2}-1} \delta_m a_m a_{n-m} + a_{\frac{n}{2}}^2 \right), & \text{for } n \equiv 0 \pmod{6}, \\ -\frac{i}{\sqrt{3}} \left(\sum_{m=1}^{\frac{n}{2}-1} \varepsilon_m a_m a_{n-m} + a_{\frac{n}{2}}^2 \right), & \text{for } n \equiv 2 \pmod{6}, \\ \frac{e^{\frac{7\pi i}{6}}}{\sqrt{3}} \sum_{m=1}^{\frac{n}{2}-1} \delta_m a_m a_{n-m}, & \text{for } n \equiv 3 \pmod{6}, \\ -\frac{i}{\sqrt{3}} \sum_{m=1}^{\frac{n}{2}-1} \varepsilon_m a_m a_{n-m}, & \text{for } n \equiv 5 \pmod{6}, \end{cases}$$

where

$$\delta_m = \rho_2^k + \rho_2^{-k} = \begin{cases} 2, & \text{for } m \equiv 0 \pmod{3}, \\ -1 & \text{for } m \equiv 1, 2 \pmod{3}, \end{cases}$$

$$\varepsilon_m = \rho_2^k + \rho_2^{2-k} = \begin{cases} 2\rho_2, & \text{for } m \equiv 0, 2 \pmod{3}, \\ -\rho_2 & \text{for } m \equiv 1 \pmod{3}. \end{cases}$$

In particular

$$a_2 = \frac{e^{\frac{\pi i}{6}}}{\sqrt{3}} \frac{a_1^2}{2}.$$

Proof. Since $q_3 \mapsto \rho_2 q_3$ as $z \mapsto z + 2$, we have

$$f(z+2) = \rho \sum_{n=0}^{\infty} \rho_2^n a_n q_3^n.$$

By transforming the equation $f(z+2) = -1 - 1/f(z)$ to $f(z)(1 + f(z+2)) = -1$, we have

$$\begin{aligned} -1 &= f(z)(1 + f(z+2)) \\ &= \left(\rho_2 \sum_{n=0}^{\infty} a_n q_3^n \right) \left(1 + \rho_2 \sum_{n=0}^{\infty} \rho_2^n a_n q_3^n \right) \\ &= \rho_2 \sum_{n=0}^{\infty} a_n q_3^n + \rho_2^2 \sum_{n=0}^{\infty} \left(\sum_{m=0}^n (-1)^m a_m a_{n-m} \right) q_3^n \\ &= \rho_2 + \rho_2^2 + \sum_{n=0}^{\infty} \left(\rho_2 a_n + \rho_2^2 \sum_{m=0}^n \rho_2^m a_m a_{n-m} \right) q_3^n. \end{aligned}$$

Then we have

$$\rho_2 a_n + \rho_2^2 \sum_{m=0}^n \rho_2^m a_m a_{n-m} = 0,$$

i.e.

$$(1 + \rho_2 + \rho_2^{n+1}) a_n = -\rho_2 \sum_{m=1}^{n-1} \rho_2^m a_m a_{n-m}$$

for $n \geq 1$. By transformaing this equation, we obtain the remaining part of Lemma 2. \square

Lemma 3.

Let $q = e^{\pi iz/k}$ ($k \in \mathbb{Z}^+$), $\rho \in \mathbb{C}$ and

$$f(z) = \rho \sum_{n=0}^{\infty} a_n q^n \quad (a_0 = 1).$$

Suppose

$$\{f; z\} = \frac{1}{4} \sum_{n=0}^{\infty} b_n q^n \quad (b_0 = 1),$$

where $\{f; z\}$ is Schwarzian derivative. Then the recurrent relation among the coefficients a_1, a_2, \dots, a_{n+1} is

$$\begin{aligned} & 2n(n-1)(n+1)a_1a_{n+1} \\ = & -a_1^2b_n + \sum_{m=1}^{n-1} \left((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \right. \\ & \left. - b_{n-m} \sum_{l=0}^m (l+1)(m-l+1)a_{l+1}a_{m-l+1} \right). \end{aligned}$$

Proof. By the definition of Schwarzian derivative we have

$$\begin{aligned} -4\{f; z\} &= \frac{2 \frac{df(z)}{dz} \frac{d^3f(z)}{dz^3} - 3 \left(\frac{d^2f(z)}{dz^2} \right)^2}{\left(\frac{df(z)}{dz} \right)^2} \\ &= \frac{2f(z)'f(z)''' - 3(f(z)'')^2}{(f(z)')^2} \\ &= - \sum_{n=0}^{\infty} b_n q^n. \end{aligned}$$

Hence we have

$$2f(z)'f(z)''' - 3(f(z)'')^2 = \left(- \sum_{n=0}^{\infty} b_n q^n \right) (f(z)')^2. \quad (3.1)$$

Since

$$\begin{aligned} f(z)' &= \rho q \sum_{n=0}^{\infty} (n+1)a_{n+1}q^n, \\ f(z)'' &= \rho q \sum_{n=0}^{\infty} (n+1)^2a_{n+1}q^n, \\ f(z)''' &= \rho q \sum_{n=0}^{\infty} (n+1)^3a_{n+1}q^n, \end{aligned}$$

we have the Fourier expansions of both sides of (3.1) as followings:

$$\begin{aligned}
& \text{(Left-hand side of (3.1))} = 2f(z)'f(z)''' - 3(f(z)'')^2 \\
& = 2 \left(\rho q \sum_{n=0}^{\infty} (n+1)a_{n+1}q^n \right) \left(\rho q \sum_{n=0}^{\infty} (n+1)^3 a_{n+1}q^n \right) \\
& \quad - 3 \left(\rho q \sum_{n=0}^{\infty} (n+1)^2 a_{n+1}q^n \right)^2 \\
& = -\rho^2 q^2 \sum_{n=0}^{\infty} \left(\sum_{m=0}^n (m+1)^2 (n-m+1)(3n-5m+1) a_{m+1} a_{n-m+1} \right) q^n
\end{aligned}$$

and

$$\begin{aligned}
& \text{(Right-hand side of (3.1))} = \left(-\sum_{n=0}^{\infty} b_n q^n \right) (f(z)')^2 \\
& = - \left(\sum_{n=0}^{\infty} b_n q^n \right) \left(\rho q \sum_{n=0}^{\infty} (n+1)a_{n+1}q^n \right) \\
& = -\rho^2 q^2 \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \left(b_{n-m} \sum_{l=0}^m (l+1)(m-l+1) a_{l+1} a_{m-l+1} \right) \right) q^n.
\end{aligned}$$

By comparing the coefficients of q^n of the above two Fourier expansions we have the equation

$$\begin{aligned}
& \sum_{m=0}^n (m+1)^2 (n-m+1) a_{m+1} a_{n-m+1} \\
& = \sum_{m=0}^n \left(b_{n-m} \sum_{l=0}^m (l+1)(m-l+1) a_{l+1} a_{m-l+1} \right).
\end{aligned}$$

This equation consists of a_1, a_2, \dots, a_n and a_{n+1} . By rewriting the equation we obtain Lemma 3. \square

Lemma 4.

