Finite Groups with Three Class Lengths

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Abstract

Many analogues between the character degrees of irreducible characters and lengths of conjugacy classes on a group G has already been studied by several authors. Our purpose here is to impose analogous conditions on the lengths of conjugacy classes of G and to describe the group structure under these conditions. Moreover, we will describe the structure of the class-length graph $\Gamma^*(G)$ corresponding to the degree graph $\Gamma(G)$.

1 Introduction

In this paper G always denotes a finite group. Let $\mathrm{cd}(G)$ be the set $\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ where $\mathrm{Irr}(G)$ is the set of all the irreducible characters of G, and $\mathrm{ccl}(G)$ be the set of the lengths of all the conjugacy classes of G.

It is clear that G is abelian if and only if $\mathrm{cd}(G) = \{1\}$ (or $\mathrm{ccl}(G) = \{1\}$). In [13], N. Ito proved that if $\mathrm{ccl}(G) = \{1, m\}$, then m is a power of a prime p and G is a direct product of a Sylow p-subgroup with an abelian group. The results on character degrees corresponding to $\mathrm{ccl}(G) = \{1, m\}$ are also known (see I. M. Isaacs [8, Chap. 12]).

The information obtained for $|\,\mathrm{cd}(G)| \geq 3$ is not as detailed as when $|\,\mathrm{cd}(G)| = 2$. However, many results for $|\,\mathrm{cd}(G)| \geq 3$ have been given by I. M. Isaacs, T. Noritzsch, D. S. Passman, B. Huppert and others. (see [2] [6] [7] [8] [9] [17] [18]). As one of their results, it is known that if G is solvable and $(m, n) = 1$ for all distinct $m, n \in \mathrm{cd}(G)$, then $|\,\mathrm{cd}(G)| \leq 3$, due to B. Huppert and T. Noritzsch (see I. M. Isaacs [8, Problems (12.3)]).

Our main purpose is to obtain the class-length analogue of this without assuming the solvability of G and to generalize the next theorem, due to D. Chillag, S. Dolfi and M. Herzog, which gives a necessary and sufficient condition on G that all the class-lengths of G are prime powers.

Theorem(D. Chillag, S. Dolfi and M. Herzog [7, 33.9]).

(1) Let $\rho^*(G)$ be the set of prime divisors of all elements of $\mathrm{ccl}(G)$. Suppose that each $m \in \mathrm{ccl}(G)$ is a prime power and $|\rho^*(G)| \geq 2$ If G does not have any abelian direct factor $\neq \{1\}$, then the following statements hold:

- *(i) G is solvable and p-nilpotent with abelian Sylow p-subgroups for some prime p.*
- *(ii) $O_{p'}(G)$ is abelian.*
- *(iii) $O_p(G) = Z(G)$.*

(iv) *$P/O_p(G)$ operates fixed point freely on $O_{p'}(G)$. In particular $P/O_p(G)$ is cyclic and $G/Z(G)$ is a Frobenius group with Frobenius kernel $O_{p'}(G) \times O_p(G)/O_p(G)$.*

(v) *$O_{p'}(G)$ is a Sylow q-subgroup of G for some prime $q \neq p$. In particular $|G| = p^a q^b$ and $\rho^*(G) = \{p, q\}$.*

(2) *Suppose conversely that G has all the properties listed in (1). Then the class-lengths of G are only the prime powers 1, $|P/O_p(G)|$, $|O_{p'}(G)|$.*

Before stating our theorems, we will introduce to some notation. The set of primes dividing an integer a is denoted by $\pi(a)$. In particular, $\pi(|G|)$ is denoted by $\pi(G)$. $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G. $Z(G)$ and G' denote the center of G and the commutator subgroup of G respectively. The rest of the notation is standard.

THEOREM 1. *Suppose $\mathrm{ccl}(G) = \{1, m_1, m_2, ..., m_s, n_1, n_2, ..., n_t\}$. Let $m = m_1 m_2 \cdots m_s$ and $n = n_1 n_2 \cdots n_t$ with $m < n$. If $(m, n) = 1$, then the following statements hold:*

(1) (i) *G is solvable and $G = MN \times A$, where (possibly after interchanging m and n) M is an abelian non-normal Hall $\pi(m)$-subgroup, N is an abelian normal Hall $\pi(n)$-subgroup and A is abelian.*

(ii) *$s = t = 1$ and $n \equiv 1 \pmod m$. In particular, $\mathrm{ccl}(G) = \{1, m, n\}$.*

(iii) *$G/Z(G)$ is a Frobenius group with Frobenius kernel $NZ(G)/Z(G)$.*

(iv) *If G does not have non-trivial abelian direct factor, then M is cyclic, $O_{\pi(m)}(G) = Z(G)$ and $N = O_{\pi(n)}(G)$ is the commutator subgroup of G.*

(2) *$\mathrm{cd}(G) = \{1, m\}$. In particular, the degree pattern of G is*

$$(\underbrace{1, ..., 1}_{mc}, \underbrace{m, ..., m}_{kc}),$$

where $n = 1 + km$ and $c = |Z(G)|$.

Remark. There are many groups with $|\,\mathrm{ccl}(G)| = 3$. For example, the symmetric group S_3 of degree 3, the alternating group A_4 of degree 4, a non-abelian group G_{pq} of order pq where p and q are distinct primes, $\mathrm{SL}(2, 3)$ and others. However, it is not true that groups for which $\mathrm{ccl}(G) = \{1, m, n\}$ with $m < n$ always satisfy $\mathrm{cd}(G) = \{1, m\}$. In fact, although

$$\mathrm{ccl}(\mathrm{S}_3) = \{1, 2, 3\}, \quad \mathrm{ccl}(\mathrm{A}_4) = \{1, 3, 4\}, \quad \mathrm{ccl}(G_{pq}) = \{1, p, q\}\ (p < q),$$
$$\mathrm{cd}(\mathrm{S}_3) = \{1, 2\}, \quad \mathrm{cd}(\mathrm{A}_4) = \{1, 3\}, \quad \mathrm{cd}(G_{pq}) = \{1, p\},$$

those of $\mathrm{SL}(2, 3)$ are

$$\mathrm{ccl}(\mathrm{SL}(2, 3)) = \{1, 4, 6\}, \qquad \mathrm{cd}(\mathrm{SL}(2, 3)) = \{1, 2, 3\}.$$

Note that $\mathrm{SL}(2, 3)$ does not satisfy the assumption $(m, n) = 1$ of Theorem 1.

