COLORINGS OF TORUS KNOTS AND THEIR TWIST-SPINS BY ALEXANDER QUANDLES OVER FINITE FIELDS

SOICHIRO ASAMI AND KEN'ICHI KUGA

Abstract. Colorings of torus knots and twist-spun torus knots by general Alexander quandles over finite fields are explicitly determined. It is then observed that, for some standard Mochizuki's cocycles, a large class of twist-spun torus knots with non-trivial colorings have trivial cocycle invariants.

1. Introduction

Coloring of knots is a classic topic in knot theory as Fox considered p-colorings of knots as early as in 1960's. It also has a new aspect of statu-sum argument based on the fruitful analogy between knot diagrams and Feynman diagrams, or 2-dimensional models of statistical mechanics. It is interesting in this respect that the state-sum argument in knot theory immediately generalizes to surface diagrams at least for the case of quandle colorings.

Finding all the colorings of a given class of knot diagrams or surface diagrams, however, is an intractable task in general. Even for the torus knots, explicit colorings found so far with some generality are those for torus knots of type $(2, n)$ by dihedral quandles in [2], and those for general torus knots by Fox's p-colorings [1].

In this paper we explicitly write down all the colorings of torus knots and their twist-spun surface knots by general Alexander quandles over finite fields.

Our original intention was to explicitly calculate all the cocycle invariants, defined by Carter et al [2] for twist-spun torus knots using Mochizuki's 3-cocycles [7] of quandle cohomology of Alexander quandles over finite fields. It turns out, however, that the colorings of torus knots are more complicated than we originally expected, and they also have some interesting features. Roughly, there are two distinct cases: $t^m = 1$ ($t^n = 1$) or $t^m \neq 1 \neq t^n$, where $t \in \mathbb{F}_q$ is the element defining Alexander quandle structure on the finite field of order q, and m, n are the coprime integers defining the (m, n)-torus knot $T(m, n)$. Fox's p-colorings are rather trivially included in the first case: they correspond to prime q and $t = -1$.

When $t^m = 1$ we observe that the cocycle invariants of twist-spun torus knots for some standard Mochizuki's cocycles are always trivial even when the surface diagrams have non-trivial colorings. As the colorings are complicated, general calculation of the cocycle invariants for the case $t^m \neq 1 \neq t^n$ seems hard. We content ourselves by calculating some simple examples using a computer (Maple). It is seen, however, that the invariants frequently turn out to be trivial. We also provide an example with non-trivial cocycle invariants.

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2. Colorings of torus knots by Alexander quandles

A set Q with a binary operation $*: Q\times Q \to Q$ is called a quandle if the following three conditions are satisfied: (i) $x * x = x$ for any $x \in Q$, (ii) for any $x, y \in Q$ there is a unique $z \in Q$ such that $x = z * y$, (iii) $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in Q$.

Let Λ be a ring with identity 1, and t an invertible element in Λ. Then a quandle structure on Λ is defined by the binary operation $x * y := tx + (1-t)y$. This is the Alexander quandle structure on Λ. When $\Lambda = \mathbb{F}_q$ the finite field of order $q = p^e$ and $t \in \mathbb{F}_q^\times$ the Alexander quandle $(\mathbb{F}_q, *)$ is denoted by $\mathbb{F}_q[X]/(X-t)$ to specify t. These are the quandles we use in this paper as the target quandle of knot colorings.

Let D be a diagram of an oriented knot K, and Σ the set of arcs of D. Given a quandle Q, a Q-*coloring* for D is a map $C : \Sigma \to Q$ which satisfies $C(\gamma) = C(\alpha) * C(\beta)$ at each crossing, where $\alpha, \gamma \in \Sigma$ are under-arcs on the right and left of the over-arc $\beta \in \Sigma$, respectively. If a Q-coloring uses only one color we say that it is trivial.