Let $q = e^{\pi iz/k}$ ($k \in \mathbb{Z}^+$), $\rho \in \mathbb{C}$ and

$$f(z) = \rho \sum_{n=0}^{\infty} a_n q^n \quad (a_0 = 1).$$

Suppose

$$\{f; z\} = \frac{1}{4} \frac{\sum_{n=0}^{\infty} \alpha_n q^n}{\sum_{n=0}^{\infty} \beta_n q^n} \quad (\alpha_0 = \beta_0 = 1),$$

where $\{f; z\}$ is Schwarzian derivative. Then the recurrent relation among the coefficients a_1, a_2, \dots, a_{n+1} is

$$\begin{aligned} & 2n(n-1)(n+1)a_1a_{n+1} \\ = & \sum_{m=1}^{n-1} \left((m+1)(n-m+1)(3n+3mn-5m^2-4m)a_{m+1}a_{n-m+1} \right. \\ & + \sum_{l=0}^m \left(-\alpha_{n-m}(l+1)(m-l+1)a_{l+1}a_{m-l+1} \right. \\ & \left. \left. + \beta_{n-m}(l+1)^2(m-l+1)(3m-5l+1) \right) a_{l+1}a_{m-l+1} \right). \end{aligned}$$

Proof. By the definition of Schwarzian derivative we have

$$-4\{f; z\} = \frac{2f(z)'f(z)''' - 3(f(z)'')^2}{(f(z)')^2} = \frac{\sum_{n=0}^{\infty} \alpha_n q^n}{\sum_{n=0}^{\infty} \beta_n q^n}.$$

Hence we have

$$\left(\sum_{n=0}^{\infty} \beta_n q^n \right) \left(2f(z)'f(z)''' - 3(f(z)'')^2 \right) = \left(-\sum_{n=0}^{\infty} \alpha_n q^n \right) (f(z)')^2. \quad (3.1)$$

Since

$$\begin{aligned} f(z)' &= \rho q \sum_{n=0}^{\infty} (n+1)a_{n+1}q^n, \\ f(z)'' &= \rho q \sum_{n=0}^{\infty} (n+1)^2 a_{n+1}q^n, \\ f(z)''' &= \rho q \sum_{n=0}^{\infty} (n+1)^3 a_{n+1}q^n, \end{aligned}$$

we have the Fourier expansions of both sides of (3.1) as followings:

$$\begin{aligned}
(\text{Left-hand side of (3.1)}) &= \left(\sum_{n=0}^{\infty} \beta_n q^n \right) \left(2f(z)'f(z)''' - 3(f(z)'')^2 \right) \\
&= -\rho^2 q^2 \left(\sum_{n=0}^{\infty} \beta_n q^n \right) \\
&\quad \times \left(\sum_{n=0}^{\infty} \left(\sum_{m=0}^n (m+1)^2 (n-m+1) (3n-5m+1) a_{m+1} a_{n-m+1} \right) q^n \right) \\
&= -\rho^2 q^2 \sum_{n=0}^{\infty} \sum_{m=0}^n \beta_{n-m} \left(\sum_{l=0}^m (l+1)^2 (m-l+1) (3m-5l+1) a_{l+1} a_{m-l+1} \right) q^n
\end{aligned}$$

and

$$\begin{aligned}
(\text{Right-hand side of (3.1)}) &= \left(-\sum_{n=0}^{\infty} \alpha_n q^n \right) (f(z)')^2 \\
&= -\rho^2 q^2 \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \left(\alpha_{n-m} \sum_{l=0}^m (l+1)(m-l+1) a_{l+1} a_{m-l+1} \right) \right) q^n.
\end{aligned}$$

By comparing the coefficients of q^n of the above two Fourier expansions we have the equation

$$\begin{aligned}
&\sum_{m=0}^n \beta_{n-m} \left(\sum_{l=0}^m (l+1)^2 (m-l+1) (3m-5l+1) a_{l+1} a_{m-l+1} \right) \\
&= \sum_{m=0}^n \left(\alpha_{n-m} \sum_{l=0}^m (l+1)(m-l+1) a_{l+1} a_{m-l+1} \right).
\end{aligned}$$

This equation consists of a_1, a_2, \dots, a_n and a_{n+1} . By rewriting the equation we obtain Lemma 3. \square

Now we recall the Eisenstein series

$$\begin{aligned}
E_2(z) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}, \\
E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n z}, \\
E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n z}.
\end{aligned}$$

Lemma 5.

We obtain following limit values:

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda^\sharp(z)}{dq_2} = -32,$$

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda^\flat(z)}{dq_2} = 32,$$

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d}{dq_2} \left(q_2 \frac{d\lambda^\flat(z)}{dq_2} \right) = 64,$$

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda_{1,2}^\sharp(z)}{dq_2} = -256,$$

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda_{1,2}^\sharp(\tau)}{d\tau} = -\frac{2}{3}\pi^2 ia_1 E_4(i), \quad \text{for } \tau = \kappa_{1,2}(z),$$

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda_{1,2}^\flat(\tau)}{d\tau} = \frac{2}{3}\pi^2 ia_1 E_4(i), \quad \text{for } \tau = \tilde{\kappa}_{1,2}(z),$$

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{dJ(\tau)}{d\tau} = -2\pi^2 ia_1 E_4(i), \quad \text{for } \tau = i(1 + a_1 q_2 + a_2 q_2^2 + a_3 q_2^3 + \dots),$$

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda_{(2)}^*(\tau)}{d\tau} = \frac{2^{\frac{5}{2}}}{(\sqrt{2}-1)^2} \pi^2 ia_1 \theta_2(\sqrt{2}i)^8, \quad \text{for } \tau = \kappa_{(2)}^*(z),$$

$$\lim_{z \rightarrow i\infty} \frac{d^2 \lambda_{(2)}^*(\tau)}{d\tau^2} = \frac{2^2}{(\sqrt{2}-1)^2} \pi^2 \theta_2(\sqrt{2}i)^8, \quad \text{for } \tau = \kappa_{(2)}^*(z).$$

Proof. Since

$$\begin{aligned} \frac{d\lambda^\sharp(z)}{dq_2} &= \frac{d}{dq_2} (1 - 16q_2^2 + 128q_2^4 - 704q_2^6 + \dots) = -32q_2 + 512q_2^3 - 4224q_2^5 + \dots, \\ \frac{d}{dq_2} \left(q_2 \frac{d\lambda^\flat(z)}{dq_2} \right) &= -64q_2 + 2048q_2^3 - 25344q_2^5 + \dots \end{aligned}$$

and $\lambda^\sharp(z) = 1 - \lambda^\flat(z)$, we obtain

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda^\sharp(z)}{dq_2} = -32 \quad \text{and} \quad \lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda^\flat(z)}{dq_2} = 32.$$

Let $\tau = \kappa_{1,2}(z)$. Since $J(\tau) = P_{1,2}^\sharp(\lambda_{1,2}^\sharp(\tau))$, we have

$$-2\pi i \frac{E_6(\tau)}{E_4(\tau)} J(\tau) = \frac{dJ(\tau)}{d\tau} = \frac{1}{27} \frac{(\lambda_{1,2}^\sharp(\tau) - 9)(\lambda_{1,2}^\sharp(\tau) + 3)^2}{(\lambda_{1,2}^\sharp(\tau) - 1)^2}.$$

And we have $\lambda_{1,2}^\sharp(\kappa_{1,2}(z)) = \lambda^\flat(z)$ and $\lambda^\flat(i\infty) = 0$. So we obtain

$$\begin{aligned} \lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda_{1,2}^\sharp(\tau)}{d\tau} &= \lim_{z \rightarrow i\infty} \frac{1}{q_2} \left(\frac{E_6(\kappa_{1,2}(z))}{E_4(\kappa_{1,2}(z))} \frac{(\lambda^\flat(z) + 3)(\lambda^\flat(z) - 1)}{\lambda^\flat(z) - 9} \right) \\ &= -\frac{2}{3}\pi^2 ia_1 E_4(i). \end{aligned}$$

Using same method we obtain

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda_{1,2}^b(\tau)}{d\tau} = \frac{2}{3} \pi^2 i a_1 E_4(i),$$

for $\tau = \tilde{\kappa}_{1,2}(z)$.

