COMMUTATIVITY OF OPERATORS

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ABSTRACT. For two bounded positive linear operators a, b on a Hilbert space, we give conditions which imply the commutativity of a, b. Some of them are related to well-known formulas for indefinite elements, e.g., $(a+b)^n = \sum_k \binom{n}{k} a^{n-k} b^k$ etc. and others are related to the property of operator monotone functions. We also give a condition which implies the commutativity of a C*-algebra.

1. INTRODUCTION

Ji and Tomiyama ([3]) give a characterization of commutativity of C*-algebra, where they also give a condition that two positive operators commute. For bounded linear operators on a Hilbert space $\mathcal{H}$, we slightly generalize their result as follows:

Theorem 1. *Let a and b be self-adjoint operators on $\mathcal{H}$. Then the following are equivalent.*

(1) $ab = ba$.
(2) $\exp(a+b) = \exp(a)\exp(b)$.
(3) *There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that*

$$(a + t_i b)^n = \sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k$$

for $i = 1, 2, \ldots, n-1$.

(4) *There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that*

$$a^n - (t_i b)^n = (a - t_i b) \sum_{k=0}^{n-1} a^{n-k-1} (t_i b)^k$$

for $i = 1, 2, \ldots, n-1$.

DePrima and Richard([2]), and Uchiyama([9],[10]) independently prove that, for any positive operators a and b, the following conditions are equivalent:

(1) $ab = ba$.

(2) $ab^n + b^n a$ is positive for all $n \in \mathbb{N}$.

We give a little weakend condition for two operators commuting.

Ji and Tomiyama, and Wu([12]) use a commutativity condition of two operators and a gap of monotonicity and operator monotonicity of functions to characterize commutativity of C*-algebras. With a similar point of view, we can get the following result:

Theorem 2. *Let A be a unital C^*-algebras. Then the following are equivalent.*

(1) *A is commutative.*

(2) *There exists a continuous, increasing functions f on $[0, \infty)$ such that f is not concave and operator monotone for A.*

(3) *Whenever positive operators a and b satisfy $ab + ba \geq 0$, $ab^2 + b^2 a \geq 0$.*

2. Proof of Theorem 1

Lemma 3. *Let a and b be self-adjoint operators on $\mathcal{H}$, and f a continuous function on the spectrum $Sp(a)$ of a. Then $ab = ba$ implies that $f(a)b = bf(a)$.*

Proof. We can choose a sequence $\{p_n\}$ of polynomials which converges to f uniformly on $\mathrm{Sp}(a)$. So we have

$$f(a)b = \lim_{n \to \infty} p_n(a)b = \lim_{n \to \infty} bp_n(a) = bf(a).$$

□

Lemma 4. *Let a, b be self-adjoint operators on $\mathcal{H}$ and k a positive integer. If $a^k ba = a^{k+1} b$, then $ab = ba$.*

Proof. We put p the orthogonal projection of $\mathcal{H}$ onto $\mathrm{Ker}(a)$. We remark that

$$\mathrm{Ker}\ a = \mathrm{Ker}\ a^2 = \cdots = \mathrm{Ker}\ a^{k+1}, \quad pa = ap = 0.$$

Since

$$0 = a^k bap = a^{k+1} bp = a^{k+1}(1-p)bp,$$

we have $(1-p)bp = 0$. The self-adjointness of b implies

$$b = pbp + (1-p)b(1-p).$$

So we have

$$\begin{aligned} ab - ba &= (p + (1-p))(ab - ba) = (1-p)(ab - ba) - pba \\ &= (1-p)(ab - ba) - pbpa = (1-p)(ab - ba). \end{aligned}$$

Since $a^k(ab - ba) = 0$, we can get $ab = ba$. □

Proo of Theorem 1. (1)⇒(2), (1)⇒(3) and (1)⇒(4) are trivial.

(2)⇒(1) The element $\exp(a+b)$ is self-adjoint, so we have

$$\exp(a)\exp(b) = \exp(b)\exp(a).$$

We apply Lemma 3 for the function $f(x) = \log x$ on $\mathrm{Sp}(a)$. Since $\log(\exp(a)) = a$, we have

$$a\exp(b) = \exp(b)a.$$

Repeated the same argument, we can show $ab = ba$.