The following corollary is an immediate conclusion from Theorem 1.

Corollary 1.1. *Let* $\mathrm{ccl}(G) = \{1, m_1^{(1)}, \ldots, m_{s_1}^{(1)}, \ldots, m_1^{(r)}, \ldots, m_{s_r}^{(r)}\}$ *and let* $m^{(k)} = m_1^{(k)} \cdots m_{s_k}^{(k)}$ *for* $k = 1, 2, \ldots, r$. *If* $(m^{(i)}, m^{(j)}) = 1$ *for* $i \neq j$, *then* $r \leq 2$.

We may obtain the next results as the special case for Corollary 1.1.

Corollary 1.2. *If* $(m, n) = 1$ *for all* $m, n \in \mathrm{ccl}(G)$ *such that* $m \neq n$, *then* $|\mathrm{ccl}(G)| \leq 3$.

This corollary is the analogous result of $\mathrm{cd}(G)$ described as above. But the solvability of G is not assumed.

Next corollary, which is one of the known results due to [4], follows as an immediate consequence of our main theorem 1 using Tschebyschev's theorem, that is, there exists a prime number between integers n and $2n$ if $n > 1$.

Corollary 1.3. *Suppose* $\mathrm{ccl}(G) = \{1, 2, \ldots, r\}$. *Then* $r \leq 3$. *In particular,* $G/Z(G) \cong \mathrm{S}_3$ *if* $r = 3$.

In the case $r = 2$, see Ishikawa [10]. Corollary 1.3 is the analogue of B. Huppert [7, §32].

Theorem 2 *Suppose* $\mathrm{ccl}(G) = \{1, m, n\}$ *with* $(m, n) = 1$ *and* $m < n$. *Suppose that* G *does not have non-trivial abelian direct factor. Put* $G^* = M^*N^*$ *where* M^* *is a non-central* $\pi(m)$*-subgroup of* G *and* N^* *is a normal* $\pi(n)$*-subgroup of* G. *Then*

$$\mathrm{ccl}(G^*) = \{1,\ |M^* : M^* \cap Z(G)|,\ |N^*|\}.$$

In particular,

$$\mathrm{ccl}(G) = \{1,\ |M : Z(G)|,\ |N|\}$$

where M *is a Hall* $\pi(m)$*-subgroup of* G *and* N *is a Hall* $\pi(n)$*-subgroup of* G.

In Section 3, using Theorem 1 and Theorem 2, we will discuss the conditions of existence on a finite group G that $\mathrm{ccl}(G) = \{1, m, n\}$ with $(m, n) = 1$ and $m < n$ in terms of integers m and n.

We have already stated in Theorem 1 a necessary condition on such a group G. The next result gives a sufficient condition for G.

Theorem 3 *Assume that* G *does not have non-trivial abelian direct factor and let* $\pi \subset \pi(G)$. *Suppose that* G *has the following properties:*

(a) G *has an abelian Hall* π*-subgroup* M.

(b) G *has an abelian normal Hall* π'*-subgroup* N.

(c) $Z(G) = O_\pi(G)$.

(d) $G/Z(G)$ *is a Frobenius group with Frobenius kernel* $NZ(G)/Z(G)$.

Then $\mathrm{ccl}(G) = \{1, |M : Z(G)|, |N|\}$.

Combining parts of Theorem 1 and Theorem 3, we obtain a generalization of the result of D. Chillag, S. Dolfi and M. Herzog which was stated first. Later we became aware that an equivalent result had already been proved by S. Dolfi [5] as the conjugacy-class version of the concept of character-π-separability introduced by O. Manz in [15]

The second section of this paper is devoted to the proofs of Theorems 1, 2 and 3. In Section 4, we will consider an application of Theorem 1 to the graph, and give a relation between the class-length graph $\Gamma^*(G)$ and the degree graph $\Gamma(G)$.

2 Proofs of our theorems

In order to prove Theorem 1, let us recall some well known facts, which can be found in B. Huppert [7].

First we need to introduce some more notation. For any element $g \in G$, let $\sigma_G(g)$ be the set of all prime divisors of class length $|g^G|$. Put $\rho^*(G) = \bigcup_{g\in G} \sigma_G(g)$, that is, the set of prime divisors of all elements of $\mathrm{ccl}(G)$.

Lemma 1 (N. Ito [12]). *Suppose that p and q are distinct primes in $\rho^*(G)$. If no length of any conjugacy class of G is divisible by pq, then (possibly after interchanging p and q) G is p-nilpotent with abelian Sylow p-subgroups.*

Remark. Suppose $\mathrm{ccl}(G) = \{1, m, n\}$ with $(m, n) = 1$. Then by Lemma 1 we may assume that

$$G \text{ is } p\text{-nilpotent with abelian Sylow } p\text{-subgroups} \qquad (*)$$

for every $p \in \pi(m)$(or for every $p \in \pi(n)$). In fact, this follows from:

(i) If G is p-nilpotent with abelian Sylow p-subgroups for every $p \in \pi(m)$, then $\pi(m)$ satisfies $(*)$.

(ii) If there exists some $p \in \pi(m)$ which does not satisfy $(*)$, then since no class length in G is divisible by pq for every $q \in \pi(n)$, it follows from Lemma 1 that G is q-nilpotent with abelian Sylow q-subgroups, and hence $\pi(n)$ satisfies $(*)$.

In the proof of Theorem 1, we may assume that $\pi(m)$ satisfies $(*)$.

We shall improve for any finite group G the result [7, Lemma 33.3] which has been proved under the assumption that G is solvable. This is used to prove Steps 2 and 3 of the proof of Theorem 1.