Let $T(m, n)$ be the (m, n)-torus knot in the 3-space $\mathbb{R}^3$. We may assume m and n are coprime positive integers by possibly changing the orientation of S^3, in other words, possibly replacing the defining element t of the Alexander quandle operation by t^{-1}. The diagram $D_{T(m,n)}$ of $T(m, n)$ is obtained by closing the m fold product of the element Δ in the n-braid group B_n, where $\Delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ with $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ the standard generators of B_n. See Figure 1. We denote the j-th crossing from left of the i-th braid Δ by x_{ij} $(1 \leqq i \leqq m, 1 \leqq j \leqq n-1)$

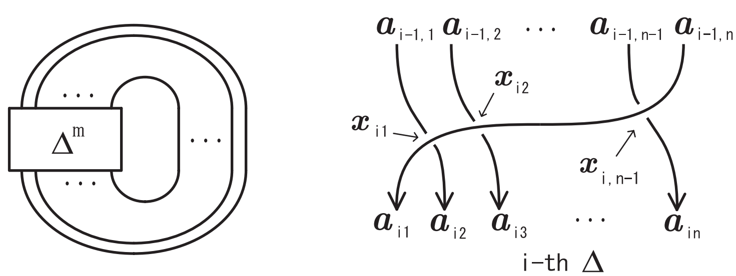

Figure 1

Let $\mathbb{F}_q = \mathbb{F}_q[X]/(X-t)$ be the Alexander quandle over the finite field of order $q = p^e$, $p = char(\mathbb{F}_q)$. As described above the quandle structure is defined by $t \in \mathbb{F}_q$ as $x * y := tx - (1-t)y,\ x, y \in \mathbb{F}_q$. We will explicitly find all $\mathbb{F}_q$-colorings of the diagram $D_{T(m,n)}$ of the (m, n)-torus knot $T(m, n)$. Beginning with the colors $a_{01}, a_{02}, \ldots, a_{0n} \in \mathbb{F}_q$ of the top over-arcs of the n strings, the color of the j-th over-arc a_{ij}, $(j = 1, 2, \ldots, n$, counting from left to right) after the i applications of Δ is uniquely determined. See the right of Figure 1. The relation between these colors are described by

$$\begin{pmatrix} a_{i+1,1} \\ a_{i+1,2} \\ \vdots \\ a_{i+1,n} \end{pmatrix} = A \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix}, \quad \text{where} \quad A = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline t & & \mathbf{0} & 1-t \\ & \ddots & & \vdots \\ \mathbf{0} & & t & 1-t \end{array}\right).$$

The i-th power A^i of A is explicitly found by decomposing A as

$$A = M + N$$

where

$$M = t\left(\begin{array}{cccc|c} 0 & \cdots & & 0 & 1 \\ \hline 1 & & & \mathbf{0} & 0 \\ & \ddots & & & \vdots \\ \mathbf{0} & & & 1 & 0 \end{array}\right) \quad \text{and} \quad N = (1-t)\left(\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline 0 & \cdots & 0 & 1 \\ & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right).$$

Lemma 2.1. $A^i = N(I + M + \cdots + M^{i-1}) + M^i,\ i = 1, 2, \ldots, m.$

Proof. The derivation of this formula is possible since the commutation relations between the matrices M and N are rather simple. Once obtained, however, verification is immediate by induction on i. □

For later calculations, it is convenient to consider the second index j of the color variables $a_{i\,j}$ modulo n, i.e., j as an element in $\mathbb{Z}/n\mathbb{Z}$, and we assume this from now on. Hence $a_{i0} = a_{in}, a_{i,-j} = a_{i,n-j}$, etc. Then in terms of the components a_{ij}, Lemma 2.1 is written as

$$a_{ij} = (1-t)\sum_{k=0}^{i-1} t^k a_{0,-k} + t^i a_{0,j-i},\ j = 1, 2, \ldots, n. \tag{2.1}$$

In particular the consistency condition of the coloring imposed on the initial colors $a_{01}, a_{02}, \ldots, a_{0n}$ are written as the equations:

$$a_{0j} = (1-t)\sum_{k=0}^{m-1} t^k a_{0,-k} + t^m a_{0,j-m},\ j = 1, 2, \ldots, n. \tag{2.2}$$

To solve the equations (2.2), we first consider the 3-term relations derived from (2.2).

$$t^{-m} a_{0,j+2m} - (t^{-m} + 1) a_{0,j+m} + a_{0j} = 0,\ j = 1, 2, \ldots, n. \tag{2.3}$$

As the eigenvalues of the transition matrix for the sequence $\{a_{0,j+\ell m}\}$, $\ell = 0, 1, \ldots, n$, we have two separate cases to consider: $t^m = 1$ or $t^m \neq 1$.