(3)⇒(1) Since $(a + t_i b)^n$ is self-adjoint, we have

$$\sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k = \sum_{k=0}^{n} \binom{n}{k} t_i^k b^k a^{n-k}, \qquad (i = 1, 2, \ldots, n-1).$$

This means that

$$\begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-2} \\ 1 & t_2 & \cdots & t_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n-1} & \cdots & t_{n-1}^{n-2} \end{pmatrix} \begin{pmatrix} \binom{n}{1}(a^{n-1}b - ba^{n-1}) \\ \binom{n}{2}(a^{n-2}b^2 - b^2a^{n-2}) \\ \vdots \\ \binom{n}{n-1}(ab^{n-1} - b^{n-1}a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So we have $a^{n-1}b = ba^{n-1}$. When n is even, we have $ab = ba$, by using Lemma 3 and the fact $a = (a^{n-1})^{1/n-1}$.

We assume that n is odd. Then we have

$$a^2 b = (a^{n-1})^{2/n-1} b = b(a^{n-1})^{2/n-1} = ba^2.$$

If we apply the same argument for the relation

$$\begin{aligned} &(a + t_i b)^n \\ =&a^n + t_i(a^{n-1}b + a^{n-2}ba + \cdots + ba^{n-1}) + t_i^2(\cdots) = \sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k, \end{aligned}$$

then we can get

$$a^{n-1}b + a^{n-2}ba + \cdots + ba^{n-1} = na^{n-1}b.$$

Using the commutativity of a^2 and b, we have

$$a^{n-1}b = a^{n-2}ba.$$

By Lemma 4, it follows that $ab = ba$.

(4)$\Rightarrow$(1) By using the same argument as (3)$\Rightarrow$(1), we can get that a coefficient of t_i^{n-1} vanishes, that is,

$$ab^{n-1} - bab^{n-2} = 0.$$

By Lemma 4, we can get $ab = ba$. $\square$

Remark 5. On the implication (2)$\Rightarrow$ (1), the following srtonger result is known for self-adjoint matrices (see [7] and [8]). If self-adjoint matrices a, b satisfy the condition

$$\mathrm{Trace}(\exp(a+b)) = \mathrm{Trace}(\exp(a)\exp(b)),$$

then $ab = ba$.

3. Operator monotone functions

Let f be a continuous function on $[0,\infty)$. We call f a matrix monotone (resp. matrix concave) function of order n if it satisfies the following condition:

$$\begin{aligned} &a, b \in M_n(\mathbb{C}), 0 \le a \le b \Rightarrow f(a) \le f(b) \\ \text{(resp. } &a, b \in M_n(\mathbb{C}), 0 \le a \le b, 0 \le t \le 1 \\ &\Rightarrow f(ta + (1-t)b) \le tf(a) + (1-t)f(b)). \end{aligned}$$

When f is matrix monotone of order n for any n, f is called operator monotone. We call a function f operator monotone for a C*-algebra A if, for $a, b \in A$, $0 \le a \le b$ implies $0 \le f(a) \le f(b)$. The following fact is well-known([5]:Theorem 2.1). Here we give a different proof of this.

Lemma 6. *If $f : [0,\infty) \longrightarrow [0,\infty)$ is continuous and matrix monotone of order $2n$, then f is matrix concave of order n.*

Proof. For $a, b \in M_n(\mathbb{C})^+$ and $0 \le t \le 1$, we put

$$X = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, Y = \begin{pmatrix} \sqrt{t} & -\sqrt{1-t} \\ \sqrt{1-t} & \sqrt{t} \end{pmatrix} \in M_{2n}(\mathbb{C}).$$

Then we have

$$
\begin{aligned}
Y^*XY &= \begin{pmatrix} ta+(1-t)b & \sqrt{t(1-t)}(b-a) \\ \sqrt{t(1-t)}(b-a) & (1-t)a+tb \end{pmatrix} \\
&\leq \begin{pmatrix} ta+(1-t)b+\epsilon & 0 \\ 0 & (1-t)a+tb+\frac{t(1-t)}{\epsilon}(a-b)^2 \end{pmatrix}
\end{aligned}
$$

for any positive number ϵ. By the assumtion for f, we can get

$$
\begin{aligned}
& Y^*f(X)Y = f(Y^*XY) \\
& \leq \begin{pmatrix} f(ta+(1-t)b+\epsilon) & 0 \\ 0 & f((1-t)a+tb+\frac{t(1-t)}{\epsilon}(a-b)^2) \end{pmatrix}.
\end{aligned}
$$

Since ϵ is arbitrary, we have

$$tf(a)+(1-t)f(b) \geq f(ta+(1-t)b).$$

$\square$

As an application of this lemma, we can see that the exponential function $\exp(\cdot)$ is increasing and convex but not matrix monotone of order 2. By Theorem 2, we can get another proof of Wu's result [12].