Lemma 2. *Let π a set of primes. Suppose that the length of any conjugacy class of G is either a π-number or a π'-number or 1. If $g \in G \setminus C_G(O_\pi(G))$, then $|g^G|$ is a π-number.*

Proof. Let $g \in G$. If $|g^G|$ is a π'-number, then by assumption $|C_G(g)|_\pi =$

$|G|_\pi$. Put $C = C_G(g)$ and $N = O_\pi(G)$. Then

$$|CN|_\pi = \frac{|C|_\pi |N|_\pi}{|C \cap N|_\pi} = \frac{|C|_\pi |N|}{|C \cap N|} \geq |C|_\pi = |G|_\pi.$$

Hence $N = C \cap N$, that is, $N \leq C = C_G(g)$. Therefore we obtain $g \in C_G(N)$. If $|g^G| = 1$, then clearly $g \in Z(G) \leq C_G(N)$. Thus the proof of Lemma 2 is complete. ∎

LEMMA 3 (B. Huppert[7, 33.4]). *$\rho^*(G)$ is the set of all prime divisors of $|G : Z(G)|$.*

Let A be a finite abelian group. We put $\hat{A} = \mathrm{Irr}(A)$, the dual group of A.

LEMMA 4 (B. Huppert[7, 18.10, 26.1, 14.5]). *Suppose that G operates on a finite abelian group A. Then the following statements hold:*

(1) If $(|G|, |A|) = 1$, then $A \cong \hat{A}$ as G-set.

(2) If G is abelian, then $\mathrm{cd}(G\hat{A}) = \{\, |a^G| \;\big|\; a \in A \,\}$.

(3) If $(|G|, |A|) = 1$, then $A = [G, A] \times C_A(G)$.

LEMMA 5 (B. Huppert[7, 33.2]). *Suppose $x, y \in G$ such that $xy = yx$ and $(o(x), o(y)) = 1$. Then $\sigma_G(xy) \supseteq \sigma_G(x) \cup \sigma_G(y)$.*

LEMMA 6 (N. Ito[11]). *Let p be a prime. Then the following are equivalent:*

(a) G has an abelian normal Sylow p-subgroup.

(b) G is p-solvable and $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

Remark. Suppose $\mathrm{ccl}(G) = \{1, m, n\}$ with $(m, n) = 1$. Then by Theorem 1 G is solvable and we have $G = MN \times A$ where M is an abelian Hall $\pi(m)$-subgroup, N is an abelian normal Hall $\pi(m)$-subgroup and A is abelian. Hence every Sylow subgroups of G are abelian and so Lemma 6 means that

$$\text{a Sylow } p\text{-subgroup of } G \text{ is normal in } G \iff p \nmid \chi(1) \text{ for all } \chi \in \mathrm{Irr}(G)$$

for any prime p. Moreover it follows from Theorem 1(2) that $\mathrm{cd}(G) = \{1, m\}$ and hence Lemma 6 implies that any Sylow p-subgroup of G is not normal in G for every $p \in \pi(m)$ and any Sylow q-subgroup of G is normal in G for every $q \in \pi(n)$.

Proof of Theorem 1. By the previous reasoning, we can assume without loss of generality that G is p-nilpotent with abelian Sylow p-subgroups for every $p \in \pi(m)$. Denote $\pi(m)$ by π and put $H = \bigcap_{p \in \pi} O_{p'}(G)$. Since G is p-nilpotent, $O_{p'}(G) \geq G'$ for all $p \in \pi$ and hence $H \geq G'$. Also H is a π'-group.

We will now prove the theorem 1 via several steps.

Step 1. *G is π-solvable (and so G has a Hall π-subgroup, say M), M is abelian and H is a unique Hall π'-subgroup of G.*

Proof. Put $\pi = \{p_1, p_2, ..., p_r\}$, $H_i = O_{p_i'}(G)$, $G_0 = G$ and $G_i = H_1 \cap H_2 \cap \cdots \cap H_i = G_{i-1} \cap H_i$ for $i = 1, 2, ..., r$. Then we obtain a normal series of G

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_r = H \supset \{1\}.$$

Since G is p_i-nilpotent for each i, $1 \leq i \leq r$

$$G_{i-1}/G_i = G_{i-1}/(G_{i-1} \cap H_i) \cong G_{i-1}H_i/H_i \leq G/H_i \cong P_i \in \mathrm{Syl}_{p_i}(G)$$

and so G_{i-1}/G_i is a π-group. Also $G_r/\{1\} \cong H$ is π'-group. Hence G is π-solvable and $|G|_{\pi'} = |H|$. Thus H is a Hall π'-subgroup of G. Since H is normal in G, H is a unique Hall π'-subgroup of G. Moreover, since G is π-solvable, G has a Hall π-subgroup M. Then

$$M' \leq M \cap G' \leq M \cap H = \{1\}.$$

Therefore M is abelian. The proof of Step 1 is now complete.

Step 2. *Let $\bar{G} = G/C_G(H)$. Then $\bar{M}$ is normal in $\bar{G}$.*

Proof. By Step 1 G is π-solvable, hence $\bar{G}$ is π-solvable. If $g \in G \setminus C_G(H)$, then by Lemma 2 $|g^G|$ is a π'-number since $H = O_{\pi'}(G)$. Hence $|\bar{g}^{\bar{G}}|$ is either a π'-number or 1 and so every class-length of $\bar{G}$ is prime to every $p \in \pi$. Thus by Lemma 3 $|\bar{G} : Z(\bar{G})|$ is not divisible by every $p \in \pi$, hence $Z(\bar{G})$ contains a Hall π-subgroup of $\bar{G}$, and so $\bar{M} \leq Z(\bar{G})$. This implies that $\bar{M}$ is normal in $\bar{G}$ as required.

Step 3. *H is abelian and G is solvable.*

Proof. Suppose that H is not abelian. Let $x \in H \setminus Z(H)$. Then by Lemma 2 $|x^G|$ is a π'-number, and hence $C_G(x) \geq M^g$ for some $g \in G$. Since $MC_G(H)$ is normal in G by Step 2 and also $C_G(x) \geq C_G(H)$, we obtain

$$M \leq MC_G(H) \leq (MC_G(H))^g = M^g C_G(H) \leq C_G(x).$$

This forces $H = Z(H) \cup C_H(M)$. Hence $H = Z(H)$ or $C_H(M)$. The first case is excluded since H is not abelian. Thus $H = C_H(M)$. Then $G = M \times H$. Hence no length of any conjugacy class of G is a π-number. This is a contradiction.