Case 1. Assume that $t^m = 1$. The relations (2.3) reduce to $a_{0,j-2m} - a_{0,j-m} = a_{0,j-m} - a_{0j}$ and yield

$$a_{0,1+\ell m} = a + \ell d \quad (\ell = 0, 1, \ldots, n), \tag{2.4}$$

where we set $a := a_{01}, d := a_{0,1+m} - a_{01}$. As $(m, n) = 1$, this coloring is consistent if and only if $a_{0,1+nm} = a_{01}$, i.e., $nd = 0$ in $\mathbb{F}_q$. Hence the above coloring (2.4) can be non-trivial only when n is a multiple of $p = char(\mathbb{F}_q)$.

To write general a_{ij} explicitly we go back to the equations 2.1. For this purpose, let $\mu = m^{-1} \in (\mathbb{Z}/n\mathbb{Z})^\times$, the multiplicative inverse of m, i.e., an integer μ such that $\nu n + \mu m = 1$ for some ν. Then taking $\ell = -(k+1)\mu$ or $\ell = -(1+i-j)\mu$ in the equation (2.4), we have explicit expressions for $a_{0,-k}$ and $a_{0,j-i}$. Substituting them in the equation (2.1) we get

$$a_{ij} = a + \left(\frac{1-t^{i+1}}{1-t} - jt^i\right)\mu d. \tag{2.5}$$

Case 2. Assume that $t^m \neq 1$. Then the relations (2.3) can be explicitly solved and yield:

$$a_{0,1-\ell m} = a + \frac{t^{-\ell m} - 1}{t^{-m} - 1} d. \tag{2.6}$$

Since $(m,n) = 1$, the above coloring (2.6) is consistent if and only if $a_{0,1-nm} = a_{01}$ and $a_{0,1-m} = a_{0,1-nm-m}$. As $t^m \neq 1$, we find from the equation (2.6) this holds if and only if $t^{mn} = 1$. In this case, the two colors a_{01} and $a_{0,1-m}$ can be chosen arbitrarily and the coloring is uniquely determined by them. Hence suppose $t^m \neq 1$ and $t^{mn} = 1$.

To write general a_{ij} explicitly, we take an integer μ such that $\nu n + \mu m = 1$ for some ν, just as in the first case. Substituting $\ell = (k+1)\mu$ or $\ell = (1+i-j)\mu$ in the equation (2.6), we have explicit expressions for $a_{0,-k}$ and $a_{0,j-i}$. Then substituting the results in the equation (2.1) we get a combination of geometric series with common ratios t and $t^{1-m\mu}$. Now we may suppose $t^n \neq 1$, since the torus knot $T(m,n)$ is symmetric with respect to m and n. Then the ratio $t^{1-m\mu} = t^{n\nu} \neq 1$, since $t^{mn} = 1$ and $(m,n) = 1$. Then after a bit tedious calculation we obtain:

$$\begin{aligned} a_{ij} =& \frac{1}{t^{-m} - 1} \Bigg\{ t^{-m} a - b \\ &+ t^{-m\mu} \left(\frac{1-t}{1-t^{1-m\mu}} - t^{i(1-m\mu)} \left(\frac{1-t}{1-t^{1-m\mu}} - t^{jm\mu} \right) \right) (b-a) \Bigg\}, \end{aligned}$$

where $a = a_{01},\ b = a_{0,n-m+1},$ and $\mu = m^{-1} \in (\mathbb{Z}/n\mathbb{Z})^\times$.

To make the structure of this result clearer we choose the variables $d = b - a,\ u = t^{\mu n}, v = t^{\nu m}$, with μ, ν satisfying $\mu m + \nu n = 1$. We have $t = uv,\ u^m = v^n = 1$. This choice of u, v comes from the isomorphism $\mathbb{Z}/(mn) \cong \mathbb{Z}/(m) \oplus \mathbb{Z}/(n)$