Let f be an operator monotone function on $(0,\infty)$, that is, f is a matrix monotone function on $(0,\infty)$ of order n for any $n \in \mathbb{N}$. Then f has the analytic continuation on the upper half plane $H_+ = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ and also has the analytic continuation on the lower half plane H_- by the reflection across $(0,\infty)$. By Pick function theory, it is known that f is represented as follows:

$$f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{\lambda + z} dw(\lambda),$$

where $\beta \geq 0$ and w is a positive measure with

$$\int_0^\infty \frac{\lambda}{1+\lambda} dw(\lambda) < +\infty$$

(see [1]:page 144). We denote by P_+ the closed right half plane $\{z \in \mathbb{C} \mid \mathrm{Re} z \geq 0\}$ and by $\overline{C}(S)$ the closed convex hull of a subset S of $\mathbb{C}$. We consider the case that $f(0) \geq 0$. Then we can easily check $f(P_+) \subset P_+$. For $a \in B(\mathcal{H})$, we denote by $W(a)$ its numerical range

$$\overline{\{(a\xi,\xi) \mid \|\xi\| = 1\}} \subset \mathbb{C}.$$

By Kato's theorem ([4]:Theorem 7), if W(a) is contained in P_+, then we have

$$W(f(A)) \subset \overline{C}(f(P_+)).$$

Proposition 7. *Let $a, b \in B(\mathcal{H})$ be positive and f, f_n be operator monotone functions from $[0,\infty)$ to $[0,\infty)$.*

(1) *If $ab + ba \geq 0$, then $af(b) + f(b)a \geq 0$.*
(2) *If $\mathrm{Sp}(b) \subset f_n([0,\infty))$, $af_n^{-1}(b) + f_n^{-1}(b)a \geq 0$ for all n and $\bigcap_n \overline{C}(f_n(P_+)) \subset \mathbb{R}$, then $ab = ba$.*

Proof. (1) We may assume that a is invertible, replacing a by $a + \epsilon$ ($\epsilon > 0$). Then we can define the new inner product on $\mathcal{H}$ by

$$\langle \xi, \eta \rangle = (a\xi, \eta), \qquad \xi, \eta \in \mathcal{H}.$$

It suffices to show that the positivity of $\mathrm{Re}b$ with respect to $\langle \cdot, \cdot \rangle$ implies the positivity of $\mathrm{Re}f(b)$ with respect to $\langle \cdot, \cdot \rangle$. Since $\mathrm{Re}b \geq 0$ is equivalent to

$$W(b) = \overline{\{\langle b\xi, \xi \rangle \mid \langle \xi, \xi \rangle = 1\}} \subset P_+$$

and $W(f(b)) \subset \overline{C}(f(P_+)) \subset P_+$, we have $\mathrm{Re}f(b) \geq 0$.

(2) In the same setting in (1), if we get $W(b) \subset \mathbb{R}$, this implies $ab = ba$. By the argument of (1) and the assumption, we have

$$W(f_n^{-1}(b)) \subset P_+ \text{ and } W(b) = W(f_n(f_n^{-1}(b)) \subset \overline{C}(f_n(P_+))$$

for any n. So we have $W(b) \subset \bigcap_n \overline{C}(f_n(P_+)) \subset \mathbb{R}$. $\square$

In [11], Uchiyama defines the function $u(t)$ on $[-a_1, \infty)$ as follows:

$$u(t) = (t + a_1)^{\gamma_1}(t + a_2)^{\gamma_2} \cdots (t + a_k)^{\gamma_k},$$

where $a_1 < a_2 < \ldots < a_k$, $\gamma_j > 0$, and he shows that the inverse function $f(x) = u^{-1}(x)$ becomes operator monotone on $[0,\infty)$ if $\gamma_1 \geq 1$. We assume that $f(0) \geq 0$ (i.e., $a_1 \leq 0$) and