Therefore H is abelian and so we obtain $G'' = \{1\}$ since $H \geq G'$. In particular, G is solvable. Hence the proof of Step 3 is complete.

Since H is abelian, we can put $H = N \times A$ where N is an abelian Hall $\pi(n)$-subgroup of G and A is an abelian Hall $\pi(mn)'$-subgroup of G.

Step 4. *M is not normal in G, N is normal in G and $G = MN \times A$.*

Proof. If M is normal in G, then $G = M \times H$ is abelian, a contradiction. Therefore M is not normal in G.

Since $(|N|, |A|) = 1$, N and A are characteristic subgroups of G and so both of them are normal in G. Thus MN is a subgroup of G and A is a unique Hall $\pi(mn)'$-subgroup of G. If $g \in MN$, then $A \leq C_G(g)$ since $p \nmid |g^G|$ for any $p \in \pi(mn)'$. Hence MN and A commute element by element. Since $(|MN|, |A|) = 1$, we obtain

$$G = MH = M(N \times A) = MN \times A.$$

The proof of Step 4 is complete.

Statement (1)(i) has been proved by Steps 1, 2, 3 and 4.

Remark. Set $G = K \times A$ where G is a finite group, K a subgroup and A an abelian subgroup of G. Then it is easy to see that $\mathrm{ccl}(G) = \mathrm{ccl}(K)$ and $\mathrm{cd}(G) = \mathrm{cd}(K)$. Also since $Z(G) = Z(K) \times A$, we have

$$G/Z(G) = (K \times A)/(Z(K) \times A) \cong K/Z(K).$$

So a unique normal Hall subgroup of $G/Z(G)$ (if exists) is mapped to a unique normal Hall subgroup of $K/Z(K)$.

Therefore, from now on, we may assume that G does not have non-trivial abelian direct factor, and that $G = MN$ where M is a non-normal abelian Hall $\pi(m)$-subgroup of G and $N = H$ is a normal abelian Hall $\pi(n)$-subgroup of G.

STEP 5. *$Z(G) = O_\pi(G)$, $G' = N$.*

Proof. Since N is abelian, by Lemma 4 it follows that $N = [M, N] \times C_N(M)$, and hence $G = MN = M[M, N] \times C_N(M)$. Since G does not have any abelian direct factor, this yields $C_N(M) = \{1\}$. Therefore $Z(G) \cap N = \{1\}$, and hence $Z(G)$ is a π-group, which implies $Z(G) \leq O_\pi(G)$. Conversely,

$$[O_\pi(G), G] \leq O_\pi(G) \cap G' \leq O_\pi(G) \cap N = \{1\},$$

and hence $Z(G) \geq O_\pi(G)$ which proves $Z(G) = O_\pi(G)$.

Furthermore, since

$$G' \leq N = [M, N] \leq [G, G] = G'.$$

we have $G' = N$ and the proof of Step 5 is complete.

STEP 6. *Let $x \in M \setminus Z(G)$ and $y \in N \setminus Z(G)$ $(= N \setminus \{1\})$. Then $|x^G| = |N|$ and $|y^G| = |M : Z(G)|$.*

Proof. If $g \in C_N(x)$, then by Lemma 5 $\sigma_G(gx) \supseteq \sigma_G(g) \cup \sigma_G(x)$. Since M is abelian, $|x^G|$ is a π'-number and so $\pi' \supseteq \sigma_G(x) \neq \emptyset$. This implies that $|(gx)^G|$ is a π'-number, and hence $\sigma_G(g) = \emptyset$. Thus $g \in N \cap Z(G) = \{1\}$, so $C_N(x) = \{1\}$. Therefore we obtain $|x^G| = |G : C_G(x)| = |N : C_N(x)| = |N|$.

Similarly, if $g \in C_M(y)$, then by Lemma 5 $\sigma_G(gy) \supseteq \sigma_G(g) \cup \sigma_G(y)$. Since N is abelian, $|y^G|$ is a π-number and so $\pi \supseteq \sigma_G(y) \neq \emptyset$. This implies that $|(gy)^G|$ is a π-number, and hence $\sigma_G(g) = \emptyset$. Thus $g \in M \cap Z(G) = Z(G)$, so

$C_M(y) \leq Z(G)$. Conversely, it is clear that $C_M(y) \geq Z(G)$. Therefore we have $C_M(y) = Z(G)$ and so $|y^G| = |G : C_G(y)| = |M : C_M(y)| = |M : Z(G)|$ as required.

Put $m_1 = |M : Z(G)|$ and $n_1 = |N|$. Since $|y^G| = m_1$ for every $y \in N \setminus \{1\}$ and N is normal in G, there exists an integer k such that $n_1 = 1 + km_1$. Therefore $n_1 \equiv 1 \pmod{m_1}$. In particular, $m_1 < n_1$ and

$$|G| = |MN| = |M||N| = cm_1(1 + km_1)$$

where $c = |Z(G)|$.

STEP 7. *$s = t = 1$, $m < n$ and $n \equiv 1 \pmod{m}$.*

Proof. Let $g \in G \setminus (M \cup N)$ and put $g = xy$ for some $x \in M \setminus \{1\}$ and $y \in N \setminus \{1\}$.

Suppose that $|g^G|$ is a π-number. Then it is clear that $|g^G| \leq |M : Z(G)| = m_1$. Also since $g^h = (xy)^h = xy^h$ for all $h \in M$, we have, by Step 6,

$$|g^G| \geq |g^M| = |y^M| = |M : Z(G)| = m_1.$$

Thus we have $|g^G| = m_1$ and hence $s = 1$.