Theorem 2.2. *Let Λ be the finite Alexander quandle $\mathbb{F}_q = \mathbb{F}_q[X]/(X-t)$, and $T(m,n)$ be the torus knot. Then non-trivial Λ-colorings are possible if and only if*

$$\begin{cases} t^m = 1, p|n & (or\ \ t^n = 1, p|m) \quad or \\ t^m \neq 1, t^{mn} = 1 & (or\ \ t^n \neq 1,\ t^{mn} = 1). \end{cases}$$

If this holds, the Λ-colorings of $T(m,n)$ correspond bijectively to arbitrary choices of two colors $(a,d) \in \Lambda \times \Lambda$ by $a_{ij} = a + R(i,j)d$ where

$$R(i,j) = \begin{cases} \left(\dfrac{1-t^{i+1}}{1-t} - jt^i \right) \mu & if \quad t^m = 1 \\ \dfrac{1}{v^{-m} - 1} \left\{ -1 + \dfrac{1}{v} \left(\dfrac{1-uv}{1-u} - u^i \left(\dfrac{1-uv}{1-u} - v^j \right) \right) \right\} & if \quad t^m \neq 1. \end{cases}$$

Let Q be a quandle. The number of the Q-colorings of a knot K is called the Q-coloring number of K. The Q-coloring number is an invariant of K. If a knot K admits the only trivial Q-colorings, the Q-coloring number of K is equal to the cardinality of Q.

Corollary 2.3. *If $T(m,n)$ admits a non-trivial Λ-coloring, the Λ-coloring number of $T(m,n)$ is q^2, and otherwise q.*

3. Colorings of twist-spun torus knots

In this section, we find the colorings of the r-twist-spun (m, n)-torus knot by finite Alexander quandles over finite fields.

A 2-knot is an oriented 2-sphere embedded in $\mathbb{R}^4$ smoothly. For a 2-knot K, we assume that the projection $p : F \to \mathbb{R}^3$ is a generic map. The singularity set of the projection consists of double points, triple points and branch points. Crossing information is indicated in $p(F)$ as follows: Along every double point curve, two sheets intersect locally, one of which is under the other relative to the projection direction of p. Then the under-sheet is broken by the over-sheet. A diagram of F is the image $p(F)$ with such crossing information. Hence, a diagram regarded as a union of disjoint compact, connected surfaces.

Let D be a diagram of a 2-knot F, Σ the set of such connected surfaces in D, and Q a quandle. A coloring of D is a map $C : \Sigma \to Q$ satisfying $C(\gamma) = C(\alpha) * C(\beta)$ at each double curve, where $\alpha, \beta, \gamma \in \Sigma$ are the three sheets meeting at the double curve such that β is the over-sheet, α, γ are the under-sheets which the normal direction of β points α to γ.

Each triple point x of D is assigned the sign $\epsilon(x) = \pm 1$ induced from the orientation in such a way that $\epsilon(x) = +1$ if and only if the ordered triple of the orientation normals of the top, middle, and bottom sheets, respectively, agree with the orientation of $\mathbb{R}^3$. The colors of the sheets near t are determined by three colors $C(\alpha)$,$C(\beta)$ and $C(\gamma)$, where γ is the top sheet, β is the middle sheet from which the orientation normal of γ points, and α is the bottom sheet from which the orientation normals of β and γ point both. The ordered triple $(C(\alpha), C(\beta), C(\gamma))$ is called the color of x and denoted by $C(x) \in Q^3$.