Let $\tau = i(1 + a_1 q_2 + a_2 q_2^2 + a_3 q_2^3 + \dots)$. Since

$$\frac{dJ(\tau)}{d\tau} = -2\pi i \frac{E_6(\tau)}{E_4(\tau)} J(\tau) = -2\pi i \frac{E_4(\tau)^2 E_6(\tau)}{E_4(\tau)^3 - E_6(\tau)^2}$$

and

$$\lim_{z \rightarrow i\infty} \frac{E_6(\tau)}{q_2} = \pi a_1 E_4(i)^2,$$

we obtain

$$\lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{dJ(\tau)}{d\tau} = -2\pi i \lim_{z \rightarrow i\infty} \frac{E_6(\tau)}{q_2} \frac{1}{E_4(\tau)} J(\tau) = -2\pi^2 i a_1 E_4(i).$$

Let $\tau = \kappa_{(2)}^*(z)$, $N(z) = 1 - 6\lambda_{(2)}^b(z) + \lambda_{(2)}^b(z)^2$ and $D(z) = 16\lambda^b(z)(1 - \lambda^b(z))$. Then we can write

$$\lambda_{(2)}^*(z) = P^*(\lambda^b(z)) = -\frac{N(z)^2}{D(z)}.$$

Since $\lambda^b(z)' = 2\theta_2(z)^8(1 - \lambda^b(z))$ and $\lambda^b(\sqrt{2}i) = 3 - 2\sqrt{2}$, we have

$$\lim_{z \rightarrow i\infty} \frac{d\lambda^b(\tau)}{\tau} = \lim_{z \rightarrow i\infty} \frac{\pi i}{2} \lambda^b(\tau)' = 2(\sqrt{2} - 1)\pi i \theta_2(\sqrt{2}i)^4.$$

Hence

$$\begin{aligned} \lim_{z \rightarrow i\infty} \frac{dN(\tau)}{d\tau} &= \lim_{z \rightarrow i\infty} (-6 + 2\lambda^b(\tau)) \frac{d\lambda^b(\tau)}{d\tau} = -2^{\frac{7}{2}}(\sqrt{2} - 1)\pi i \theta_2(\sqrt{2}i)^4, \\ \lim_{z \rightarrow i\infty} \frac{1}{q_2} N(\tau) &= \lim_{z \rightarrow i\infty} \frac{\frac{dN(\tau)}{d\tau}}{\frac{\pi i}{2} q_2} = \lim_{z \rightarrow i\infty} \frac{\frac{\pi i}{2} \tau' \frac{dN(\tau)}{d\tau}}{\frac{\pi i}{2} q_2} = 2^4(\sqrt{2} - 1)\pi a_1 \theta_2(\sqrt{2}i)^4. \end{aligned}$$

So we obtain following two limit values with noting that $N(\sqrt{2}i) = 0$:

$$\begin{aligned} \lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d\lambda_{(2)}^*(\tau)}{d\tau} &= \lim_{z \rightarrow i\infty} \frac{1}{q_2} \frac{d}{d\tau} \left(N(\tau)^2 \frac{1}{D(\tau)} \right) \\ &= \lim_{z \rightarrow i\infty} \left(-\frac{1}{q_2} 2N(\tau) \frac{dN(\tau)}{d\tau} \frac{1}{D(\tau)} + \frac{1}{q_2} N(\tau)^2 \frac{1}{D(\tau)^2} \frac{dD(\tau)}{d\tau} \right) \\ &= 2^{\frac{5}{2}}(\sqrt{2} - 1)^{-2} \pi^2 i a_1 \theta_2(\sqrt{2}i)^8 \end{aligned}$$

and

$$\begin{aligned} \lim_{z \rightarrow i\infty} \frac{d^2 \lambda_{(2)}^*(\tau)}{d\tau^2} &= \lim_{z \rightarrow i\infty} \left(N(\tau) \left(-\frac{d^2 N(\tau)}{d\tau^2} \frac{1}{D(\tau)} + 2 \frac{dN(\tau)}{d\tau} \frac{1}{D(\tau)^2} \frac{dD(\tau)}{d\tau} \right. \right. \\ &\quad \left. \left. + N(\tau) \frac{d}{d\tau} \left(\frac{1}{D(\tau)^2} \frac{dD(\tau)}{d\tau} \right) \right) - 2 \frac{1}{D(\tau)} \left(\frac{dN(\tau)}{d\tau} \right)^2 \right) \\ &= 2^2 (\sqrt{2} - 1)^{-2} \pi^2 \theta_2(\sqrt{2}i)^8. \end{aligned}$$

□

Lemma 6.

We obtain following limit values:

$$\begin{aligned} \lim_{z \rightarrow i\infty} \frac{1}{q_3^2} \frac{d\lambda^b(z)}{dq_3} &= 48, \\ \lim_{z \rightarrow i\infty} \frac{1}{q_3^2} \frac{d\lambda_{1,2}^b(z)}{dq_3} &= 192, \\ \lim_{z \rightarrow i\infty} \frac{1}{q_3^2} \frac{dJ(\tau)}{d\tau} &= -\frac{8}{9} \pi^3 i \rho_3^2 a_1^2 E_6(\rho_2), \quad \text{for } \tau = \rho_2(1 + a_1 q_3 + a_2 q_3^2 + a_3 q_3^3 + \dots). \end{aligned}$$

Proof. Since

$$\lambda^b(z) = 16q_3^3 - 128q_3^6 + 704q_3^9 - \dots,$$

we obtain

$$\lim_{z \rightarrow i\infty} \frac{1}{q_3^2} \frac{d\lambda^b(z)}{dq_3} = 48.$$

Since

$$\lambda_{1,2}^b(z) = P^b(\lambda^b(z)) = 64q_3^3 - 1536q_3^6 + 19200q_3^{12} + \dots,$$

we obtain

$$\lim_{z \rightarrow i\infty} \frac{1}{q_3^2} \frac{d\lambda_{1,2}^b(z)}{dq_3} = 192.$$

Let $\tau = \rho_2(1 + a_1 q_3 + a_2 q_3^2 + a_3 q_3^3 + \dots)$.

Since

$$\frac{dJ(\tau)}{d\tau} = -2\pi i \frac{E_6(\tau)}{E_4(\tau)} J(\tau) = -2\pi i \frac{E_4(\tau)^2 E_6(\tau)}{E_4(\tau)^3 - E_6(\tau)^2}$$

and

$$\lim_{z \rightarrow i\infty} \frac{E_4(\tau)}{q_3} = -\frac{2}{3} \pi i \rho_3 a_1 E_6(\rho_2),$$

we obtain

$$\lim_{z \rightarrow i\infty} \frac{1}{q_3^2} \frac{dJ(\tau)}{d\tau} = -2\pi i \lim_{z \rightarrow i\infty} \left(\frac{E_4(\tau)}{q_3} \right)^2 \frac{E_6(\tau)}{E_4(\tau)^3 - E_6(\tau)^2} = -\frac{8}{9} \pi^3 i \rho_3^2 a_1^2 E_6(\rho_2).$$

□

Proof of Theorem 1.