Next suppose that $|g^G|$ is a π'-number. If $x \in G \setminus Z(G)$, then similarly as above, we have $|g^G| = |x^N| = |N| = n_1$. If $x \in Z(G)$, then we have $|g^G| = |y^G| = m_1$ since $g^h = xy^h$ for all $h \in G$. This is a contradiction. Therefore we obtain $t = 1$. In particular, $m = m_1 < n_1 = n$ and so $n \equiv 1 \pmod{m}$ as required.

Statement (1)(ii) follows from Step 7.

STEP 8. *Let $\bar{G} = G/Z(G)$. Then $\bar{G}$ is a Frobenius group with Frobenius kernel $\bar{N}$ and M is cyclic.*

Proof. Let $g \in G$. First we claim that $|g^G| = m$ if and only if $g \in (Z(G) \times N) \setminus Z(G)$.

If $|g^G| = m$, then clearly $C_G(g) \geq Z(G) \times N$. But since

$$|C_G(g)| = \frac{|G|}{|g^G|} = \frac{|MN|}{|M : Z(G)|} = |Z(G)||N|,$$

we have $C_G(g) = Z(G) \times N$, which implies $g \in Z(G) \times N \setminus Z(G)$.

Conversely, if $g \in (Z(G) \times N) \setminus Z(G)$, then $|g^G|$ is a π-number since $C_G(g) \geq N$, and so $|g^G| = m$ as claimed.

Similarly, we obtain that $|g^G| = n$ if and only if $g \in M^h \setminus Z(G)$ for some $h \in G$. Therefore we have a partition of G such that

$$\begin{aligned} G &= \big(Z(G)\big) \cup \big((Z(G) \times N) \setminus Z(G)\big) \cup \big(\bigcup_{h \in G} M^h \setminus Z(G)\big) \\ &= \big(Z(G) \times N\big) \cup \big(\bigcup_{h \in G} M^h\big). \end{aligned}$$

Next we claim that if $M \neq M^h$ for $h \in G$, then $M \cap M^h = Z(G)$.

For, suppose $M \neq M^h$ so that $M \cap M^h \geq Z(G)$ since $Z(G) = O_\pi(G)$. On the other hand, if $g \in M \cap M^h$, then $|g^G| = 1$ or n. Since M and M^h are abelian, $C_G(g)$ contains M and M^h, hence $M \subset MM^h \subseteq C_G(g)$, and so $|g^G| = |G : C_G(g)| < |G : M| = n$. This implies $|g^G| = 1$, that is, $g \in Z(G)$ and thus $M \cap M^h \leq Z(G)$. Therefore we have $M \cap M^h = Z(G)$ as claimed.

Hence $\bar{G}$ is a Frobenius group. In particular, $\bar{N}$ is a Frobenius kernel of $\bar{G}$. Moreover, since $\bar{M} = M/Z(G)$ is an abelian Frobenius complement, so $\bar{M}$ is cyclic.

We can now choose generators $x_1, ..., x_d$ of M and an integer e such that

$$M = <x_1> \times <x_2> \times \cdots \times <x_d>,$$
$$Z(G) = <x_1^e> \times <x_2> \times \cdots \times <x_d>.$$

Then $<x_2> \times \cdots \times <x_d>$ is an abelian direct factor of G and hence $<x_2> \times \cdots \times <x_d> = \{1\}$. Therefore M is cyclic and the proof of Step 8 is complete.

Statement (1)(iii) and (iv) are immediate from Steps 5, 6 and 8.

STEP 9. $\mathrm{cd}(G) = \{1, m\}$. *In particular, the degree pattern of G is*

$$(\underbrace{1, ..., 1}_{cm}, \underbrace{m, ..., m}_{ck})$$

where $c = |Z(G)|$ and $n = 1 + km$.

Proof. Put $\hat{N} = \mathrm{Irr}(N)$. Then it follows from Lemma 4 that

$$\mathrm{cd}(G) = \mathrm{cd}(MN) = \mathrm{cd}(M\hat{N}) = \{\, |y^M| \;\big|\; y \in N\} = \{1, m\}.$$

Since $G' = N$ by Step 5, the number of linear characters of G is $|G : G'| = |M| = cm$ and hence the number of irreducible characters of G with degree m is ck since

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G| = cm(1 + km).$$

as desired.

Statement (2) follows from Step 9 and therefore the proof of Theorem 1 is now complete. ∎

Proof of Theorem 2. Let M be a Hall $\pi(m)$-subgroup of G which contains M^*, and N be a unique Hall $\pi(n)$-subgroup of G so that $N^* \leq N$. Let $g \in G^*$.

(i) Suppose $|g^G| = m$, then since $C_G(g) = Z(G) \times N$, we have

$$C_{G^*}(g) = G^* \cap C_G(g) = M^*N^* \cap (Z(G) \times N).$$

Let $h \in C_{G^*}(g)$ and put $h = x_1y_1 = x_2y_2$ for some $x_1 \in M^*$, $y_1 \in N^*$, $x_2 \in Z(G)$, $y_2 \in N$. Then

$$x_2^{-1}x_1 = y_2y_1^{-1} \in Z(G)M^* \cap NN^* \leq M \cap N = \{1\},$$

and so $x_1 = x_2 \in M^* \cap Z(G)$ and $y_1 = y_2 \in N^* \cap N = N^*$. Hence $h = x_1 y_1 \in (M^* \cap Z(G)) \times N^*$, so $C_{G^*}(g) \leq (M^* \cap Z(G)) \times N^*$. Conversely, since $M^* \cap Z(G) \leq Z(G) < C_G(g)$ and $N^* \leq N < C_G(g)$, we have $(M^* \cap Z(G)) \times N^* \leq G^* \cap C_G(g) = C_{G^*}(g)$. Therefore $C_{G^*}(g) = (M^* \cap Z(G)) \times N^*$ and hence

$$|g^{G^*}| = |M^* N^* : (M^* \cap Z(G)) \times N^*| = |M^* : M^* \cap Z(G)|.$$

(ii) Suppose $|g^G| = n$. Then since $C_G(g) = M^h$ for some $h \in G$, g is a π-element. Also G is solvable and so G^* is solvable. Thus there exists a Hall $\pi(m)$-subgroup M_g^* of G^* such that $M_g^* \ni g$. Since M^* is a Hall $\pi(m)$-subgroup of G^*, M^* and M_g^* are conjugate. Hence M_g^* is abelian and $M_g^* \leq C_{G^*}(g)$. On the other hand, since

$$|C_{G^*}(g)| = |G^* \cap M^h| \leq |G^*|_{\pi(m)} = |M^*| = |M_g^*|,$$

this implies $C_{G^*}(g) = M_g^*$. Therefore we have

$$|g^{G^*}| = |M^* N^* : M_g^*| = |N^*|.$$

In particular, since $Z(G) = O_{\pi(m)}(G)$ by Theorem 1, we have $M > Z(G)$ and hence $\mathrm{ccl}(G) = \{\, 1,\ |M : Z(G)|,\ |N| \,\}$. The proof of Theorem 2 is complete. ∎