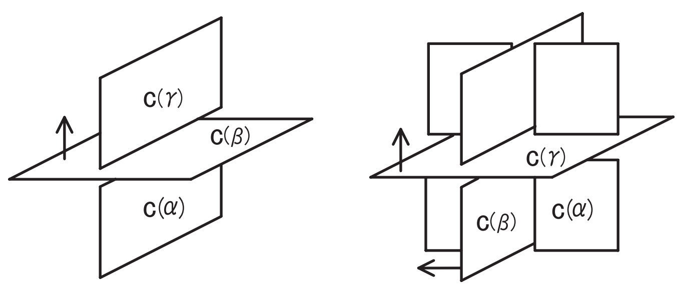

Figure 2

The diagrams we use for the twist-spun torus knots in this paper are standard for this purpose and described as follows. We begin with the diagram $D_{T(m,n)}$ of the torus knot $T(m, n)$ used in the previous section. We cut the right most arc of the arcs closing the braid diagram, i.e., the one whose color variable is a_{01}. See the left of Figure 1. The result is a tangle diagram, say T, with two external endpoints. Place T on the right half of the xy-plane in such a way that the end points e_+ and e_- are at $(0, 1, 0)$ and $(0, -1, 0)$ respectively. By shrinking the body of T, keeping the two external strings long, we can assume the tangle looks like the right half unit circle on the xy-plane with a tiny disk δ near the point $(1, 0)$ containing the body of T . If we rotate this T around the y axis, we get a diagram for the spinning torus knot. This diagram looks almost like a unit sphere, whose equator being the trace of the center of δ. Then we add k twists in this process of rotation in such a way that the sphere becomes an immersed sphere intersecting with itself in r disjoint arcs, say $s_1, \ldots, s_r$, each being almost parallel to the y axis, and the trace of the center of δ becomes a simple closed curve meeting each s_i twice at two distinct points, say p_i and q_i, in a consecutive manner as it goes round the kink corresponding to each twisting. See Figure 3. This diagram is given by Satoh in [1], called Satoh's diagram, and effective in not only the torus knots but also the general knots.

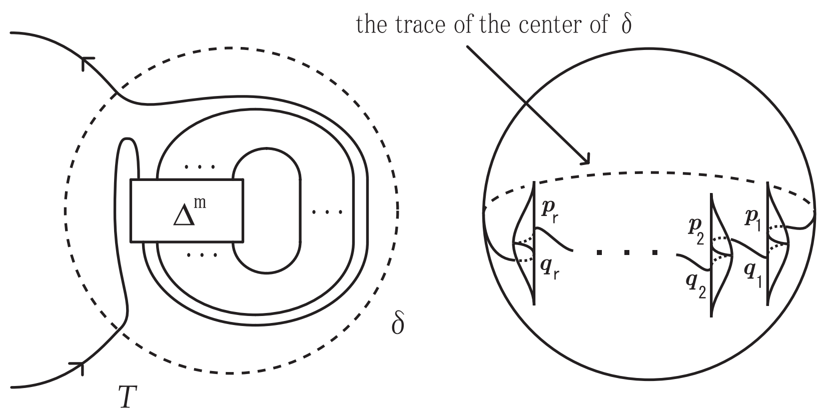

Figure 3

We place the surface knot $\tau^r T(m, n)$ in the 4-dimensional $xyzt$-coordinate space so that the t-coordinates of the points corresponding to the diagram points p_1, q_1, p_2, q_2, $\ldots$, p_r, q_r are decreasing in this order. We fix an orientation of the surface such that the normal direction points to the inside except kinks.

In this diagram, there are two big sheets which are the traces of the two external strings of T. Other sheets are either small sheets near the intersection p_i or narrow rectangles which are traces of the internal arcs of T starting near q_i and going past p_{i+1} and ending near q_{i+1} (or from q_r through p_1 to q_1).

Now we color this diagram using a finite quandle $\mathbb{F}_q$. First important observation is that the colors of two external sheets must be the same. This follows from the explicit calculation using the results of section, but can be shown immediately using the trick in [8] which we reproduce here with some generality.

Lemma 3.1. *Suppose $x * y = x * z$ always implies $y = z$ in a quandle Q. Then for any Q coloring of a tangle T with two external strings, the two external strings have the same color.*

Proof. Let T_1 be the tangle diagram obtained from that of T and a small oriented circle component disjoint from T. Let T_2 be another diagram of the same tangle where the added circle component is large, containing the body of T and cut in two points by the two external strings of T. Fix any coloring of T and trivially extend it to a coloring of T_1. Moving the added small circle under T and making it engulf the body of T, we get a coloring of T_2. During the process, the consistency of the coloring is preserved by changing the colors of the added circle corresponding to the Reidemeister moves of types II and III where the coloring of T remains unchanged. The large added circle component of T_2 is colored with two colors $x, z \in Q$ such that $z = x * y_1 = x * y_2$, where $y_1, y_2 \in Q$ are the colors of the external strings. Hence $y_1 = y_2$ under the assumption of the quandle Q. □

As a result of the above lemma, the almost spherical diagram described above of $\tau^r T(m,n)$ is colored for the most part by a single color, say a, except for the small or narrow sheets in the trace of δ. The large background sheets colored by a will be referred to as shadows in the following.