(0) Since

$$\kappa^\sharp(z+2) = -\frac{1}{\kappa^\sharp(z)} \quad , \quad \kappa^\sharp(z+4) = \kappa^\sharp(z),$$

and $\kappa^\sharp(i\infty) = i$, we can identify $\kappa^\sharp(z)$ with the Fourier expansion

$$i(1 + a_1q_2 + a_2q_2^2 + a_3q_2^3 + \cdots).$$

Let $x = \lambda^\sharp(z)$. Since the properties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^\sharp)^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{J^{-1}; x\} &= \frac{1 - (\frac{1}{3})^2}{x^2} + \frac{1 - (\frac{1}{2})^2}{(1-x)^2} + \frac{1 - (\frac{1}{3})^2 - (\frac{1}{2})^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\kappa^\sharp; z\} &= -4\{\lambda^\sharp; z\} - 4\{J^{-1}; x\} \left(\frac{dx}{dz}\right)^2 \\ &= \left(-(-4\{(\lambda^\sharp)^{-1}; x\}) + (-4\{J^{-1}; x\})\right) \left(\frac{dx}{dz}\right)^2 \\ &= -\frac{1}{36} \frac{4+5x}{x^2(1-x)^2} \left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $(\frac{2}{\pi i})^2$, we have

$$\begin{aligned} \frac{2\kappa^\sharp(z)'\kappa^\sharp(z)''' - 3(\kappa^\sharp(z)''^2)}{(\kappa^\sharp(z)')^2} &= -\frac{1}{36} \frac{4+5\lambda^\sharp(z)}{\lambda^\sharp(z)^2(1-\lambda^\sharp(z))^2} (\lambda^\sharp(z)')^2 \\ &= -\frac{1}{9} (5\theta_0(z)^4\theta_3(z)^4 + 4\theta_3(z)^8). \end{aligned}$$

(Note that $\lambda^\sharp(z)' = 2\theta_2(z)(1-\lambda^\sharp(z))$ for q_2 .)

Put

$$\frac{1}{9} (5\theta_0(z)^4\theta_3(z)^4 + 4\theta_3(z)^8) = \sum_{n=0}^{\infty} b_n q_2^n.$$

By the fomulas

$$\begin{aligned} \theta_0(z)^4\theta_3(z)^4 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_2^{4n}, \\ \theta_3(z)^8 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_2^{2n}, \end{aligned}$$

we have

$$b_n = \begin{cases} 1, & n = 0 \\ 0, & \text{for } n:\text{odd}, \\ (-1)^{\frac{n}{2}} \frac{64}{9} \sigma_3^- \left(\frac{n}{2} \right), & \text{for } n \equiv 2 \pmod{4}, \\ (-1)^{\frac{n}{2}} \frac{64}{9} \sigma_3^- \left(\frac{n}{2} \right) + \frac{80}{9} \sigma_3^- \left(\frac{n}{4} \right), & \text{for } n \equiv 0 \pmod{4}, \end{cases}$$

By Lemma 1 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (0).

Now we calculate a_1 . Differentiating the identity $J(\kappa^\sharp(z)) = \lambda^\sharp(z)$ and multiplying both sides by $\frac{2}{\pi i}$, we have

$$\kappa^\sharp(z)' \frac{dJ(\tau)}{d\tau} = \lambda^\sharp(z)'$$

where $\tau = \kappa^\sharp(z)$. Multiplying both sides by $1/q_2^2$ and using Lemma 5, we have

$$ia_1 \times (-2\pi^2 ia_1 E_4(i)) = -32$$

as $z \rightarrow i\infty$, and since we have

$$E_4(i) = \frac{3\Gamma\left(\frac{1}{4}\right)^8}{2^6\pi^6},$$

we obtain (0) in Theorem 1. This result is obtained by KANEKO and YOSHIDA [KY].

(1) $\tilde{\kappa}_{(2)}$ is bijection and the automorphism on $\mathbb{H} \curvearrowright \Gamma(2)$. So it is linear transformation. Since $\tilde{\kappa}_{(2)}$ maps 0, 1 and $i\infty$ to $i\infty$, 1 and 0, respectively, we obtain

$$\tilde{\kappa}_{(2)}(z) = -\frac{1}{z}.$$

(2) Since

$$\kappa^b(z+2) = -1 - \frac{1}{\kappa^b(z)}, \quad \kappa^b(z+6) = \kappa^b(z),$$

and $\kappa^b(i\infty) = \rho_2$, we can identify $\kappa^b(z)$ with the Fourier expansion

$$\rho_2(1 + a_1q_3 + a_2q_3^2 + a_3q_3^3 + \cdots).$$

Let $x = \lambda^b(z)$. Since the properties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^b)^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{J^{-1}; x\} &= \frac{1 - \left(\frac{1}{3}\right)^2}{x^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned}
-4\{\kappa^b; z\} &= -4\{\lambda^b; z\} - 4\{J^{-1}; x\} \left(\frac{dx}{dz}\right)^2 \\
&= \left(-(-4\{(\lambda^b)^{-1}; x\}) + (-4\{J^{-1}; x\})\right) \left(\frac{dx}{dz}\right)^2 \\
&= -\frac{1}{36} \frac{4+5x}{x^2(1-x)^2} \left(\frac{dx}{dz}\right)^2.
\end{aligned}$$

Multiplying both sides by $\left(\frac{3}{\pi i}\right)^2$, we have

$$\begin{aligned}
\frac{2\kappa^b(z)'\kappa^b(z)''' - 3(\kappa^b(z)''')}{(\kappa^b(z)')^2} &= -\frac{1}{36} \frac{4+5\lambda^b(z)}{\lambda^b(z)^2(1-\lambda^b(z))^2} (\lambda^b(z)')^2 \\
&= -\frac{1}{4} (9\theta_3(z)^8 - 5\theta_0(z)^4\theta_3(z)^4).
\end{aligned}$$

(Note that $\lambda^b(z)' = 3\theta_2(z)(1-\lambda^b(z))$ for q_3 .)

Put

$$\frac{1}{4} (9\theta_3(z)^8 - 5\theta_0(z)^4\theta_3(z)^4) = \sum_{n=0}^{\infty} b_n q_3^n.$$

By the fomulas

$$\begin{aligned}
\theta_0(z)^4\theta_3(z)^4 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_3^{6n}, \\
\theta_3(z)^8 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_3^{3n},
\end{aligned}$$

we have

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 4, 5 \pmod{6}, \\ 36(-1)^n \sigma_3^-\left(\frac{n}{3}\right), & \text{for } n \equiv 3 \pmod{6}, \\ 36(-1)^n \sigma_3^-\left(\frac{n}{3}\right) - 20\sigma_3^-\left(\frac{n}{6}\right), & \text{for } n \equiv 0 \pmod{6}, \end{cases}$$

By Lemma 2 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (2).