Proof of Theorem 3. Put $\bar{G} = G/Z(G)$. As $N, G/N$ are abelian, so G is solvable, hence so is $\bar{G}$. The order of a Frobenius complement of $\bar{G}$ is equal to $|\bar{G} : \bar{N}| = |\bar{M}|$, so $\bar{M}$ is a Frobenius complement of $\bar{G}$ since any subgroup of order $|\bar{M}|$ is conjugate to $\bar{M}$. Hence

$$\bar{G} = \bar{N} \cup (\bigcup_{\bar{h} \in \bar{G}} \bar{M}^{\bar{h}}),$$

and we obtain a partition of G

$$G = (N \times Z(G)) \cup (\bigcup_{h \in G} M^h).$$

Take $g \in G$.

Suppose $g \in (N \times Z(G)) \setminus Z(G)$. Since $N \times Z(G)$ is abelian, we have $C_G(g) \geq N \times Z(G)$. Now $|g^G| = |g^M| = |M : C_M(g)|$ as $g^G = g^{NM} = g^M$. Since $\bar{M}$ operates $\bar{N}$ fixed point free, $C_{\bar{M}}(\bar{g}) = \{1\}$ and hence $C_M(g) \leq Z(G)$. Otherwise $Z(G) \leq M$ as $Z(G) = O_\pi(G)$, hence $C_M(g) \geq Z(G)$. Therefore we have $C_M(g) = Z(G)$ and so $|g^G| = |g^M| = |M : C_M(g)| = |M : Z(G)|$.

Suppose $g \in M^h \setminus Z(G)$ for some $h \in G$. Since M^h is abelian, $C_G(g) \geq M^h$. Now $|g^G| = |g^N| = |N : C_N(g)|$ since $g^G = g^{M^h N} = g^N$. As $\bar{M}$ operates $\bar{N}$ fixed point free, $C_{\bar{N}}(\bar{g}) = \{1\}$ and hence $C_{N \times Z(G)}(g) \leq Z(G)$. As $Z(G) = O_\pi(G)$, we get

$$C_N(g) = C_{N \times Z(G)}(g) \cap N \leq Z(G) \cap N = \{1\},$$

and hence $C_N(g) = \{1\}$. Therefore

$$|g^G| = |N : C_N(g)| = |N|.$$

Thus we obtain $\mathrm{ccl}(G) = \{1,\ |M : Z(G)|,\ |N|\}$ and the proof is complete. ∎

3 Existence of a group with three class-lengths

Throughout this section, we will assume that G does not have non-trivial abelian direct factor, and that $\mathrm{ccl}(G) = \{1, m, n\}$ with $(m, n) = 1$ and $m < n$.

Let M be a Hall $\pi(m)$-subgroup of G and let N be a Hall $\pi(n)$-subgroup of G. Then M is abelian non-normal in G and N is abelian normal in G, as stated in Theorem 1.

Take $q \in \pi(n)$ and let $Q \in \mathrm{Syl}_q(G)$. Then it follows from Lemma 6 that Q is normal in G, and so we have $\mathrm{ccl}(MQ) = \{1, m, |Q|\}$ and $|Q| \equiv 1 \pmod{m}$ by Theorem 1 and Theorem 2. Therefore, in studying how the structure of N and $\mathrm{ccl}(G)$ are related, it seems to be important to consider the case that n is a prime power.

From now on, we may assume that $N = Q \in \mathrm{Syl}_q(G)$ and $n = q^e$ where q is prime and $e > 0$.

3.1 We shall now construct an example of a group G with $\mathrm{ccl}(G) = \{1, m, q^e\}$, as a semi-direct product of an elementary abelian group Q of order q^e by a cyclic group M of order m. Here since $q^e \equiv 1 \pmod{m}$, we can put $q^e = 1 + km$ for some integer $k > 0$.

Let F be a finite field with q^e elements. Then the additional group F^+ is elementary abelian of order q^e. Also since the multiplicative group $F^\times$ is a cyclic group of order $q^e - 1$, $F^\times$ contains the subgroup $<\sigma>$ of order m. We define the action of $<\sigma>$ on F^+ by $a^x = xa$ for all $a \in F^+$ and every $x \in <\sigma>$. Let G be the semi-direct product of F^+ by $<\sigma>$. Then, for $(x, a), (y, b) \in G$ where $x, y \in <\sigma>$ and $a, b \in F^+$, $(y, b)^{-1}(x, a)(y, b) = (y^{-1}, -b^{y^{-1}})(xy, a^y + b) = (x, -b^x + a^y + b) = (x, ya + (1 - x)b)$, and hence $|(x, a)^G| \le |F^+| = q^e$. If $x \ne 1$, then $|(x, a)^G| \ge |(x, a)^{F^+}| = |F^+| = q^e$, and thus $|(x, a)^G| = q^e$. Also it is clear that $|(1, a)^G| = m$ or 1. Therefore we have $\mathrm{ccl}(G) = \{1, m, q^e\}$.

By the mentioned above, we obtain the following result.

Corollary 3.1. Let $n = \prod_{i=1}^{r} {q_i}^{e_i}$ be the prime number decomposition of n. There exists a finite group G satisfying $\mathrm{ccl}(G) = \{1, m, n\}$ with $(m, n) = 1$ and $m < n$ if and only if ${q_i}^{e_i} \equiv 1 \pmod{m}$ for $i = 1, 2, \cdots, r$.