Near the intersection point p_k, the body of the tangle T goes through the other sheet and breaks it into small sheets corresponding to the bounded regions on the copy of δ centered at p_k. We denote the color of the region described in the diagram below by $s_{i,j,k}$. The arcs appearing in this diagram correspond to the narrow rectangular sheets parallel to the trace of the center of δ whose colors we denote by $a_{i,j,k}$ as in the diagram.

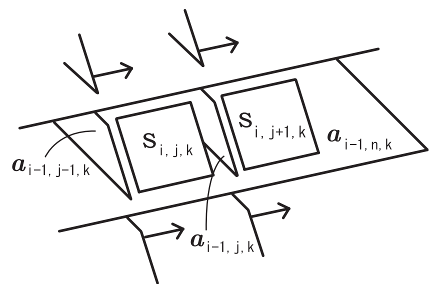

FIGURE 4

Note that the colors $s_{i,j,1}$ are uniquely determined once the colors $a_{i,j,1}$ are specified according to the relations

$$\begin{aligned} s_{i,j+1,1} &= ts_{i,j,1} + (1-t)a_{i-1,j,1} \\ s_{i,1} &= a \end{aligned}$$

From above relations and theorem 2, we have the following.

Lemma 3.2. *When $a_{i,j,1}$ are given by the $a_{i,j}$ in the theorem 2, then $s_{i,j,1}$ are given by the following formula.*

$$s_{i,j,1} = a + C(i,j)d$$

where

$$C(i,j) = \begin{cases} \left(\dfrac{1-t^{j-1}}{1-t} - (j-1)t^{i-1} \right) \mu & \text{if} \quad t^m = 1 \\ \dfrac{u^{i-1}(u-v^{-1})(1-v^{j-1}) + (u^{j-1}v^{j-1}-1)(1-v^{-1})}{(v^{-m}-1)(1-u)} & \text{if} \quad t^m \neq 1. \end{cases}$$

Near the intersection point q_k, the rectangular sheets along the trace of δ are cut by the big sheet colored by a. This forces the coloring appearing the other side to be multiplied by a: $x \mapsto x * a$, without destroying the consistency of the colorings. Then noting

$$(\dots((x * \overbrace{a) * a) * \dots) * a}^{k} = t^k x + (1-t^k)a$$

we see the coloring above is consistent if and only if $t^r = 1$, unless the original coloring of the torus knot was trivial.

Theorem 3.3. *Let Λ be the finite Alexander quandle $\mathbb{F}_q = \mathbb{F}_q[X]/(X-t)$, and $\tau^r T(m,n)$ be the r-twist spun torus knot of type (m,n) in the 4-sphere. Then non-trivial Λ-colorings are possible if and only if*

$$\begin{cases} t^r = 1, t^m = 1, p|n, & (or\ \ t^r = 1, t^n = 1, p|m) \quad or \\ t^r = 1, t^m \neq 1, t^{mn} = 1 & (or\ \ t^r = 1, t^n \neq 1,\ t^{mn} = 1). \end{cases}$$

If this holds, the Λ-colorings of $\tau^r T(m,n)$ correspond bijectively to arbitrary choices of two colors $(a,d) \in \Lambda \times \Lambda$ by $a_{i,j,k} = a + t^k R(i,j)d$, $s_{i,j,k} = a + t^k C(i,j)d$, $(1 \leqq k \leqq r)$.

For a 2-knot also, its Q-coloring number is defined and an invariant of a 2-knot.

Corollary 3.4. *If $\tau^r T(m,n)$ admits a non-trivial Λ-coloring, the coloring number of $\tau^r T(m,n)$ is q^2, and otherwise q.*

Corresponding to each crossing x_{ij} of $D_{T(m,n)}$, the above diagram of $\tau^r T(m,n)$ has $2r$ triple points near p_k and q_k $(1 \leqq k \leqq r)$ which are denoted by $t_k^+(x_{ij})$ and $t_k^-(x_{ij})$ respectively. By the definition of the above diagram of $\tau^r T(m,n)$, the color and the sign of the triple points $t_k^+(x_{ij})$ and $t_k^-(x_{ij})$ are given by

$$\begin{aligned} c(t_k^+(x_{ij})) &= (s_{i,j,k-1}, a_{i-1,j,k-1}, a_{i-1,n,k-1}), \\ c(t_k^-(x_{ij})) &= (a_{i-1,j,k-1}, a_{i-1,n,k-1}, a), \\ \epsilon(t_k^+(x_{ij})) &= +1, \epsilon(t_k^-(x_{ij})) = -1 \end{aligned}$$

respectively.