Now we calculate a_1 . Differentiating the identity $J(\kappa^b(z)) = \lambda^b(z)$ and multiplying both sides by $\frac{3}{\pi i}$, we have

$$\kappa^b(z)' \frac{dJ(\tau)}{d\tau} = \lambda^b(z)'$$

where $\tau = \kappa^b(z)$. Multiplying both sides by $1/q_3^3$ and using Lemma 6, we have

$$\rho_2 a_1 \times \left(-\frac{8}{9} \pi^3 i \rho_3^2 a_1^2 E_6(\rho_2) \right) = 48$$

as $z \rightarrow i\infty$, and since we have

$$E_6(\rho_2) = \frac{2^5}{3^3} \frac{\pi^3}{\Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{2}{3}\right)},$$

we obtain (2) in Theorem 1.

(3) Since

$$\kappa_{1,2}(z+2) = -\frac{1}{\kappa_{1,2}(z)}, \quad \kappa_{1,2}(z+4) = \kappa_{1,2}(z),$$

and $\kappa_{1,2}(i\infty) = i$, we can identify $\kappa_{1,2}(z)$ with the Fourier expansion

$$i(1 + a_1 q_2 + a_2 q_2^2 + a_3 q_2^3 + \cdots).$$

Let $x = \lambda^b(z)$. Since the properties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^b)^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{(\lambda_{1,2}^\#)^{-1}; x\} &= \frac{1 - \left(\frac{1}{2}\right)^2}{x^2} + \frac{1}{(1-x)^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\kappa_{1,2}; z\} &= -4\{\lambda^b; z\} - 4\{(\lambda_{1,2}^\#)^{-1}; x\} \left(\frac{dx}{dz}\right)^2 \\ &= \left(-(-4\{(\lambda^b)^{-1}; x\}) + (-4\{(\lambda_{1,2}^\#)^{-1}; x\})\right) \left(\frac{dx}{dz}\right)^2 \\ &= -\frac{1}{4} \frac{1}{x^2(1-x)} \left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $\left(\frac{2}{\pi i}\right)^2$, we have

$$\begin{aligned} \frac{2\kappa_{1,2}(z)' \kappa_{1,2}(z)''' - 3(\kappa_{1,2}(z)'')^2}{(\kappa_{1,2}(z)')^2} &= -\frac{1}{4} \frac{1}{\lambda^b(z)^2(1-\lambda^b(z))} (\lambda^b(z)')^2 \\ &= -\theta_0(z)^4 \theta_3(z)^4. \end{aligned}$$

(Note that $\lambda^b(z)' = 2\theta_2(z)(1-\lambda^b(z))$ for q_2 .)

Put

$$\theta_0(z)^4 \theta_3(z)^4 = \sum_{n=0}^{\infty} b_n q_2^n.$$

By the fomula

$$\theta_0(z)^4 \theta_3(z)^4 = 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_2^{4n},$$

we have

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 3 \pmod{4}, \\ 16\sigma_3^-\left(\frac{n}{4}\right), & \text{for } n \equiv 0 \pmod{4}, \end{cases}$$

By Lemma 1 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (3).

Now we calculate a_1 . Differentiating the identity $\lambda_{1,2}^\sharp(\kappa_{1,2}(z)) = \lambda^b(z)$ and multiplying both sides by $\frac{2}{\pi i}$, we have

$$\kappa_{1,2}(z)' \frac{d\lambda_{1,2}^\sharp(\tau)}{d\tau} = \lambda^b(z)'$$

where $\tau = \kappa_{1,2}(z)$. Multiplying both sides by $1/q_2^2$ and using Lemma 5, we have

$$ia_1 \times \left(-\frac{2}{3}\pi^2 ia_1 E_4(i) \right) = 32$$

as $z \rightarrow i\infty$, and since we have

$$E_4(i) = \frac{3\Gamma\left(\frac{1}{4}\right)^8}{2^6\pi^6},$$

we obtain (3) in Theorem 1.

(4) Since

$$\tilde{\kappa}_{1,2}(z+2) = -\frac{1}{\tilde{\kappa}_{1,2}(z)}, \quad \tilde{\kappa}_{1,2}(z+4) = \tilde{\kappa}_{1,2}(z),$$

and $\tilde{\kappa}_{1,2}(i\infty) = i$, we can identify $\tilde{\kappa}_{1,2}(z)$ with the Fourier expansion

$$i(1 + a_1q_2 + a_2q_2^2 + a_3q_2^3 + \cdots).$$

Let $x = \lambda_{1,2}^\sharp(z) = P^\sharp(\lambda^b(z))$. Since the properties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda_{1,2}^\sharp)^{-1}; x\} &= \frac{1 - \left(\frac{1}{2}\right)^2}{x^2} + \frac{1}{(1-x)^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{x(1-x)}, \\ -4\{(\lambda_{1,2}^b)^{-1}; x\} &= \frac{1}{x^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\tilde{\kappa}_{1,2}; z\} &= -4\{\lambda_{1,2}^\sharp; z\} - 4\{(\lambda_{1,2}^b)^{-1}; x\} \left(\frac{dx}{dz}\right)^2 \\ &= \left(-\left(-4\{(\lambda_{1,2}^\sharp)^{-1}; x\}\right) + \left(-4\{(\lambda_{1,2}^b)^{-1}; x\}\right) \right) \left(\frac{dx}{dz}\right)^2 \\ &= \frac{1}{4} \frac{1-2x}{x^2(1-x)^2} \left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $(\frac{2}{\pi i})^2$, we have

$$\begin{aligned}
& \frac{2\tilde{\kappa}_{1,2}(z)\tilde{\kappa}_{1,2}(z)''' - 3(\tilde{\kappa}_{1,2}(z)'')}{(\tilde{\kappa}_{1,2}(z)')^2} \\
&= -\frac{1}{4} \frac{1}{\lambda_{1,2}^\#(z)^2(1 - \lambda_{1,2}^\#(z))} \left(\lambda_{1,2}^\#(z)'\right)^2 \\
&= \frac{1}{4} \frac{1 - 2(2\lambda^b(z) - 1)^2}{(2\lambda^b(z) - 1)^2(1 - (2\lambda^b(z) - 1))^2} (4(2\lambda^b(z) - 1)\lambda^b(z)') \\
&= -\frac{\theta_3(z)^8(8\theta_0(z)^8 - 8\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8)}{(\theta_0(z)^4 - 2\theta_3(z)^4)^2}.
\end{aligned}$$

(Note that $\lambda^b(z)' = 2\theta_2(z)(1 - \lambda^b(z))$ for q_2 .)

Put

$$\begin{aligned}
\theta_3(z)^8(8\theta_0(z)^8 - 8\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8) &= \sum_{n=0}^{\infty} \alpha_n q_2^n, \\
(\theta_0(z)^4 - 2\theta_3(z)^4)^2 &= \sum_{n=0}^{\infty} \beta_n q_2^n.
\end{aligned}$$

By the fomula

$$\begin{aligned}
\theta_0(z)^4\theta_3(z)^4 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_2^{4n}, \\
\theta_3(z)^8 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_2^{2n},
\end{aligned}$$

we have

$$\begin{aligned}
\alpha_n &= \sum_{m=0}^n s_{n-m}(s_m - 8t_m - 8u_m), \\
\beta_n &= 4s_n - 4t_n + u_n,
\end{aligned}$$

where s_n , t_n and u_n satisfy

$$\begin{aligned}
\theta_3(z)^8 &= \sum_{n=0}^{\infty} s_n q_2^n, \\
\theta_0(z)^4\theta_3(z)^4 &= \sum_{n=0}^{\infty} t_n q_2^n, \\
\theta_0(z)^8 &= \sum_{n=0}^{\infty} u_n q_2^n,
\end{aligned}$$

i.e.

$$s_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n : \text{ odd}, \\ 16(-1)^{\frac{n}{2}}\sigma_3^- \left(\frac{n}{2}\right), & \text{for } n : \text{ even}, \end{cases}$$

$$t_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2, 3 \pmod{4}, \\ 16\sigma_3^- \left(\frac{n}{4}\right), & \text{for } n \equiv 0 \pmod{4}, \end{cases}$$

$$u_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n : \text{ odd}, \\ 16\sigma_3^- \left(\frac{n}{2}\right), & \text{for } n : \text{ even}, \end{cases}$$

By Lemma 1 and 4, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (4).