$$\gamma = \sum_{j:a_j \leq 0} \gamma_j > 1.$$

Then $f(z)$ is a holomorphic function from D into D, where $D = \mathbb{C} \setminus (-\infty, 0] = \{z \in \mathbb{C} \setminus \{0\} \mid -\pi < \mathrm{arg}z < \pi\}$. For $z = re^{i\theta}$ ($0 < \theta < \pi/2$), we set $z + a_j = r_j e^{i\theta_j}$ ($j = 1, 2, \ldots, k$). Then we have

$$0 < \theta_k < \cdots < \theta_1 < \pi \text{ and } \mathrm{arg}u(z) = \sum_{j=1}^{k} \gamma_j \theta_j \geq \gamma\theta.$$

This means that $|\mathrm{arg} f(z)| < \frac{1}{\gamma}|\mathrm{arg} z|$ if $0 < |\mathrm{arg} z| < \pi/2$. Since

$$\overline{C}(f(P_+)) \subset \overline{C}(\{z \in D \mid |\mathrm{arg} z| < \frac{\pi}{2\gamma}\}) \subset \{z \in D \mid |\mathrm{arg} z| \leq \frac{\pi}{2\gamma}\}$$

$$\overline{C}(f^2(P_+)) \subset \overline{C}(f(\{z \in D \mid |\mathrm{arg} z| \leq \frac{\pi}{2\gamma}\})) \subset \{z \in D \mid |\mathrm{arg} z| \leq \frac{\pi}{2\gamma^2}\}$$

$$\cdots$$

$$\overline{C}(f^n(P_+)) \subset \overline{C}(f(\overline{C}(f^{n-1}(P_+))) \subset \{z \in D \mid |\mathrm{arg} z| \leq \frac{\pi}{2\gamma^n}\},$$

we can get

$$\bigcap_{n=1}^{\infty} \overline{C}(f^n(P_+)) \subset \mathbb{R}.$$

Corollary 8. *Let $a, b \in B(\mathcal{H})$ be positive and the function u have the following form:*

$$u(t) = (t + a_1)^{\gamma_1}(t + a_2)^{\gamma_2} \cdots (t + a_k)^{\gamma_k},$$

where $a_1 < a_2 < \ldots < a_k$, $\gamma_j > 0$, $a_1 \leq 0$, $\gamma_1 \geq 1$ and $\sum_{j:a_j \leq 0} \gamma_j > 1$. If $au^n(b) + u^n(b)a \geq 0$ for all $n \in \mathbb{N}$, then we have $ab = ba$.

Proof of Theorem 2. (1)⇒(2) and (1)⇒(3) are trivial.

(2)⇒(1) If A is not commutative, then there exists a irreducible representation π of A on a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} > 1$. Let $\mathcal{K}$ be a 2-dimensional subspace of $\mathcal{H}$. By Kadison's transitivity theorem(see [6]), for any positive operator $T \in B(\mathcal{K})(\cong M_2(\mathbb{C}))$, we can choose a positive element $a \in A$ such that $\pi(a)|_{\mathcal{K}} = T$. By the assumption and Lemma 5, f is not matrix monotone of order 2. This means that we can choose $S, T \in B(\mathcal{K})$ such that

$$0 \leq S \leq T \text{ and } f(S) \not\leq f(T).$$

So there exist $a, b \in A$ such that

$$0 \leq a \leq b \text{ and } \pi(a) = S, \pi(b) = T.$$

Since $f(S) = f(\pi(a)) = \pi(f(a))$ and $f(T) = f(\pi(b)) = \pi(f(b))$, this contradicts to the operator monotonicity of f for A.

(3)⇒(1) Let a, b be positive in A. For a sufficiently large positive number t, $(a + t)b + b(a + t)$ becomes positive. By the assumption, we have

$$(a + t)^{2^n} b + b(a + t)^{2^n} \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

By Corollary 7, we have $(a + t)b = b(a + t)$, i.e., $ab = ba$. Therefore A is commutative. □

Using the same method as the proof of (3)$\Rightarrow$(1), we can see the following condition (4) also becomes an equivalent condition in Theorem 2:

(4) Whenever positive operators a and b satisfy $au(b) + u(b)a \geq 0$ for a function u as in Corollary 7, $au^2(b) + u^2(b)a \geq 0$.

Acknowledgement. The authors express their thanks to Professors M. Uchiyama, J. Tomiyama and F. Hiai for giving many useful comments.

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