4. Cocycle invariants of twist-spun torus knots

In this section we calculate the cocycle invariants of some twist-spun torus knots using theorem 3.3 and 3-cocycles for the quandle cohomology groups found by Mochizuki [7].

We recall the cocycle invariants of 2-knots in 4-space $\mathbb{R}^4$ (see [2] for details) and Mochizuki's 3-cocycles. Let Q be a quandle, G a abelian group. We may define the cohomology group $H^*(Q,G)$ for a quandle Q (cf. [2]). A map $\theta : Q^3 \to G$ is called a 3-*cocycle* of Q if it satisfies

(i) $\theta(a,b,c) = 0$ if $a = b$ or $b = c$,
(ii) for any $a,b,c,d \in Q$,

$$\begin{aligned} &\theta(a,c,d) - \theta(a,b,d) + \theta(a,b,c) \\ &= \theta(a*b,c,d) - \theta(a*c,b*c,d) + \theta(a*d,b*d,c*d). \end{aligned}$$

Let D be a diagram of a 2-knot F, x a triple point of D. For a 3-cocycle θ, we define the weight $W_\theta(t,C) = \epsilon(t)\theta(C(t)) \in G$, and also define $W_\theta(C) = \sum_x W_\theta(t,C) \in G$. The cocycle invariant $\Phi_\theta(F)$ of a 2-knot F is defined by

$$\Phi_\theta(F) = \sum_C W_\theta(C),$$

which take value in the group ring $\mathbb{Z}[G]$. It is proved in [2] that $\Phi_\theta(F)$ is an invariant of a 2-knot F which does not depend on the choice of a diagram D of F, and $\Phi_\theta(F) = \Phi_{\theta'}(F)$ if θ and θ' are cohomologous. The cocycle invariant $\Phi_\theta(F)$

of F is called trivial if it is equal to the coloring number. For example, if F is a ribbon 2-knot or admits only trivial colorings, then $\Phi_\theta(F)$ is trivial.

Let Λ be the finite Alexander quandle $\mathbb{F}_q[X]/(X-t)$, G the Abelian group $\mathbb{F}_q$. For non-negative integers a, b, c, the map $F(a,b,c)$ form Λ^3 to G is defined by

$$F(a,b,c) : (x,y,z) \mapsto (x-y)^a(y-z)^b z^c.$$

In [7], Mochizuki proved the following lemma.

Lemma 4.1 ([7])**.** *Let q_i denote powers of the prime p.*

1. *Assume that $t^{q_1+q_2+q_3} = 1$. Then $F(q_1,q_2,q_3)$ is a 3-cocycle.*
2. *Assume that $t^{q_1+q_2} = 1$. Then $F(q_1,q_2,0)$ is a 3-cocycle.*

Case 1. Assume that $t^m = 1$. Using the 3-cocycles $F(q_1,q_2,0)$ and $F(q_1,q_2,q_3)$, we calculate the cocycle invariants of the r-twist-spun torus knot $\tau^r T(m,n)$. We consider the case that $t^r = 1$, $p|n$, that is, $\tau^r T(m,n)$ admit a non-trivial coloring. Since $n = 0$ in $\mathbb{F}_q$, it holds that $\sum_{j=1}^n j^{q_l} = (\sum_{j=1}^n j)^{q_l} = 0$ $(l = 1,2)$. Then we have

$$\sum_{j=1}^{n}\sum_{i=1}^{m} F(q_1,q_2,0)(t_k^+(x_{ij})) = \sum_{j=1}^{n} \frac{-j^{q_2}(\mu d)^{q_1+q_2}}{1-t^{q_1}} m = 0$$

and

$$\sum_{j=1}^{n}\sum_{i=1}^{m} F(q_1,q_2,0)(t_k^-(x_{ij})) = \sum_{j=1}^{n} \frac{j^{q_1}(\mu d)^{q_1+q_2}}{1-t^{q_1}} t^{-q_1} m = 0.$$