Now we calculate a_1 . Differentiating the identity $\lambda_{1,2}^b(\tilde{\kappa}_{1,2}(z)) = \lambda_{1,2}^\sharp(z)$ and multiplying both sides by $\left(\frac{2}{\pi i}\right)$, we have

$$\tilde{\kappa}_{1,2}(z)' \frac{d\lambda_{1,2}^b(\tau)}{d\tau} = \lambda_{1,2}^\sharp(z)'$$

where $\tau = \tilde{\kappa}_{1,2}(z)$. Multiplying both sides by $1/q_2^2$ and using Lemma 5, we have

$$ia_1 \times \frac{2}{3}\pi^2 ia_1 E_4(i) = -256$$

as $z \rightarrow i\infty$, and since we have

$$E_4(i) = \frac{3\Gamma\left(\frac{1}{4}\right)^8}{2^6\pi^6},$$

we obtain (4) in Theorem 1.

(5) Since

$$\check{\kappa}_{1,2}(z+2) = -1 - \frac{1}{\check{\kappa}_{1,2}(z)}, \quad \check{\kappa}_{1,2}(z+6) = \check{\kappa}_{1,2}(z),$$

and $\check{\kappa}_{1,2}(i\infty) = \rho_2$, we can identify $\check{\kappa}_{1,2}(z)$ with the Fourier expansion

$$\rho_2(1 + a_1q_3 + a_2q_3^2 + a_3q_3^3 + \cdots).$$

Let $x = \lambda_{1,2}^b(z) = P^b(\lambda^b(z))$. Since the properties of Schwarzian derivative, we have

$$-4\{(\lambda_{1,2}^b)^{-1}; x\} = \frac{1}{x^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{x(1-x)},$$

$$-4\{J^{-1}; x\} = \frac{1 - \left(\frac{1}{3}\right)^2}{x^2} + \frac{1 - \left(\frac{1}{2}\right)^2}{(1-x)^2} + \frac{1 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^2}{x(1-x)}$$

and

$$\begin{aligned}
-4\{\tilde{\kappa}_{1,2}; z\} &= -4\{\lambda_{1,2}^b; z\} - 4\{J^{-1}; x\} \left(\frac{dx}{dz}\right)^2 \\
&= \left(-(-4\{(\lambda_{1,2}^b)^{-1}; x\}) + (-4\{J^{-1}; x\})\right) \left(\frac{dx}{dz}\right)^2 \\
&= -\frac{1}{9} \frac{1}{x^2(1-x)} \left(\frac{dx}{dz}\right)^2.
\end{aligned}$$

Multiplying both sides by $\left(\frac{3}{\pi i}\right)^2$, we have

$$\begin{aligned}
\frac{2\tilde{\kappa}_{1,2}(z)'\tilde{\kappa}_{1,2}(z)''' - 3(\tilde{\kappa}_{1,2}(z)'')^2}{(\tilde{\kappa}_{1,2}(z)')^2} &= -\frac{1}{9} \frac{1}{\lambda_{1,2}^b(z)^2(1-\lambda_{1,2}^b(z))} (\lambda_{1,2}^b(z)')^2 \\
&= -\frac{1}{9} \frac{1}{\lambda^b(z)^2(1-\lambda^b(z))} (4(1-2\lambda^b(z))\lambda^b(z)')^2 \\
&= -\theta_3(z)^8.
\end{aligned}$$

(Note that $\lambda^b(z)' = 3\theta_2(z)(1-\lambda^b(z))$ for q_3 .)

Put

$$\frac{1}{4} (9\theta_3(z)^8 - 5\theta_0(z)^4\theta_3(z)^4) = \sum_{n=0}^{\infty} b_n q_3^n.$$

By the fomula

$$\theta_3(z)^8 = 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_3^{3n},$$

we have

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 2 \pmod{3}, \\ 16(-1)^n \sigma_3^-\left(\frac{n}{3}\right), & \text{for } n \equiv 0 \pmod{3}, \end{cases}$$

By Lemma 2 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (5).

Now we calculate a_1 . Differentiating the identity $J(\tilde{\kappa}_{1,2}(z)) = \lambda_{1,2}^b(z)$ and multiplying both sides by $\frac{3}{\pi i}$, we have

$$\tilde{\kappa}_{1,2}(z)' \frac{dJ(\tau)}{d\tau} = \lambda_{1,2}^b(z)'$$

where $\tau = \tilde{\kappa}_{1,2}(z)$. Multiplying both sides by $1/q_3^3$ and using Lemma 6, we have

$$\rho_2 a_1 \times \left(-\frac{8}{9} \pi^3 i \rho_3^2 a_1^2 E_6(\rho_2)\right) = 192$$

as $z \rightarrow i\infty$, and since we have

$$E_6(\rho_2) = \frac{2^5}{3^3} \frac{\pi^3}{\Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{2}{3}\right)},$$

we obtain (5) in Theorem 1.

(6) Since

$$\kappa_{(2)}^*(z+2) = -\frac{2}{\kappa_{(2)}^*(z)}, \quad \kappa_{(2)}^*(z+4) = \kappa_{(2)}^*(z),$$

and $\kappa_{(2)}^*(i\infty) = \sqrt{2}i$, we can identify $\kappa_{(2)}^*(z)$ with the Fourier expansion

$$\sqrt{2}i(1 + a_1q_2 + a_2q_2^2 + a_3q_2^3 + \cdots).$$

Let $x = \lambda^b(z)$. Since the properties of Schwarzian derivative, we have

$$\begin{aligned} -4\{(\lambda^b)^{-1}; x\} &= \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{x(1-x)}, \\ -4\{(\lambda_{(2)}^*)^{-1}; x\} &= \frac{1 - (\frac{1}{2})^2}{x^2} + \frac{1 - (\frac{1}{4})}{(1-x)^2} + \frac{1 - (\frac{1}{2})^2 - (\frac{1}{4})}{x(1-x)} \end{aligned}$$

and

$$\begin{aligned} -4\{\kappa_{(2)}^*; z\} &= -4\{\lambda^b; z\} - 4\{(\lambda_{(2)}^*)^{-1}; x\} \left(\frac{dx}{dz}\right)^2 \\ &= \left(-(-4\{(\lambda^b)^{-1}; x\}) + (-4\{(\lambda_{(2)}^*)^{-1}; x\})\right) \left(\frac{dx}{dz}\right)^2 \\ &= -\frac{1}{16} \frac{4-3x}{x^2(1-x)^2} \left(\frac{dx}{dz}\right)^2. \end{aligned}$$

Multiplying both sides by $(\frac{2}{\pi i})^2$, we have

$$\begin{aligned} \frac{2\kappa_{(2)}^*(z)' \kappa_{(2)}^*(z)''' - 3(\kappa_{(2)}^*(z)'')^2}{(\kappa_{(2)}^*(z)')^2} &= -\frac{1}{16} \frac{4-3\lambda^b(z)}{\lambda^b(z)^2(1-\lambda^b(z))^2} (\lambda^b(z)')^2 \\ &= -\frac{1}{4} (3\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8). \end{aligned}$$

(Note that $\lambda^b(z)' = 2\theta_2(z)(1-\lambda^b(z))$ for q_2 .)