3.2 Suppose that G has a cyclic Sylow q-subgroup Q. Since Q is the only Sylow q-subgroup of G, every q-element of G lies in Q. In particular, every element of order q lies in Q. Since Q is cyclic, the number of elements of order q is equal to $q - 1$, and these $q - 1$ elements form an union of conjugacy classes of the same length m. This implies $q \equiv 1 \pmod{m}$, which is a necessary condition for that G satisfying our assumption has a cyclic Sylow q-subgroup.

We can show that $q \equiv 1 \pmod{m}$ is also a sufficient condition. First we want to show the following proposition.

Proposition 3.2. *Let A be a finite cyclic group and let $\sigma \in \mathrm{Aut}(A) \setminus \{1\}$. If $(o(\sigma), |A|) = 1$, then $<\sigma>$ operates fixed point freely on A. In particular, if $H = <\sigma>A$ is the semi-direct product of A by $<\sigma>$, then* $\mathrm{ccl}(H) = \{1, o(\sigma), |A|\}$.

Proof. If $<\sigma>$ does not operate fixed point freely on A, then there exists an element of $<\sigma>$ such that it does not fixed point freely on A and its order is prime. Hence it suffices to prove that every element of prime order of $<\sigma>$ operates fixed point freely on A. Therefore we may assume that $m = o(\sigma)$ is prime.

Put $M = <\sigma>$ and $\bar{A} = A/C_A(M)$. We define the action of M on $\bar{A}$ by $\bar{a}^\tau = \overline{a^\tau}$ for all $a \in A$ and for every $\tau \in M$. Clearly this is well-defined. Let $\tau \in M$ and $a \in A$. If $\bar{a} \in C_{\bar{A}}(\tau)$, then $a^{-1}a^\tau \in C_A(M)$ and hence

$$a^{-1}a^\tau = (a^{-1})^\tau a^{\tau^2} = \cdots = (a^{-1})^{\tau^{m-1}} a^{\tau^m}.$$

This implies $(a^{-1}a^\tau)^m = 1$ and hence $a^\tau = a$. Thus we conclude that $\bar{a} \in C_{\bar{A}}(\tau)$ if and only if $a \in C_A(\tau)$.

We next claim that M operates fixed point freely on $\bar{A}$. Take any $\tau \in M \setminus \{1\}$. Then since $M = <\tau>$, $C_A(M) = C_A(\tau)$ and hence it follows from above argument that $\bar{a}^\tau = \bar{a}$ if and only if $\bar{a} = 1$. Therefore τ operates fixed point freely on $\bar{A}$, as claimed.

Thus, by Theorem 3, we have $\mathrm{ccl}(M\bar{A}) = \{1, |M| = m, |\bar{A}|\}$, and $M\bar{A} = (\bigcup_{\bar{g} \in M\bar{A}} M^{\bar{g}}) \cup \bar{A}$. Hence since $M\bar{A} \cong H/C_A(M)$, we obtain $H = (\bigcup_{g \in H} M^g \times C_A(M)) \cup A$. Now, since $H = MA$ and A is abelian, so $C_A(M) \leq Z(H)$.

If $x \in M^g \times C_A(M)$ for some $g \in H$, then $C_H(x) \geq M^g$, and hence $|x^H|$ is either a $\pi(A)$-number or 1. Also if $x \in A$, then $C_H(x) \geq A$, and so $|x^H|$ is either a $\pi(M)$-number or 1. Thus we have $\mathrm{ccl}(H) = \{1, m', n'\}$ where m' is a $\pi(M)$-number and n' is a $\pi(A)$-number. Again since $M\bar{A} \cong H/C_A(M)$, so $m' \geq m = |M|$ and hence $m' = m$. Moreover it follows from Theorem 2 that $n' = |A|$ ∎

We now suppose that $q \equiv 1 \pmod{m}$ and put $q = 1 + km$ for some integer k. Then the automorphism group of a cyclic group Q of order q^e $(e = 1, 2, ...)$ is a cyclic group of order $\varphi(q^e) = q^{e-1}(q-1) = q^{e-1}km$. Hence $\mathrm{Aut}(Q)$ has an element of order m, and so, by Proposition 3.2, we obtain $\mathrm{ccl}(<\sigma>Q) = \{1, m, q^e\}$. Therefore we conclude that $q \equiv 1 \pmod{m}$ is a sufficient condition for that G has a cyclic Sylow q-subgroup.

3.3 Now we give a rule with respect to Sylow q-subgroups G. Let $Q \in \mathrm{Syl}_q(G)$ and let

$$Q = \underbrace{\mathbb{Z}/q\mathbb{Z} \times \cdots \times \mathbb{Z}/q\mathbb{Z}}_{c_1} \times \underbrace{\mathbb{Z}/q^2\mathbb{Z} \times \cdots \times \mathbb{Z}/q^2\mathbb{Z}}_{c_2} \times \cdots \times \underbrace{\mathbb{Z}/q^k\mathbb{Z} \times \cdots \times \mathbb{Z}/q^k\mathbb{Z}}_{c_k}.$$

Let Q_{k-1} be a subset of Q such that orders of elements are at most q^{k-1}. Then Q_{k-1} is a characteristic subgroup of Q, and hence Q_{k-1} is normal in G. By Theorem 2 $\mathrm{ccl}(MQ_{k-1}) = \{1, m, |Q_{k-1}| = q^{e-c_k}\}$, hence $q^{e-c_k} \equiv 1 \pmod{m}$ and so $q^{c_k} \equiv q^{c_k}q^{e-c_k} = q^e \equiv 1 \pmod{m}$. Similarly we obtain $q^{c_j} \equiv 1 \pmod{m}$ $(j = 1, 2, ..., k)$. Hence $q^d \equiv 1 \pmod{m}$ where $d = (c_1, c_2, ..., c_k, e)$ is the greatest common divisor of $c_1, c_2, ..., c_k$ and exponent e.