If $\tau^r T(m,n)$ admits only trivial colorings then $\Phi_{F(q_1,q_2,0)}(\tau^r T(m,n)) = q$. These results lead us to the conclusion that $\Phi_{F(q_1,q_2,0)}(\tau^r T(m,n))$ is equal to the Λ-coloring number of $\tau^r T(m,n)$. Similarly, it hold that $\Phi_{F(q_1,q_2,q_3)}(\tau^r T(m,n))$ is equal to the Λ-coloring number of $\tau^r T(m,n)$. Therefore, for the 3-cocycles $F(q_1,q_2,0)$ and $F(q_1,q_2,q_3)$, the cocycle invariant of $\tau^r T(m,n)$ is trivial.

Case 2. Assume that $t^m \neq 1$. The calculation easily becomes too time and memory consuming and the following examples are among those rare cases where we can actually finish computing. We may find the definition of the following 3-cocycles in [7] (and our Maple worksheets). The Maple worksheets of our calculation are found in `http://www.math.s.chiba-u.ac.jp/~xasami/Maple.html` .

Let $\Lambda = \mathbb{F}_{2^4}$ be the Alexander quandle over the finite field of order 2^4. Explicitly we take the finite group $\mathbb{F}_{2^4} = \mathbb{Z}_2[X]/(X^4+X^3+1)$, and the quandle structure defined by $t = X$. Then this t generates the multiplicative group $\mathbb{F}_{2^4}^\times$ which is cyclic of order 15. As $t^{mn} = 1, t^m \neq 1, t^n \neq 1$ are required to have non-trivial coloring, the simplest choice is $(m,n) = (3,5)$. Also the twisting number k is required to satisfy $t^k = 1$, the simplest choice is $k = 15$. So we take the surface knot $\tau^{15}T(3,5)$. For this case all the cocycle invariants (in the notation of Mochizuki [6] we have $\Psi(13,2)$, $\Psi(11,4)$, $\Psi(7,8)$) are trivial.

Similarly trivial results are obtained for $\Lambda = \mathbb{F}_{7^2} = \mathbb{F}_7[X]/(X^2+5X+5)$ with the choice $t = 3$. This t has multiplicative order $2^4 3$. Then all the cocycle invariants of the simplest possible choice $\tau^6 T(2,3)$ are trivial.

Some non-trivial results are obtained for the same surface knot $\tau^6 T(2,3)$using the Alexander quandle $\Lambda = \mathbb{F}_{5^2} = \mathbb{F}_5[X]/(X^2+3X+3)$ with the choice $t = X+2$. This t has multiplicative order $2^3 3$. Then all the cocycle invariants of the simplest possible choice $\tau^6 T(2,3)$ are trivial except for the cocycles $\Gamma 1(1,5,5,25)$, $\Gamma 1(5,1,25,5)$, $\Gamma 3(1,1,5,5)$, $\Gamma 3(5,5,25,25)$. For those cocycles the cocycle invariants of $\tau^6 T(2,3)$ are of the form $25I + 300f + 300g \in \mathbb{Z}[\mathbb{F}_{5^2}]$ where (f,g) are respectively (e^1, e^4),

(e^1, e^4), (e^{2X+2}, e^{3X+3}), (e^{2X+4}, e^{3X+1}). Here the exponential e^x represents an element x of the additive group of $\mathbb{F}_{5^2}$ multiplicatively.

For the same finite field $\mathbb{F}_{5^2} = \mathbb{F}_5[X]/(X^2 + 3X + 3)$, If we set the Alexander quandle structure by $t = 2X + 2$. All the cocycle invariants of the simplest possible choice $\tau^{12}T(4,3)$ are trivial.

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Division of Mathematical Sciences and Physics, Graduate School of Science and Technology, Chiba University, 1–33 Yayoi-cho, Inage-ku, Chiba, 263–8522, Japan

E-mail address: xasami@g.math.s.chiba-u.ac.jp

Department of Mathematics and Informatics, Faculty of Science, Chiba University, 1–33 Yayoi-cho, Inage-ku, Chiba, 263–8522, Japan

E-mail address: kuga@math.s.chiba-u.ac.jp