Put

$$\frac{1}{4} (3\theta_0(z)^4\theta_3(z)^4 + \theta_3(z)^8) = \sum_{n=0}^{\infty} b_n q_2^n.$$

By the fomula

$$\begin{aligned} \theta_0(z)^4\theta_3(z)^4 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_2^{4n}, \\ \theta_3(z)^8 &= 1 + 16 \sum_{n=1}^{\infty} \sigma_3^-(n) q_2^{2n}, \end{aligned}$$

we have

$$b_n = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \equiv 1, 3 \pmod{4}, \\ 12\sigma_3^-\left(\frac{n}{2}\right), & \text{for } n \equiv 2 \pmod{4}, \\ 12\sigma_3^-\left(\frac{n}{2}\right) + 4\sigma_3^-\left(\frac{n}{4}\right), & \text{for } n \equiv 0 \pmod{4}, \end{cases}$$

By Lemma 1 and 3, we obtain the recurrent relation of $\{a_n\}$ in Theorem 1 (6).

Now we calculate a_1 . Differentiating the identity $\lambda_{(2)}^*(\kappa_{(2)}^*(z)) = \lambda^b(z)$ twice and multiplying both sides by $\left(\frac{z}{\pi i}\right)^2$, we have

$$\kappa_{(2)}^*(z)'' \frac{d\lambda_{(2)}^*(\tau)}{d\tau} + (\kappa_{(2)}^*(z)')^2 \frac{d^2\lambda_{(2)}^*(\tau)}{d\tau^2} = \lambda^b(z)''$$

where $\tau = \kappa_{(2)}^*(z)$. Multiplying both sides by $1/q_2^2$ and using Lemma 5, we have

$$\sqrt{2}ia_1 \times \frac{2^{\frac{5}{2}}}{(\sqrt{2}-1)^2} \pi^2 ia_1 \theta_2(\sqrt{2}i)^8 + (\sqrt{2}ia_1)^2 \times \frac{2^2}{(\sqrt{2}-1)^2} \pi^2 \theta_2(\sqrt{2}i)^8 = 64$$

as $z \rightarrow i\infty$, and since we have

$$\theta_2(\sqrt{2}i)^8 = \frac{(\sqrt{2}-1)^2}{2^3} \frac{\pi^2}{\Gamma\left(\frac{5}{8}\right)^4 \Gamma\left(\frac{7}{8}\right)^4},$$

we obtain (6) in Theorem 1. □

4. Appendix

In this section we give the special values of the elliptic theta functions θ_0 , θ_2 and θ_3 , and the Eisenstein series E_4 and E_6 used in this paper.

4.1. Elliptic theta function

Let $k = k(z) = \theta_2(z)^2/\theta_3(z)^2$. Since

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2} \theta_3(z)^2$$

and

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

we have the identity

$$\theta_3(z)^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda^b(z)\right).$$

In the case $z = i$:

Since the inversion fourmula of gamma function and the equation

$${}_2F_1\left(2a, 1 - 2a; 2c; \frac{1}{2}\right) = \frac{2^{1-2c}\Gamma(2c)\Gamma\left(\frac{1}{2}\right)}{\Gamma(a+c)\Gamma\left(c-a+\frac{1}{2}\right)}, \quad ([\text{EMOT}], 2.1.5, \text{p68}),$$

we have

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)^2} = \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\pi^{\frac{3}{2}}}$$

as $a = \frac{1}{4}$ and $c = \frac{1}{2}$. So we obtain

$$\begin{aligned} \theta_3(i)^4 &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda^b(i)\right)^2 \\ &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}\right)^2 \\ &= \frac{\Gamma\left(\frac{1}{4}\right)^4}{2^2\pi^3}. \end{aligned}$$

Since we have

$$\begin{aligned} \frac{\theta_2(i)^4}{\theta_3(i)^4} &= \lambda^b(i) = \frac{1}{2}, \\ \frac{\theta_0(i)^4}{\theta_3(i)^4} &= 1 - \lambda^b(i) = \frac{1}{2}, \end{aligned}$$

we obtain

$$\theta_0(i)^4 = \theta_2(i)^4 = \frac{\Gamma\left(\frac{1}{4}\right)^4}{2^3\pi^3}.$$

In the case $z = \rho_1 = e^{\frac{\pi i}{3}}$:

Since the equation

$${}_2F_1\left(a + \frac{1}{3}, 3a; 2a + \frac{2}{3}; e^{\frac{\pi i}{3}}\right) = 2\pi e^{\frac{\pi i a}{2}} 3^{-\frac{3a+1}{2}} \frac{\Gamma\left(2a + \frac{2}{3}\right)}{\Gamma\left(a + \frac{1}{3}\right)\Gamma\left(a + \frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)},$$

([\text{EMOT}], 2.9 (55), p105),

we obtain

$$\theta_3(\rho_1)^4 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; e^{\frac{\pi i}{3}}\right)^2 = 2^2 3^{-\frac{3}{2}} \pi e^{\frac{\pi i}{6}} \frac{1}{\Gamma\left(\frac{2}{3}\right)^2 \Gamma\left(\frac{5}{6}\right)^2}$$

as $a = \frac{1}{6}$. Since we have

$$\begin{aligned}\frac{\theta_2(\rho_1)^4}{\theta_3(\rho_1)^4} &= \lambda^b(\rho_1) = \rho_1, \\ \frac{\theta_0(\rho_1)^4}{\theta_3(\rho_1)^4} &= 1 - \lambda^b(\rho_1) = e^{\frac{5\pi i}{3}},\end{aligned}$$

we obtain

$$\begin{aligned}\theta_0(\rho_1)^4 &= 2^2 3^{-\frac{3}{2}} \pi e^{\frac{11\pi i}{6}} \frac{1}{\Gamma\left(\frac{2}{3}\right)^2 \Gamma\left(\frac{5}{6}\right)^2}, \\ \theta_2(\rho_1)^4 &= 2^2 3^{-\frac{3}{2}} \pi e^{\frac{\pi i}{2}} \frac{1}{\Gamma\left(\frac{2}{3}\right)^2 \Gamma\left(\frac{5}{6}\right)^2}.\end{aligned}$$

In the case $z = \rho_2 = e^{\frac{2\pi i}{3}}$:

Since the properties of elliptic theta functions

$$\begin{aligned}\theta_0(z) &= \theta_3(1+z), \\ e^{\frac{\pi i}{4}} \theta_2(z) &= \theta_2(1+z), \\ \theta_3(z) &= \theta_0(1+z),\end{aligned}$$

we obtain

$$\begin{aligned}\theta_0(\rho_2)^4 &= \theta_3(\rho_1)^4 = 2^2 3^{-\frac{3}{2}} \pi e^{\frac{\pi i}{6}} \frac{1}{\Gamma\left(\frac{2}{3}\right)^2 \Gamma\left(\frac{5}{6}\right)^2}, \\ \theta_2(\rho_2)^4 &= e^{-\pi i} \theta_2(\rho_1)^4 = 2^2 3^{-\frac{3}{2}} \pi e^{-\frac{\pi i}{2}} \frac{1}{\Gamma\left(\frac{2}{3}\right)^2 \Gamma\left(\frac{5}{6}\right)^2}, \\ \theta_3(\rho_2)^4 &= \theta_0(\rho_1)^4 = 2^2 3^{-\frac{3}{2}} \pi e^{\frac{11\pi i}{6}} \frac{1}{\Gamma\left(\frac{2}{3}\right)^2 \Gamma\left(\frac{5}{6}\right)^2}.\end{aligned}$$