We now suppose that $q^f \equiv 1 \pmod{m}$ and $q^j \not\equiv 1 \pmod{m}$ for some integer f and every $j\,(= 1, 2, ..., f-1)$.

Then $c_1, c_2, ..., c_k, e$ are multiples of f. Hence we have

$$\begin{aligned}Q = &\underbrace{\mathbb{Z}/q\mathbb{Z} \times \cdots \times \mathbb{Z}/q\mathbb{Z}}_{f} \times \cdots \times \underbrace{\mathbb{Z}/q\mathbb{Z} \times \cdots \times \mathbb{Z}/q\mathbb{Z}}_{f} \\ &\times \underbrace{\mathbb{Z}/q^2\mathbb{Z} \times \cdots \times \mathbb{Z}/q^2\mathbb{Z}}_{f} \times \cdots \times \underbrace{\mathbb{Z}/q^2\mathbb{Z} \times \cdots \times \mathbb{Z}/q^2\mathbb{Z}}_{f} \\ &\times \cdots \times \underbrace{\mathbb{Z}/q^k\mathbb{Z} \times \cdots \times \mathbb{Z}/q^k\mathbb{Z}}_{f} \times \cdots \times \underbrace{\mathbb{Z}/q^k\mathbb{Z} \times \cdots \times \mathbb{Z}/q^k\mathbb{Z}}_{f}.\end{aligned}$$

Hence if a cyclic group M of order m operates fixed point freely on

$$\underbrace{\mathbb{Z}/q^i\mathbb{Z} \times \cdots \times \mathbb{Z}/q^i\mathbb{Z}}_{f}$$

for each $i\,(= 1, 2, ...)$, then we can define the action of M on the above Q operating fixed point freely.

Remark Let f be as 3.3. If e is a prime, then we can determine the structure of the Sylow q-subgroup Q of G by 3.3. Suppose that

$$Q = \mathbb{Z}/q^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/q^{e_r}\mathbb{Z},$$

so that $e = e_1 + \cdots + e_r$. Since e is a prime, f is equal to e or 1.

(i) Suppose that $f = e$. Then since r is a multiple of f by 3.3 and since $r \leq e$, we have $r = e$. This implies that $e_1 = \cdots = e_r = 1$. Therefore Q is an elementary abelian group. In particular, G is a semi-direct product of Q with a cyclic group M of order m.

(ii) Suppose that $f = 1$, so that $q \equiv 1 \pmod{m}$. Then by 3.2 M operates fixed point freely on $\mathbb{Z}/q^{e_i}\mathbb{Z}$ for any $i = 1, ..., r$, so we can define an action of M on

$$Q = \mathbb{Z}/q^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/q^{e_r}\mathbb{Z}$$

such that fixed point freely. So all abelian groups of order q^e are able to be the Sylow q-subgroup of G such that $\mathrm{ccl}(G) = \{1, m, q^e\}$.

4 The class-length graph

In Section 2 we introduced the notation $\rho^*(G)$ which is the set of prime divisors of all elements of $\mathrm{ccl}(G)$. Similarly we put $\rho(G)$ be the set of prime divisors of all elements of $\mathrm{cd}(G)$.

We define the class-length graph $\Gamma^*(G)$, whose vertices are the primes in $\rho^*(G)$ and $p, q \in \rho^*(G)$ are connected by an edge in $\Gamma^*(G)$ if there exists some $g \in G$ such that pq divides $|g^G|$. Similarly we define the degree graph $\Gamma(G)$, whose vertices are the primes in $\rho(G)$ and $p, q \in \rho(G)$ are connected in $\Gamma(G)$ if there exists some $\chi \in \mathrm{Irr}(G)$ such that $pq \mid \chi(1)$. Let $n(\Gamma^*(G))$ and $n(\Gamma(G))$ denote the number of connected components of the class-length graph $\Gamma^*(G)$ and the degree graph $\Gamma(G)$ respectively.

In [14] and [16], O. Manz, R. Staszewski and W. Willems proved that $n(\Gamma(G)) \leq 3$ and if G is solvable, then $n(\Gamma(G)) \leq 2$. G. Alfandary [1] proved the following theorem for the class-length graph analogue of this result. We give another short proof using Corollary 1.1.

THEOREM 4(G. Alfandary [1]). *Let G be a finite group. Then the following statements hold:*

(1) $n(\Gamma^*(G))$ is at most 2.

(2) If $n(\Gamma^*(G)) = 2$, then G is solvable.

(3) If G is a non-abelian simple group, then $n(\Gamma^*(G)) = 1$.

Proof. If $n(\Gamma^*(G)) = 3$, then we can write

$$\mathrm{ccl}(G) = \{1, a_1, ..., a_r, b_1, ..., b_s, c_1, ... c_t\}, \\ (a, b) = (b, c) = (c, a) = 1$$

where $a = a_1 \cdots a_r$, $b = b_1 \cdots b_s$, $c = c_1 \cdots c_t$. However, by Corollary 1.1, there exists no such a group, a contradiction. By a similar argument it is impossible to be $n(\Gamma^*(G)) \geq 4$. Therefore (1) holds.

If $n(\Gamma^*(G)) = 2$, it follows from Theorem 1 that G is solvable. So (2) holds.

(3) follows from (1) and (2). Thus the proof of Theorem 4 is complete. ∎

Combining Theorem 1 and Theorem 4, we obtain a correspondence between $\mathrm{ccl}(G)$ and $\mathrm{cd}(G)$.

COROLLARY 4.1 *Let G be a finite group. Then the following statements hold:*

(1) If $n(\Gamma^*(G)) = 2$, then $n(\Gamma(G)) = 1$.

(2) If $n(\Gamma(G)) \geq 2$, then $n(\Gamma^*(G)) = 1$.

Proof. (1) If $n(\Gamma^*(G)) = 2$, then $\mathrm{cd}(G) = \{1, m\}$ by Theorem 1(2), which implies $n(\Gamma(G)) = 1$.

(2) Since $n(\Gamma^*(G)) \leq 2$ by Theorem 4, the statement (2) follows the statement (1). ∎

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