In the case $z = \sqrt{2}i$:

Since the equation

$${}_2F_1(a, b; a - b + 1; z) = (1 - z)^{-a} {}_2F_1\left(\frac{a}{2}, \frac{a + 1 - 2b}{2}; a - b + 1; -\frac{4z}{(1 - z)^2}\right),$$

([EMOT], 2.11 (32), p113),

we have

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 3 - 2\sqrt{2}\right) = 2^{-\frac{1}{2}} (\sqrt{2} - 1)^{-\frac{1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; -1\right)$$

as $a = b = \frac{1}{2}$. And since the equation

$${}_2F_1(a, b; 1 + a - b; -1) = 2^{-a} \frac{\Gamma(1 + a - b) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 - b + \frac{a}{2}\right) \Gamma\left(\frac{1}{2} + \frac{a}{2}\right)}, \quad ([EMOT], 2.8 (47), p104),$$

we have

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -1\right) = 2^{-\frac{1}{4}} \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{5}{8}\right) \Gamma\left(\frac{7}{8}\right)}.$$

By the above two equations we obtain

$$\begin{aligned}
\theta_3(\sqrt{2}i)^4 &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda^b(\sqrt{2}i)\right)^2 \\
&= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 3 - 2\sqrt{2}\right)^2 \\
&= 2^{-1}(\sqrt{2} - 1)^{-1} {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; -1\right)^2 \\
&= 2^{-\frac{3}{2}}(\sqrt{2} - 1)^{-1} \pi \frac{1}{\Gamma\left(\frac{5}{8}\right)^2 \Gamma\left(\frac{7}{8}\right)^2}.
\end{aligned}$$

Since we have

$$\begin{aligned}
\frac{\theta_2(\sqrt{2}i)^4}{\theta_3(\sqrt{2}i)^4} &= \lambda^b(\sqrt{2}i) = 3 - 2\sqrt{2} = (\sqrt{2} - 1)^2, \\
\frac{\theta_0(\sqrt{2}i)^4}{\theta_3(\sqrt{2}i)^4} &= 1 - \lambda^b(\rho_1) = 2(\sqrt{2} - 1),
\end{aligned}$$

we obtain

$$\begin{aligned}
\theta_0(\sqrt{2}i)^4 &= 2^{-\frac{1}{2}} \pi \frac{1}{\Gamma\left(\frac{5}{8}\right)^2 \Gamma\left(\frac{7}{8}\right)^2}, \\
\theta_2(\sqrt{2}i)^4 &= 2^{-\frac{3}{2}}(\sqrt{2} - 1) \pi \frac{1}{\Gamma\left(\frac{5}{8}\right)^2 \Gamma\left(\frac{7}{8}\right)^2}.
\end{aligned}$$

4.2. Eisenstein series

Let $\tau \in \mathbb{H}$, $L = \mathbb{Z} \oplus \mathbb{Z}\tau$ and $\mathcal{P}(z)$ is the Weierstrass \mathcal{P} function with the lattice L . By the equation

$$\mathcal{P}'(z)^2 = 4\mathcal{P}^3(z) - g_2(\tau)\mathcal{P}(z) + g_3(\tau),$$

we define $g_2(\tau)$ and $g_3(\tau)$. Since we know the equations

$$g_2(\tau) = \frac{4}{3}\pi^4 E_4(\tau), \quad g_3 = \frac{8}{27}\pi^6 E_6(\tau),$$

we have

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = \frac{2^6}{3^3}\pi^{12} (E_4(\tau)^3 - E_6(\tau)^2).$$

And we know the equation

$$\begin{aligned}
\Delta(\tau) &= (2\pi)^{12} e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{\pi i n\tau})^{24} \\
&= (2\pi)^4 \theta_1'(\tau)^8 \\
&= 2^4 \pi^{12} (\theta_0(\tau)\theta_2(\tau)\theta_3(\tau))^8 \\
&= 2^4 \pi^{12} \theta_2(\tau)^8 \theta_3(\tau)^8 (\theta_3(\tau)^4 - \theta_2(\tau)^4)^2.
\end{aligned}$$

So we obtain

$$\frac{2^6}{3^3}\pi^{12} (E_4(\tau)^3 - E_6(\tau)^2) = 2^4 \pi^{12} (\theta_0(\tau)\theta_2(\tau)\theta_3(\tau))^8.$$

On the otherhand we know two expressions for J as followings:

$$\begin{aligned} J(\tau) &= \frac{4 (1 - \lambda^b(\tau) + \lambda^b(\tau)^2)^3}{27 \lambda^b(\tau)^2 (1 - \lambda^b(\tau))^2} \\ &= \frac{4 \theta_2(\tau)^8 - \theta_2(\tau)^4 \theta_3(\tau)^4 + \theta_3(\tau)^8}{27 \theta_2(\tau)^8 \theta_3(\tau)^8 (\theta_3(\tau)^4 - \theta_2(\tau)^4)^2} \end{aligned}$$

and

$$J(\tau) = \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}.$$

Thus we have

$$\begin{cases} E_4(\tau) &= \theta_2(\tau)^8 - \theta_2(\tau)^4 \theta_3(\tau)^4 + \theta_3(\tau)^8, \\ E_6(\tau) &= \frac{1}{2} (\theta_2(\tau)^4 - 2\theta_3(\tau)^4) (2\theta_2(\tau)^4 - \theta_3(\tau)^4) (\theta_2(\tau)^4 + \theta_3(\tau)^4). \end{cases}$$

Using 4.1, we obtain

$$\begin{aligned} E_4(i) &= \frac{3}{2^6} \frac{\Gamma\left(\frac{1}{4}\right)^8}{\pi^6}, \\ E_4(\rho_1) &= 0, \\ E_4(\rho_2) &= 0, \\ E_4(\sqrt{2}i) &= \frac{5}{2^3} \frac{\pi^2}{\Gamma\left(\frac{5}{8}\right)^4 \Gamma\left(\frac{7}{8}\right)^4}, \\ E_6(i) &= 0, \\ E_6(\rho_1) &= \frac{2^5}{3^3} \frac{\pi^3}{\Gamma\left(\frac{2}{3}\right)^6 \Gamma\left(\frac{5}{6}\right)^6}, \\ E_6(\rho_2) &= \frac{2^5}{3^3} \frac{\pi^3}{\Gamma\left(\frac{2}{3}\right)^6 \Gamma\left(\frac{5}{6}\right)^6}, \\ E_6(\sqrt{2}i) &= \frac{7}{2^4} \frac{\pi^3}{\Gamma\left(\frac{5}{8}\right)^6 \Gamma\left(\frac{7}{8}\right)^6}. \end{aligned}$$

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Correspondences concerning these Technical Reports should be addressed to
Koji Nishida
Graduate School of Science and Technology
Chiba University
Yayoi-cho 1-33, Inage-ku, Chiba-shi, 263-8522 JAPAN.
e-mail: nishida@math.s.chiba-u.ac.jp